The mixed problem in Lipschitz domains

Russell Brown

Department of Mathematics
University of Kentucky

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This talk reports joint work with several collaborators: Wright, Taylor, Sykes, Ott, M. Mitrea, I. Mitrea, Lanzani, and Capogna.

The mixed problem

We let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain. Suppose that D and N are subsets of the boundary with $D \cup N = \partial\Omega$ and $D \cap N = \emptyset$.
We consider the mixed boundary value problem or Zaremba's problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega) & \end{cases} \qquad (MP)$$

where $(\nabla u)^*$ is the non-tangential maximal function.
Recall that if v is defined in Ω and $x \in \partial\Omega$, we may define the non-tangential maximal function by

$$v^*(x) = \sup\{|v(y)| : y \in \Gamma(x)\}$$

where $\Gamma(x) = \{y \in \Omega : |y - x| < 2dist(y, \partial\Omega)\}$.

Defining boundary values for the solutions

Homogeneity considerations dictate that the trace map should map the Besov space $B^2_{1/2}(\Omega)$ into $L^2(\partial\Omega)$.

This trace theorem does not hold. However, we have the following:

Theorem (Dahlberg and ...)

Suppose that u is harmonic. We have $u \in B^2_{1/2}(\Omega)$ if and only if $u^* \in L^2(\partial\Omega)$.

If $u^* \in L^2(\partial\Omega)$, then u has boundary values in the non-tangential sense,

$$u(x) = \lim_{\Gamma(x)\ni y \to x} u(y), a.e.$$

This theorem implies that the gradient of a solution for the L^2-mixed problem will have boundary values. There are also results in L^p.

The Neumann and regularity problems

The extreme case $D = \emptyset$ is commonly known as the Neumann problem and the case $N = \emptyset$ is referred to as the regularity (in the Dirichlet) problem.

Building on work of Jerison and Kenig [JK82] for $p = 2$, Dahlberg and Kenig obtained results in the optimal range of L^p-spaces.

Theorem (Dahlberg and Kenig [DK87])

We may find an $\epsilon > 0$ so that for $1 < p < 2 + \epsilon$, we have a solution to the Neumann problem which satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f_N\|_{L^p(\partial\Omega)}$$

and a solution of the regularity problem which satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq \|u\|_{L^p_1(\partial\Omega)}.$$

We use L^p_1 to denote the Sobolev space of functions having one derivative in L^p.

Ingredients of the proof

The L^2 results of Jerison and Kenig [JK82] are proven using the Rellich identity and earlier, deep work of Dahlberg on the Dirichlet problem.

When $p = 1$, non-tangential estimates for the Neumann problem fail when f_N is in L^1. However, there are results when the data lies in a Hardy space, H^1. Interpolation between the L^2 and H^1 give the main results of the theorem. A real variable argument extends to $p > 2$.

The key point in the solution of the problem in a Hardy space is to give decay of the solution as we move away from the support of the data. This decay follows from estimates for the Green function.

Both the Rellich identity and Dahlberg and Kenig's techniques for obtaining decay of solutions with data in a Hardy space will reappear in the work below.

The main question.

Our goal is to find conditions on N, D, Ω, p so that we may solve the mixed problem in L^p. This means that when f_N is in $L^p(N)$ and f_D is in $L^p_1(\partial\Omega)$, we may find a unique solution of (MP) which satisfies the estimate

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{L^p(N)} + \|f_D\|_{L^p_1(\partial\Omega)}).$$

Note that we assume that our Dirichlet data f_D is defined on all of the boundary and lies in the Sobolev space $L^p_1(\partial\Omega)$. The set D may not be an extension domain for Sobolev spaces, but a solution of (MP) will give an extension, thus we need some assumption in this direction.

