

# 1 Lecture 24: Linearization

## 1.1 Outline

- The linearization of a function at a point  $a$ .
- Linear approximation of the change in  $f$ . Error, absolute error.
- Examples

## 1.2 Linearization

Functions can be complicated. Often, it is useful to replace a function by a simpler function. Today we will discuss one way to approximate a function and look at how to use this linearization to approximate functions and also when this is a bad idea.

Given a differentiable function  $f$  defined near  $a$ , the *linearization of  $f$  at  $a$*  is the linear function given by

$$L(x) = f(a) + f'(a)(x - a).$$

Thus, the graph of this function is the tangent line to the graph of  $f$ . We expect that the linearization will be a good approximation to  $f$  near  $a$ , but not a good approximation when we are far away from  $a$ .

*Example.* Let  $f(x) = \sin(x)$ . Find the linearization of  $f$  at  $x = 0$ . Use the linearization to approximate  $f(0.1)$  and  $f(100)$ . Compare these approximations with the approximations from your calculator.

*Solution.* To find the linearization at 0, we need to find  $f(0)$  and  $f'(0)$ . If  $f(x) = \sin(x)$ , then  $f(0) = \sin(0) = 0$  and  $f'(x) = \cos(x)$  so  $f'(0) = \cos(0)$ . Thus the linearization is

$$L(x) = 0 + 1 \cdot x = x.$$

Using this linearization, we obtain

$x$	$\sin(x)$	$L(x)$	$L(x) - \sin(x)$
0.001	0.0009999998	0.001	$1.67 \times 10^{-10}$
0.01	0.00999983	0.01	$1.67 \times 10^{-7}$
0.05	0.04998	0.05	$2.08 \times 10^{-5}$
0.1	0.099384	0.1	$1.67 \times 10^{-4}$
0.2	0.19867	0.2	$1.33 \times 10^{-3}$
100	-0.50637	100	100.51

As expected the linearization is pretty good near 0. It is interesting to note that the error decreases very rapidly as we approach 0. It appears that the error in approximating  $\sin(x)$  is approximately  $x^3$ . Why is this? ■

We give a few other uses of the linear approximation.

1. In studying the behavior of a pendulum, the angular displacement  $\theta$  as a function of time satisfies the equation

$$\theta'' = -k^2 \sin(\theta).$$

This is a difficult equation to solve and it is common to replace the term  $\sin(\theta)$  on the right by the simpler term  $\theta$ , the linearization of  $\sin(\theta)$ . This gives the equation

$$\theta'' = -k^2\theta$$

and it is easy to see that  $\sin(kt)$  and  $\cos(kt)$  are solutions. Observations suggest that these functions have the right behavior and that the approximation is useful, when  $\theta$  is small.

2. In the examples below, we will use linearization to give an easy way to compute approximate values of functions that cannot be computed by hand. Next semester, we will look at ways of using higher degree polynomials to approximate a function.
3. In a week or two, we will use a linear approximation to help solve an equation such as  $f(x) = 0$ .

### 1.3 Approximating the change in a function.

Since the linear approximation is only good near a point  $a$ , it often makes more sense to talk about the change in  $f$ . This if we are given a number  $a$  and a nearby number  $x$ , we denote the *change in  $f$*  by

$$\Delta f = f(x) - f(a).$$

If we replace  $f$  by its linearization,  $L$ , then we obtain the *linear approximation to the change in  $f$* ,

$$f(x) - f(a) \approx L(x) - L(a) = f'(a)(x - a).$$

It is also common to use the notation  $\Delta x = x - a$  for the change in  $x$  and the notation  $\Delta f = f(x) - f(a)$  for the change in  $f$ .

We can use the linearization to estimate values functions such as  $\sqrt{9.1}$ . The point is that we are near a value 9 where it is easy to compute the square root. While this is not terribly useful when we have a calculator, it is good practice and good fun.

