1 Lecture 04: The tangent and velocity problem, informal treatment of limits

- Estimating the slope of a tangent line.
- Instantaneous velocity
- A limit that does not exist, one-sided limits
- Limits that approach infinity

1.1 The tangent problem

It is a well-known fact from geometry that the tangent to a circle is line that is perpendicular to the radius. One of the problems that can be studied using calculus is finding tangent lines to more general curves. Suppose we have a function $y = f(x)$ and want to define a tangent line to the graph.

We know several ways to write the equation of a line. The approach that is most useful in this problem is the "point-slope form of a line", the line through (x_0, y_0) with slope m is

$$y - y_0 = m(x - x_0).$$

If we want to find the tangent line to $f(x)$ at $x = x_0$, then we know the line should pass through $(x_0, f(x_0))$. The only mystery is what is the appropriate value for the slope. The technique we will use is to pick a point $(x, f(x))$ that is near $(x_0, f(x_0))$ and compute the slope of the line joining $(x, f(x))$ and $(x_0, f(x_0))$. This line which meets the graph of f at least twice will be called a *secant line*. We try various values of x that are close to $f(x)$ and hope that we can guess the value of the slope when the distance between x and x_0 vanishes.

Example. Consider the function $f(x) = e^x$. What is the slope of the tangent line to the graph of f at $x = 0$?

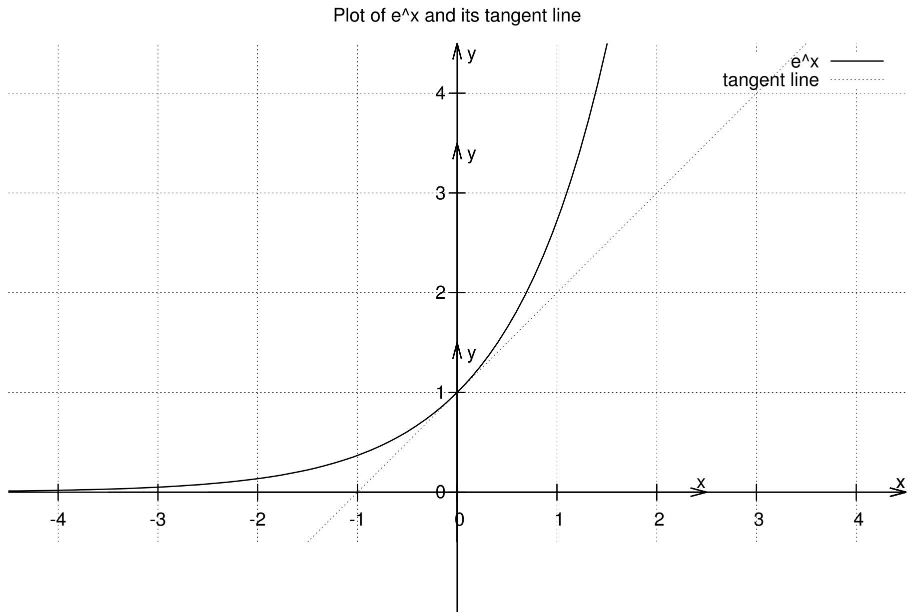
Solution. If x is a point near 0, the slope of the line joining $(0, f(0))$ to $(x, f(x))$ is

$$m = \frac{f(x) - f(0)}{x - 0} = \frac{e^x - 1}{x}.$$

If we compute this for several values of x, we obtain

Value of x	Slope of secant line
0.1	1.05
0.01	1.005
-0.002	0.999
$1/\pi^3$	1.0163

A moment's reflection might lead us to guess that the slope is 1. Thus the tangent line to the graph of e^x is $y = x + 1$. The graph below suggests that this correct.



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Example. As a second example, we consider the function $f(x) = \sin(x)$ and find the tangent line at 0.