A simple example

Let $\Omega = \{(x, y) : |(x, y)| < 1, y > 0\} \subset \mathbf{R}^2$, put $D = \partial\Omega \cap \{x < 0\}$, and $N = \partial\Omega \setminus D$. If we set $u(x, y) = \mathrm{Re}\,\sqrt{x + iy}$, then u satisfies

$$\begin{aligned} u(x,0) &= 0, \qquad && x < 0 \\ \frac{\partial}{\partial y} u(x,0) &= 0 && x > 0 \end{aligned}$$

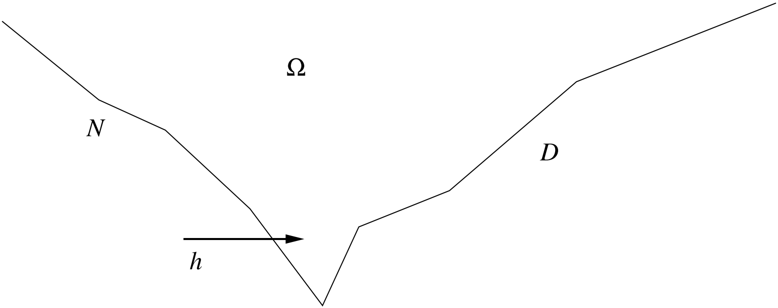
and u is smooth on the curved part of $\partial\Omega$. It is not hard to see that $(\nabla u)^*(x, 0) \approx 1/|x|^{1/2}$ and that u is the unique weak solution of (MP) with nice data. Thus, we cannot hope to prove a theorem on the mixed problem in L^2 that holds in all Lipschitz domains.

A special vector field

A creased domain is a domain Ω with a smooth vector field h which satisfies for some $\delta > 0$,

$$\begin{aligned} h \cdot \nu &> \delta, && \text{a.e. on } D \\ h \cdot \nu &< -\delta, && \text{a.e. on } N. \end{aligned}$$

We need an additional technical condition that implies that the boundary between D and N has Lipschitz regularity and the domain forms an angle of less than π at boundary between D and N.



The Rellich identity

If u is harmonic and sufficiently regular to allow the use of the divergence theorem, we have

$$\int_{\partial\Omega} |\nabla u|^2 h \cdot \nu - 2h \cdot \nabla u \frac{\partial u}{\partial \nu} \, d\sigma = \int_{\Omega} |\nabla u|^2 \operatorname{div} h - 2Dh\nabla u \cdot \nabla u \, dy.$$

Using that $h \cdot \nu$ has opposite signs on D and N, energy estimates, and some manipulations we obtain an *a priori* estimate for the mixed problem

$$\int_{\partial\Omega} |\nabla u|^2 \, dy \leq C(\int_N f_N^2 \, d\sigma + \int_D f_D^2 + |\nabla_t f_D|^2 \, d\sigma)$$

In the case of a graph domain, a homotopy argument allows to reduce to the case where D is flat and then reflection allows us to obtain existence. This uses that the angle between D and N is less than π. From this we can obtain results in bounded domains.

Results

This gives

Theorem (Brown 1994)

Let Ω, N, and D be a creased domain. If f_N is in $L^2(N)$ and f_D in $L^2_1(D)$, then the mixed problem (MP) for L^2 has a unique solution which satisfies the estimate

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \leq C(\|f_D\|_{L^2_1(D)} + \|f_N\|_{L^2(N)}).$$

Similar results were obtained for the Lamé system by Brown and I. Mitrea (2009) [BM09] and for the Stokes system by Brown, I. Mitrea, M. Mitrea, and Wright (2010) [BMMW10].

Reductions

In studying the boundary value problem (MP), we may reduce to the case $f_D = 0$ using Dahlberg and Kenig's result on the regularity problem.
To handle (MP) in the case $f_D = 0$, we will begin with a weak solution of the mixed problem and establish that this weak solution has a gradient at the boundary in the sense of non-tangential limits. We will need several additional facts about these weak solutions including:

- Weak solutions with nice data have gradient in $L^p(\Omega)$ for some $p > 2$.
- Local Hölder continuity of weak solutions
- The existence of Green functions

Assumption on the decomposition

In this section, we assume that the decomposition of the boundary satisfies an *Ahlfors regularity* or density condition

- For $x \in D$ and $0 < r < r_0$, we have $\sigma(\Delta_r(x)) \approx r^{n-1}$. Here, we are using $\Delta_r(x)$ to denote a certain family of balls on the boundary of $\partial\Omega$.

