1 Derivatives of trigonometric functions

1.1 Outline

- Preliminaries
- The derivatives of $\sin(t)$ and $\cos(t)$.
- Derivatives of the remaining trigonometric functions
- Examples

1.2 Preliminaries

We recall that the functions sin and cos are continuous and also the basic limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

We will also need the addition formula for sin and cos.

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned}$$

1.3 The derivatives of $\sin(x)$ and $\cos(x)$

Our first goal is to find the derivatives of sin and cos. We have

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \text{and} \qquad \frac{d}{dx}\cos(x) = \sin(x).$$

To establish the formula for the derivative of sin, we write the difference quotient and use the addition formula for sin to find

$$\begin{aligned} \frac{\sin(x+h)-\sin(x)}{h} &= \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \end{aligned}$$

Using our basic trig limits and the rules for sums and products of limits, we obtain

$$\begin{aligned} \frac{d}{dx}\sin(x) &= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x). \end{aligned}$$

Exercise. Carry out a similar computation to find the derivative of $\cos(x)$.

Example. Find the tangent line to $f(x) = \sin(x)\cos(x)$ at $x = 0$.

Solution. The tangent line passes through the point $(0, f(0)) = (0, 0)$ and has derivative $f'(0)$ and has the equation $y - f(0) = f'(0)(x - 0)$.

We compute $f'(x)$ using the product rule,

$$f'(x) = \sin'(x)\cos(x) + \sin(x)\cos'(x) = \cos^2(x) - \sin^2(x).$$

Substituting $x = 0$ gives $f'(0) = 1$. Using that $(0, f(0)) = 0$ and $f'(0) = 1$ gives the tangent line is

$$y = x.$$

∎

Example. When does the graph of the function $f(x) = x + 2\sin(x)$ have a horizontal tangent line?

Solution. The graph will have a horizontal tangent line at values x which satisfy $f'(x) = 0$. We compute $f'(x) = 1 + 2\cos(x)$. We will have $f'(x) = 0$ if $\cos(x) = -1/2$. The solutions of this equation in the interval $[0, 2\pi]$ are $x = 2\pi/3$ and $x = 4\pi/3$. To obtain all solutions, we add an arbitrary multiple of 2π. Thus solutions are

$$x = 2\pi/3 + 2k\pi,\ 4\pi/3 + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

∎

1.4 Derivatives of the remaining trigonometric functions

In this section, we find the derivatives of the remaining trigonometric functions. To find the derivatives we express the function in terms of sin and cos and then using the quotient or reciprocal rule.

Example. Find the derivative of $\tan(x)$.

Solution. We recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Using the quotient rule, we have

$$\begin{aligned} \frac{d}{dx}\frac{\sin(x)}{\cos(x)} &= \frac{\cos(x)\frac{d}{dx}\sin(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= 1/\cos^2(x) = \sec^2(x). \end{aligned}$$

Thus, we have

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

∎

Exercise. Establish the differentiation formulae:

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\cot(x) = -\sec^2(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x).$$

1.5 Examples

We close with a couple of examples:

Example. Find the derivative $f'(0)$ if f is defined by

$$f(x) = \frac{\cos(x)}{2+\sin(x)}.$$

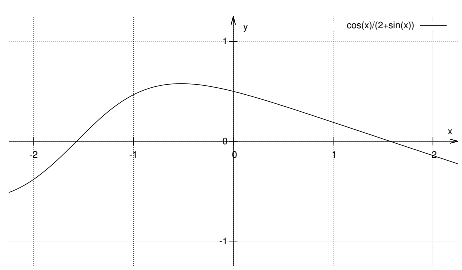
Check your answer by graphing the function and estimating the rate of change at 0.

Solution. We use the quotient rule to find the derivative:

$$\begin{aligned}
\frac{d}{dx}\frac{\cos(x)}{2+\sin(x)} &= \frac{(2+\sin(x))\frac{d}{dx}\cos(x) - \cos(x)\frac{d}{dx}(2+\sin(x))}{(2+\sin(x))^2} \\
&= \frac{(2+\sin(x))(-\sin(x)) - \cos(x)\cos(x)}{(2+\sin(x))^2} \\
&= \frac{-2\sin(x) - 1}{(2+\sin(x))^2}
\end{aligned}$$

Thus when $x = 0$, $f'(0) = -1/4$.

The graph below suggests this is a reasonable value.



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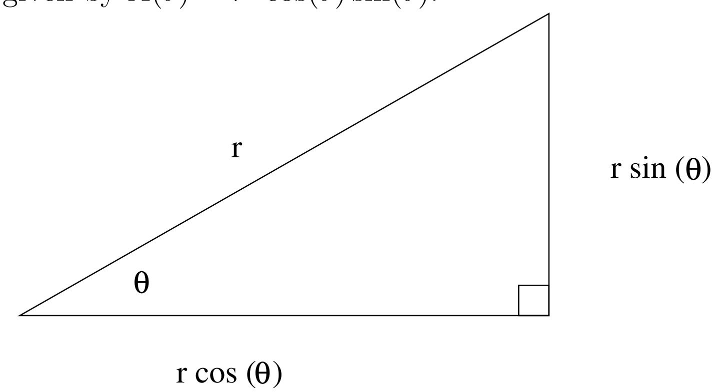
Example. Let T be a right-triangle with hypotenuse r and one of the acute angles θ. For which value of θ is the rate of change of area of T with respect to θ zero?

See if you can guess the answer and then compute the answer using the tools of calculus.

Solution. We first need to find an expression for the area of the triangle T in terms of the angle θ. The two legs of the triangle will have length $r\cos(\theta)$ and $r\sin(\theta)$. Since one leg may serve as the base and the other as the height, we have the area A is given by $A(\theta) = r^2 \cos(\theta)\sin(\theta)$.



We compute the derivative or rate of change with the product rule,

$$A'(\theta) = r^2(\cos'(\theta)\sin(\theta) + \cos(\theta)\sin'(\theta) = r^2(-\sin^2(\theta) + \cos^2(\theta)).$$

This derivative will be zero when $\cos^2(\theta) = \sin^2(\theta)$ or $\sin(\theta) = \pm\cos(\theta)$. Since θ is an acute angle, the solution must be $\theta = \pi/4$. ∎

Example. If $f(x) = \sin(x) + \cos(x)$, find the derivative $f^{(401)}(x)$.

Can you find a function f which satisfies

$$f'' + f = 0?$$

Can each member of the class find a different, correct answer?

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