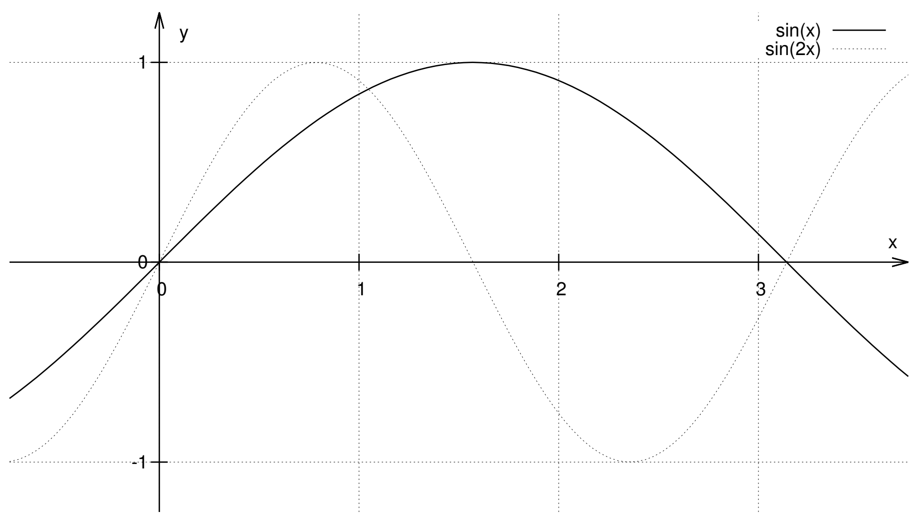
1 Lecture 18: The chain rule

1.1 Outline

- Comparing the graphs of $\sin(x)$ and $\sin(2x)$.
- The chain rule.
- The derivative of a^x.
- Some examples.

1.2 Comparing the graphs of $\sin(x)$ and $\sin(2x)$

We graph $f(x) = \sin(x)$ and $f(x) = \sin(2x)$ on the same axes and ask what happens to the slope of the tangent line at the origin.



The graph of $\sin(2x)$ is obtained by compressing the graph of $\sin(x)$ by a factor of 2. Thus, a tangent line to the graph of $\sin(2x)$ at a point $(a/2, \sin(a))$ will be twice as steep as the corresponding tangent line to the graph of $\sin(x)$ at the corresponding point $(a, \sin(a))$.

Example. Let $f(x) = mx + b$ and $g(x) = nx + d$. What is the slope of the composite function $f \circ g$?

Solution. We have $f \circ g(x) = m(nx + d) + b = mnx + md + b$. The slope of the composition is mn. ∎

1.3 Chain rule

The chain rule provides a way to compute the derivatives of composite functions in general and gives that the rate of change of $f \circ g$ at x is given by the products of the rates of change of f at $g(x)$ and the rate of change of f at x.

The precise statement is

Theorem 1 *If g is a function that is differentiable at x and f is a function that is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and we have*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Not a proof. To prove the chain rule, we are tempted to multiply and divide by $g(y) - g(x)$ to obtain

$$\begin{aligned} \lim_{y \to x} \frac{f(g(y)) - f(g(x))}{y - x} &= \lim_{y \to x} \frac{f(g(y)) - f(g(x))}{g(y) - g(x)} \frac{g(y) - g(x)}{y - x} \\ &= f'(g(x))g'(x). \end{aligned}$$

Where we have used the rule for a product of limits at the second step.

This argument is right if we know that $g(y) - g(x)$ is not zero for all y near x, but not equal to x. However, there is no way to guarantee that $g(y) - g(x)$ is not zero in general. A correct argument is given as an exercise on page 177 of our text. ∎

The chain rule is particularly easy to remember in the Leibniz notation. We let $u = g(x)$, then we have

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Thus, one can imagine canceling du on the right-hand side to obtain the left-hand side. However, this is nonsense. We have not defined du to mean anything. The symbol df/du is only defined as a whole by our limit definition of the derivative. But sometimes nonsense is useful.

In applying the chain rule, one can either start at the inside and work out or at the outside and work in. Either approach is fine, but it pays to be consistent.

Example. Let $f(x) = e^{-2x^2}$. Write f as the composition of two functions $f = g \circ h$. Can you find another solution?