Solution. Again we look at secant lines that touch the graph of $\sin(x)$ at two points near 0. If we compute the slope of the secant line or the rate of change of the function on the intervals $[0, 0.2]$, $[-0.003, 0]$, and $[0, 0.0001]$ we obtain the following slopes

Interval $[a, b]$	$(\sin(b) - \sin(a))/(b - a)$
$[0, 0.2]$	0.99335
$[-0.003, 0]$	0.999998500000675
$[0, 0.0001]$	0.999999998333333

Figure 1: Estimating the slope of the curve $y = \sin(x)$ at 0.

The slope of these lines appears to approach 1 as the interval shrinks to 0. Thus, we guess that the slope of the tangent line is 1. Since the tangent should pass through the point $(0, \sin(0))$, the equation is $y = 1(x - 0)$ or $y = x$.

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1.2 Limits

The process we used in the previous section to find the tangent line is of fundamental importance. We give an informal definition.

Definition. Suppose $f(x)$ is a function that is defined an interval containing a number a, except possibly at a. If the values $f(x)$ become close to a number L when we let the distance between x and a approaches 0, then we call L the *limit of f as x approaches a* and write

$$\lim_{x\to a} f(x) = L.$$

In the previous example where we found the slope of the tangent line, we were trying to find:

$$\lim_{x\to 0} \frac{e^x - 1}{x}.$$

The function $(e^x - 1)/x$ is defined near $x = 0$, but not at 0. We are interested in trying to determine the behavior near 0.

1.3 Velocity

To describe the motion of an object moving along a line, for example an object that is thrown straight up in the air, we use a position function $p(t)$ which gives the height at a time t. Suppose, we want to describe the velocity of the object. We recall the fundamental relation

$$\text{distance} = \text{rate} \times \text{time}.$$

In our case, we want to compute a velocity at $t = t_0$. Our solution will look similar (identical) to our solution of the tangent problem.

We fix a small interval $[t_0, t]$ with one endpoint t_0 and a nearby time t. The change in position or displacement in this interval is $p(t)-p(t_0)$ and the time it takes to travel this distance is $t - t_0$. Thus the *average velocity* on this interval is

$$\frac{p(t) - p(t_0)}{t - t_0}.$$

If we let t approach t_0, and the average velocities cluster around one number, then we call this number the *instantaneous velocity* at t_0. This instantaneous velocity is given by the limit

$$\lim_{t\to t_0} \frac{p(t) - p(t_0)}{t - t_0}.$$

Example. We give a simple numerical example. A ball thrown up in the air and its height in meters at time t seconds is given by $p(t) = -5t^2 + 20t$. Find the average velocity on the interval $(3, 3 + h)$ and guess the instantaneous velocity at 3.

Solution. On the interval $3 \le t \le 3+h$, the change in position is $p(3+h)-p(3)$ meters and the time interval is of length $3+h-3=h$ seconds. Thus the average velocity is

$$\frac{p(3+h)-p(3)}{h}.$$

We could again try numerical values of h, but this problem we can simplify algebraically:

$$\begin{aligned}\frac{p(3+h)-p(3)}{h} &= \frac{-5(3+h)^2+60+20h-15}{h} && (1)\\ &= \frac{-10h-5h^2}{h} && (2)\\ &= -10-5h && (3)\end{aligned}$$

Using the last expression it is easy to see that this expression approaches -10 as h gets close to zero.

Using our new notation, we would write

$$\lim_{h\to 0}\frac{p(3+h)-p(3)}{h} = -10.$$

and that the instantaneous velocity at 3 is -10 meters/second. ∎

Exercise. Find the tangent line to the graph of $f(x)=\frac{1}{x}$ at $x=2$.

Exercise. A ball is thrown so that its height at time t is

$$h(t) = -5t^2+20t$$

meters after t seconds. Find the instantaneous velocity at time $t=2$ seconds. What are the units for this velocity?

Find the instantaneous velocity at an arbitrary time $t=a$.