Our published work uses a stronger condition, the corkscrew condition. To state this condition, we introduce $\Lambda = \bar{D} \cap \bar{N}$ to denote the boundary between D and N and we will use $\Delta_r(x) = B_r(x) \cap \partial\Omega$ to denote a ball in $\partial\Omega$. With these notations we can state the *corkscrew condition*:

- If $x \in \Lambda$ and $0 < r < r_0$, we may find $\hat{x} \in D$ so that $|x - \hat{x}| \leq Mr$ and $\Delta_r(\hat{x}) \subset D$.

An example

Our Ahlfors condition allows the set D to be quite general. For example if Ω is the unit disk, then in polar coordinates, our condition will allow

$$D = \bigcup_{k=1}^{\infty} \{(r, \theta) : r = 1, \frac{1}{2^k} < \theta < \frac{1}{2^k} + \frac{1}{2^{k+1}}\},$$

but not

$$D = \bigcup_{k=1}^{\infty} \{(r, \theta) : r = 1, \frac{1}{2^k} < \theta < \frac{1}{2^k} + \frac{1}{k^{k+1}}\}.$$

The weak formulation

For the weak problem, we will want to include a nonzero right-hand side in the equation, but we only consider the case when $f_D = 0$. Thus, we consider the problem

$$(WMP) \qquad \begin{cases} \Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N, & \text{on } N \end{cases}$$

We let $L^2_1(\Omega)$ be the standard Sobolev space in Ω. We let $L^2_{1,D}(\Omega)$ to be the closure in $L^2_1(\Omega)$ of the collection of smooth functions which vanish in a neighborhood of $\bar{D}$.

The weak formulation II

We say that u is a solution of (WMP) if

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v, dy = -\langle f, v\rangle + \langle f_N, v\rangle_{\partial\Omega}, & v \in L^2_{1,D}(\Omega) \\ u \in L^2_{1,D}(\Omega) \end{cases}$$

Here f is in the dual of $L^2_{1,D}(\Omega)$, f_N is in the dual of the traces of $L^2_{1,D}(\Omega)$ and $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are pairings of duality.

Hölder continuity of solutions

Theorem (Taylor [Tay11, TOB13])

If u is a solution of (WMP), $x \in \Omega$, $0 < r < r_0$, and f and f_N vanish on $B_r(x) \cap \Omega$ and $\Delta_r(x) \cap \partial\Omega$, respectively, then we have

$$|u(x)-u(y)| \leq C\left(\frac{|x-y|}{r}\right)^{\gamma} \fint_{B_r(x)\cap\Omega} |u-\bar{u}_{x,r}|\, dy, \qquad x, y \in B_{r/2}(x)\cap\Omega.$$

The constant C and the Hölder exponent γ depend only the Lipschitz constant in Ω and the Ahlfors regularity assumption on D. The parameter r_0 is comparable to the diameter of Ω.

The elements of the proof

The proof uses the well-known de Giorgi [DG57] technique for establishing Hölder continuity of weak solutions to elliptic equations. A similar result for the mixed problem was established by Stampacchia [Sta60] under additional restrictions on the domain. The key point is to establish Poincaré and Sobolev inequalities in a family of starshaped domains. The Ahlfors regularity assumption is needed for these inequalities.