We will also compute the error and the percentage error. If a number  $a$  is an approximation to the number  $A$ , then the *error in approximating  $A$*  is the absolute value of the difference,

$$|a - A|.$$

The percentage error in the approximation of  $A$  by  $a$  is the quantity

$$100 \times \frac{|a - A|}{|A|}.$$

Often the percentage error is more interesting.<sup>1</sup>

*Example.* Let  $f(x) = \sqrt{1 + 2x}$  and use the linearization to approximate  $f(4.3)$ .

Find the error in the approximation of  $f(4.3)$ , the percentage error in the approximation of  $f(4.3)$  and the percentage error in the approximation of  $\Delta f$ .

*Solution.* Note that it is easy to compute  $f(4) = \sqrt{9} = 3$ . We compute the linearization of  $f$  at 4. To do this we need  $f(4) = 9$  and we need to find  $f'(4)$ .

We compute the derivative by the power rule and chain rule,

$$\begin{aligned} \frac{d}{dx} \sqrt{1 + 2x} &= \frac{d}{dx} (1 + 2x)^{1/2} \\ &= \frac{1}{2} (1 + 2x)^{-1/2} \cdot 2 \\ &= \frac{1}{\sqrt{1 + 2x}}. \end{aligned}$$

Evaluating at  $x = 4$ , give  $f'(4) = 1/3$ . Thus the linearization of  $f$  at 4 is

$$L(x) = f(4) + f'(4)(x - 4) = 3 + \frac{1}{3}(x - 4).$$

To make the approximation we write

$$f(4.3) \approx L(4.3) = f(4) + \frac{1}{3} \cdot 0.3 = 3.1.$$

A calculator gives the more accurate approximation,

$$f(4.3) \approx 3.0984.$$

The errors are

$$|f(4.3) - 3.1| \approx 0.00161, \quad 100\% \times \frac{|f(4.3) - 3.1|}{|f(4.3)|} \approx 0.0521\%.$$

Finally the error in the change of  $f$  is given by

$$100\% \times \frac{|f(4.3) - f(4) - f'(4)(4.3 - 4)|}{|f(4.3) - f(4)|} \approx 1.6\%.$$

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<sup>1</sup>Be careful on the homework. Some of the questions ask you to compute the percentage error in *the change in  $f$* . It is important to read the question carefully to get the right denominator. ■

We close with another reason to consider linear approximation. For a curve which is given implicitly as the solution of an equation such as  $x^2 + y^3 = xy$ , it is difficult to find points on the curve. If one can find one point on the curve, we can use linear approximation to approximate nearby points.

*Example.* Suppose that a curve is given by the equation  $x^2 + y^3 = 2x^2y$ . Verify that the point  $(x, y) = (1, 1)$  lies on the curve. Assume that the curve is given by a function  $y = y(x)$  for  $x$  near 1 and approximate  $y(1.2)$ .

*Solution.* To verify that  $(x, y) = (1, 1)$  lies on the curve, we need to know that  $1^3 + 1^2 = 2 \cdot 1^2 \cdot 1$  which is true.

To find the linearization, we use that  $y(1) = 1$  and find the derivative of  $y$  at  $x = 1$ . Differentiating

$$(x^2 + y^3)' = (2x^2y)'$$

gives

$$2x + 3y^2y' = 4y + 2x^2y'.$$

Solving for  $y'$  gives

$$y' = \frac{4y - 2x}{3y^2 - 2x^2}$$

and that  $y'(1) = 2$ . Thus the linearization of  $y$  is  $L(x) = 1 + 2(x - 1)$  and  $L(1.2) \approx 1.4$ . Thus the point  $(1, 1.2)$  should be close to the curve.

If we substitute this point into the equation  $x^2 + y^3 = 2x^2y$ , we find that  $1.2^2 + 1.4^3 = 4.184$  and  $2 \cdot 1.2^2 \cdot 1.4 = 4.0320$ . As these values are close, we expect that  $(1.2, 1.4)$  is close to a point on the curve. ■

We conclude with a function where this type of approximation is not very helpful.