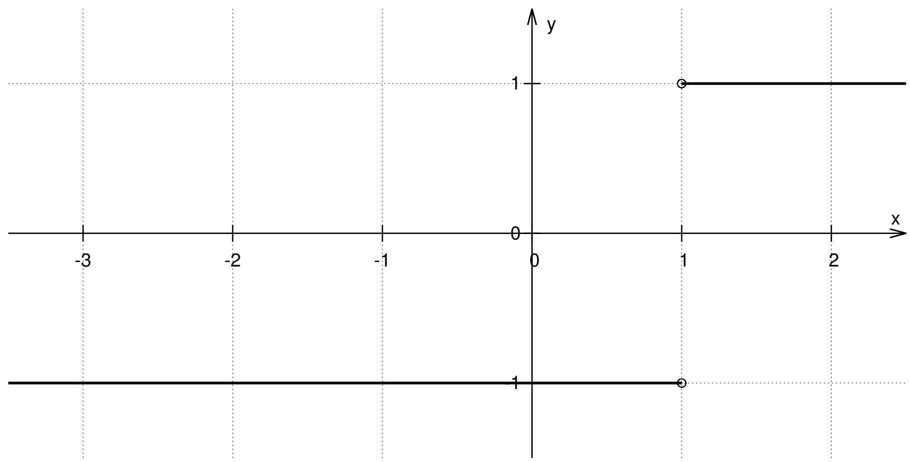
1.4 One sided limits

Example. Can you find the tangent line to $f(x)=|x-1|$ at $x=1$?

Solution. In this case, we would want to consider the slope

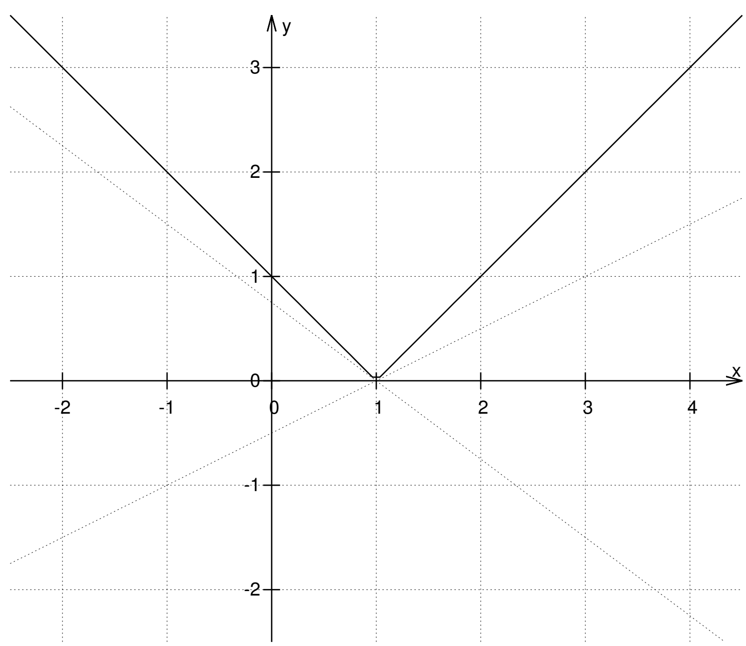
$$\frac{f(x)-f(1)}{x-1} = \frac{|x-1|}{x-1}.$$

Let us call this a new function $g(x)=|x-1|/(x-1)$ and consider the graph of g,



Examining the graph, we see that the function g does not have a limit. When $x > 1$, the value of g is 1 and when $x < 1$, the value of g is -1. As a result there is no single value which g approaches when x approaches 1.

Returning to the graph of f, we see that there is a corner at $x = 1$ and there is no clear way to define a single tangent line. The graph includes several lines with touch the graph at one point.



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The previous example serves to introduce one-sided limits.