Existence of a Green function

Following arguments of Grüter and Widman [GW82], Taylor, Ott, and Brown showed that there exists a Green function for (WMP) with the estimates for $n \geq 3$:

$$
\begin{aligned}
|G(x,y)| &\leq C|x-y|^{2-n}, \\
|G(x,y) - G(x,z)| &\leq \frac{C}{|x-y|^{n-2}} \left(\frac{|y-z|}{|x-y|} \right)^{\gamma} \text{ if } 2|y-z| < |x-y|.
\end{aligned}
$$

The second estimate holds when $n = 2$ and the Green function has a logarithmic singularity when $n = 2$. The results for $n \geq 3$ follow using standard arguments. The results for $n = 2$ took a little more work.

Reverse Hölder estimates

Theorem (Ott and Brown)

Suppose that Ω is a Lipschitz domain and D satisfies the Ahlfors regularity condition. There exists $q_0 > 2$ so that if $2 < q < q_0$ and u is a solution of (WMP) with $f = 0$, then for each ball $B_r(x)$ with a center in $\bar{\Omega}$, we have

$$\begin{aligned}\left(⨍_{B_r(x)\cap\Omega} |\nabla u|^q \, dy\right)^{1/q} \leq \; & C ⨍_{B_{2r}(x)\cap\Omega} |\nabla u| \, dy \\ & + \left(\frac{C}{r^{n-1}} \int_{B_{2r}(x)\cap\partial\Omega} |f_N|^{q(n-1)/n} \, d\sigma\right)^{n/(q(n-1))}.\end{aligned}$$

This follows using the well-known reverse Hölder technique of Giaquinta [Gia93]. The point which might be new is handling the Neumann data in a Caccioppoli inequality.

Hardy spaces on subsets of $\partial\Omega$

We recall that an *atom for* $H^1(\partial\Omega)$ is a function a which is supported in a surface ball Δ and satisfying

$$\|a\|_\infty \le C/\sigma(\Delta), \qquad \int_{\partial\Omega} a\, d\sigma = 0.$$

We say that a is an *atom for* $H^1(N)$ if a is the restriction to N of an atom for $\partial\Omega$.
A function f_N is in the space $H^1(N)$ if we can write $f_N = \sum_j \lambda_j a_j$ for some sequence of atoms $\{a_j\}$ and numerical sequence $\{\lambda_j\}$ in ℓ^1. The norm in $H^1(N)$ is

$$\|f_N\|_{H^1(N)} = \inf \sum_j |\lambda_j|$$

where the infimum is taken over all representations of f_N.

Estimates near the diagonal

To establish an estimate for (MP) when $f_D = 0$ and $f_N \in H^1(N)$, we only need to consider when $f_N = a$, an atom supported in $\Delta_r(x_a)$. We let u be a solution of (WMP) with data $f_N = a$ and $f_D = 0$.
Let $\delta(x) = \mathrm{dist}(x, \Lambda)$. We use Jerison and Kenig's L^2 estimates and a covering argument to obtain for $\delta(x) < 4r$ that

$$\int_{\Delta_r(x)} |\nabla u|^2 \delta^{1-\rho}\, d\sigma \leq C\Big(\int_{\Delta_{2r}(x)} |f_N|^2 \delta^{1-\rho} d\sigma + \int_{B_{2r}(x)\cap\Omega} |\nabla u|^2 \delta^{-\rho}\, dy.\Big)$$

We chose ρ small, and use Hölder's inequality, the reverse Hölder estimate, and an energy estimate for weak solutions to obtain the estimate for ∇u near the support of a,

$$\left(\fint_{\Delta_{8r}(x_a)} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C r^{1-n}$$

Assumption on the boundary

For this argument to work, we need to know that negative powers of δ are integrable. This will follow if the dimension of Λ is not too large. More precisely, our assumption on Λ is that for some $\epsilon > 0$, we have

$$\mathcal{H}^{n-2+\epsilon}(\Delta_r(x) \cap \Lambda) \leq M r^{n-2+\epsilon}, \qquad x \in \Lambda, 0 < r < r_0.$$

Under this assumption we have

Lemma

If $x \in \partial\Omega$ and r satisfies $0 < r < r_0$, then for $-1+\epsilon < s < \infty$, $x \in \partial\Omega$, and $0 < r < r_0$, we have

$$\int_{\Delta_r(x)} \delta(y)^s \, d\sigma \approx r^{n-1} \max(r, \delta(x))^s.$$

The exponent p in the previous slide satisfies $p < q_0(1-\epsilon)/(2-\epsilon)$.