Solution. One choice is $g(x) = e^x$ and $h(x) = -2x^2$. A second choice is $g(x) = e^{-2x}$ and $h(x) = x^2$. Other answers are possible. ∎

Example. Use the chain rule to find the derivatives

$$\frac{d}{dx}e^{2x} \qquad \text{and} \qquad \frac{d}{dx}\sin(x^3+2x)$$

Solution. We can write $e^{2x} = f(g(x))$ with $f(u) = e^u$ and $g(x) = 2x$. We have that $f'(u) = e^u$ and $g'(x) = 2$. According to the chain rule

$$(f \circ g)'(x) = f'(g(x))g'(x) = 2e^{2x}$$

For the function $\sin(x^3+2x)$ we can write $\sin(x^3+2x) = h(k(x))$ with $h(u) = \sin(u)$ and $k(x) = x^3 + 2x$. We have $h'(u) = \cos(u)$ and $k'(x) = 3x^2 + 1$. Thus

$$(h \circ k)'(x) = h'(k(x))k'(x) = \cos(x^3 + x)(3x^2 + 2).$$

∎

Example. Find the derivative dy/dx if $y = 3^x$ and if $y = x^3$.

Solution. If we let $y = 3^x$, then $\ln(y) = x\ln(3)$ and applying the exponential function

$$y = e^{\ln(y)} = e^{x\ln(3)}.$$

Thus by the chain rule,

$$\frac{dy}{dx} = \ln(3)e^{x\ln(3)}.$$

The second one does not require the chain rule,

$$\frac{d}{dx}x^3 = 3x^2.$$

∎

Example. Find the derivative of $f(x) = \frac{e^x+1}{2x^2+2}$ without using the quotient rule.

Solution. We can rewrite the function f as

$$f(x) = (e^x + 1)(2x^2 + 2)^{-1}.$$

Now we may use the product rule and chain rule to find the derivative.

$$\begin{aligned} f'(x) &= e^x(2x^2+2)^{-1} + (e^x+1)(-1)(2x^2+2)^{-2}(2x) \\ &= e^x(2x^2+2)^{-1} - 2x(e^x+1)(2x^2+2)^{-2}. \end{aligned}$$

∎

Example. Find the derivative

$$\frac{d}{dx}\sqrt{5+\cos^3(x^2)}.$$

Solution. We first rewrite the radical using an exponent. We start at the outside and use the chain rule twice.

$$\begin{aligned}
\frac{d}{dx}(5+\cos^3(x^2))^{1/2} &= \frac{1}{2}(5+\cos^3(x^2))^{-1/2}\frac{d}{dx}(5+\cos^3(x^2)) \\
&= \frac{1}{2}(5+\cos^3(x^2))^{-1/2}\cdot 3\cos^2(x^2)\cdot\frac{d}{dx}\cos(x^2) \\
&= \frac{1}{2}(5+\cos^3(x^2))^{-1/2}\cdot 3\cos^2(x^2)\cdot(-\sin(x^2))\frac{d}{dx}x^2 \\
&= \frac{1}{2}(5+\cos^3(x^2))^{-1/2}\cdot 3\cos^2(x^2)\cdot(-\sin(x^2))2x \\
&= \frac{-3x\cos^3(x^2)\sin(x^2)}{\sqrt{5+\cos^2(x^2)}}.
\end{aligned}$$

∎

Example. One leg of a right triangle is fixed and of length 7 meters. The other leg is decreasing and at the time when its length is 24 meters, it is decreasing at a rate of 0.3 meters/second. Find the derivative of the hypotenuse at the time when the legs are 7 meters and 24 meters.

Solution. We let $x(t)$ give the length of the leg which is changing. By Pythagoras's theorem, the length in meters of the hypotenuse is

$$h(t)=\sqrt{49+x(t)^2}.$$

We express the square root as raising to the 1/2 power and use the power rule and chain rule to differentiate

$$h'(t)=\frac{d}{dt}(49+x(t)^2)^{1/2}=\frac{1}{2}(49+x(t)^2)^{-1/2}2x(t)x'(t).$$

We let t_0 be the time when $x=24$ and $x'=-0.3$. Note that since the length is decreasing, the derivative is negative. Substituting these values gives

$$h'(t_0)=\frac{1}{\sqrt{7^2+24^2}}24(-0.3)=-36/125 \text{ m/s}.$$

Note that if we work out the units, they turn out to be m/s as expected.

∎

Example. If the radius r of a circle is increasing and $dr/dt = 5$ when $r = 6$, find the rate of change of the area of the circle when $r = 6$. Give the units if the radius is measured in meters and time is measured in minutes.

Solution. We recall that $A = \pi r^2$ and thus $\frac{dA}{dr} = 2\pi r$. Using the chain rule,

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

The units for area are meter^2 and time is measured in minutes, so the units for dA/dt are meter^2/minute. ∎

February 26, 2015