Definition. Suppose $f(x)$ is a function that is defined on an interval (a, b) for some $b > a$, except possibly at a. If the values $f(x)$ become close to a number L when the distance between x and a approaches 0 and $x > a$, then we call L the *limit of f as x approaches a from above* and write

$$\lim_{x \to a^+} f(x) = L.$$

Definition. Suppose $f(x)$ is a function that is defined on an interval containing (b, a) for some $b < a$. If the values $f(x)$ become close to a number L when the distance between x and a approaches 0 and $x < a$, then we call L the *limit of f as x approaches a from below* and write

$$\lim_{x \to a^-} f(x) = L.$$

The following theorem gives the relation between one and two-sided limits.

Theorem 4 *Suppose that f is a function defined on an open interval containing a, except possibly at a. Then we have $\lim_{x \to a} f(x)$ exists if and only if both of the one-sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal.*

Example. Consider the graph of $f(x) = \frac{x^2 - x - 2}{x - 2}$. Find $\lim_{x \to 2} f(x)$.

Solution. In figure 1.4 we graph the function $(x^2 - x - 2)/(x - 2)$. Note that if $x \neq 2$, then f simplifies to the linear function $(x + 1)$ and $x = 2$ is not in the domain of f. From the graph in figure 1.4 it is clear that the limit is 3. ∎

1.5 Limits that are infinite

Recall that if $\lim_{x \to a} f(x) = L$, the values of f become arbitrarily close to L, but we may never have $f(x) = L$. We want to describe the behavior of a function like $f(x) = 1/x^2$ near $x = 0$. As x small, the reciprocal $1/x^2$ becomes large and positive. We say that $\lim_{x \to 0} \frac{1}{x}^2 = +\infty$. But there is no number ∞ so that f never reaches ∞.

We try to give a definition of this behaviour.

Definition. We say that the *limit of f as x approaches a is* $+\infty$ and write

$$\lim_{x \to a} f(x) = \infty$$

if the values of f become arbitrarily large and positive as the distance between x and a approaches 0.

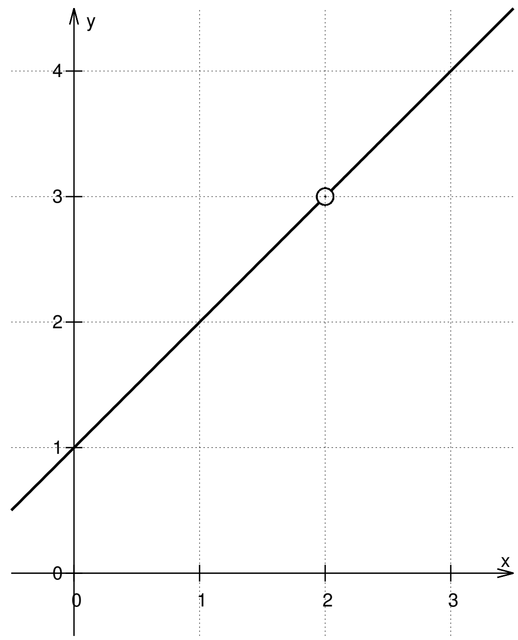

Figure 2: Graph of $f(x) = (x^2 - x - 2)/(x - 2)$.

We leave it to the reader to define what it means for a limit to be $-\infty$ and one-sided limits which approach $\pm\infty$.

Example. Discuss the limit $\lim_{x \to -4} \frac{x-8}{x-4}$.

Solution. If we consider values of $x > 4$, then $x - 4 > 0$, but becomes small as x approaches 4. Thus the reciprocal $1/(x - 4)$ approaches $+\infty$ as x approaches 4 from the left. Also $x - 8 < 0$ for x near 4. Together we have

$$\lim_{x \to 4^+} \frac{x - 8}{x - 4} = -\infty.$$

Similar reasoning with $x < 4$, but close to 4 gives that

$$\lim_{x \to 4^-} \frac{x - 8}{x - 4} = +\infty.$$

Since the left and right limits are not the same, the limit does not exist and is not $+\infty$ or $-\infty$.

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