Off-diagonal estimates for solutions with atomic data

To study our solution u away from the support of a, we use the Green function to represent u,

$$u(x) = \int_N G(x,y)a(y)\,d\sigma(y).$$

Set $\Sigma_k = \Delta_{2^k r}(x_a) \setminus \Delta_{2^{k-1} r}(x_a)$. Using that a has mean value zero or that $G(x,\cdot)$ vanishes near a, the Hölder estimates for G, a Caccioppoli inequality, and the arguments used near the diagonal, we obtain

$$\left(\fint_{\Sigma_k} |\nabla u|^p \, d\sigma \right)^{1/p} \leq C 2^{-k\beta} (2^k r)^{1-n}.$$

Existence in Hardy spaces

The estimates for a solution with atomic data are the main step needed to prove our main theorem in Hardy spaces.

Theorem (Taylor, Ott, and Brown)

If f_N lies in $H^1(N)$ and $f_D = 0$, then we may find a unique solution of (MP) with $(\nabla u)^* \in L^1(\partial\Omega)$ that satisfies

$$\int_{\partial\Omega} (\nabla u)^* \, d\sigma \leq C \|f_N\|_{H^1(N)}.$$

Extrapolation

The next step is to use a real variable argument of Shen [She07] (see also earlier work of Caffarelli and Peral [CP98]) that will allow us to use the Hardy space estimate and the reverse Hölder estimate to obtain a result in L^p spaces. Shen's result is:

Assume that F is a function on a cube $4Q_0$ and that there are exponents p and q with $1 < p < q$ so that for every cube Q we may find functions F_Q and R_Q so that $|F| \leq |F_Q| + |R_Q|$,

$$
\begin{aligned}
⨍_{2Q} |F_Q| \, d\sigma &\leq C(⨍_{4Q} |f|^p d\sigma)^{1/p} \\
(⨍_{2Q} |R_Q|^q \, d\sigma)^{1/q} &\leq C[⨍_{4Q} |F| \, d\sigma + (⨍_{4Q} |f|^p \, d\sigma)^{1/p}]
\end{aligned}
$$

With these assumptions, we can conclude that $F \in L^r$ for r in the interval (p, q) and we have

$$
(⨍_{Q_0} |F|^r \, d\sigma)^{1/r} \leq C(⨍_{4Q_0} |F| \, d\sigma + (⨍_{4Q_0} |f|^r \, d\sigma)^{1/r}).
$$

The main theorem

This gives our main result:

Theorem (Taylor, Ott, Brown, 2013)

Let Ω be a Lipschitz domain and suppose that D satisfies the corkscrew condition and that the dimension of Λ is at most $n-2+\epsilon$. Then for p with $1 < p < q_0(1-\epsilon)/(2-\epsilon)$, we may find a solution of the L^p mixed problem (MP) which satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{L^p(N)} + \|f_N\|_{L^p_1(\partial\Omega)}).$$

Further results

1. The technique of Shen also gives weighted results. See Taylor, Ott, and Brown.
2. In work of Lanzani, Capogna, and Brown (2008) [LCB08], an extension of the Rellich identity gives weighted results in certain two-dimensional domains. This result helped to motivate the argument we use in the general case.
3. Work of Ott and Brown establish existence for the Lamé system in two dimensions [OB13].

What's next?

1. Systems in dimensions three and larger.
2. Stokes in higher dimensions. We have made a few steps in dimension two.
3. Are there special geometries where we can identify the range of p for which the mixed problem in L^p may be solved?

Thanks

Thanks for your attention.
Eventually the slides will be at

`http://www.math.uky.edu/~rbrown/conferences/`

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