1 Lecture 40: Exponential growth and decay

- A model for exponential growth and decay
- Fitting our solution to data, doubling time and half-life
- Examples: Population growth, carbon dating, estimating time of death.

1.1 Warmup question

If $f'(x) = 2f(x)$, what is f?

1.2 A model for population growth

A simple model for a population is to assume that the rate of change of the population is directly proportional to the total population. If y is the total population and b is the birth rate or the fraction of the population that give birth each unit of time, then the rate of change due to births is by. Similarly if d is the death rate, then dy is the number of deaths per year. The rate of change of the population y with respect to time is $by - dy$. We can express this using a derivative as

$$\frac{dy}{dt} = ky \qquad (1)$$

where the constant $k = b - d$. Since y is positive, we have that y is increasing if $k > 0$ and y is decreasing if $k < 0$. It is easy to see that the family of functions $y(t) = P_0 e^{kt}$ are solutions of (1). Here, P_0 is a constant and each choice of P_0 gives a different solution of (1). In fact, these are the only solutions.

Theorem 2 *If y solves (1) on an interval I, then there is a constant P_0 so that $y(t) = P_0 e^{kt}$ on I.*

Proof. If we want to show $y(t) = P_0 e^{kt}$, then we expect that $e^{-kt}y(t)$ is constant. One way to show a function is constant is to show the derivative is zero. We consider the function $f(t) = e^{-kt}y(t)$ and differentiate f using the product rule,

$$f'(t) = (-k)e^{-kt}y(t) + e^{-kt}y'(t) = -ke^{-kt}y + e^{-kt}ky = 0.$$

We have used (1) for the second equality. Since $f' = 0$ on an interval, f is constant. If we call the constant P_0, then we have $f(t) = P_0 e^{kt}$. ∎

1.3 Model fitting

If a function y is given by $y(t) = P_0 e^{kt}$ and $k > 0$ we say that y grows exponentially. If $k < 0$, then we say that y decays exponentially. In this case we will often replace k by $-k$ and write $y(t) = P_0 e^{-kt}$. Thus the constant k is positive. Once we know that we have exponential growth and decay, we need two additional bits of data to determine the constant k and the value of P_0. Note we have $y(0) = P_0$ so P_0 is the initial value of y.

Example. Suppose that y obeys (1), $y(1) = 2$ and $y(2) = 5$. Find $y(t)$.

Solution. We know that $y(t) = P_0 e^{kt}$. The given information tells us that

$$P_0 e^k = 2, \qquad P_0 e^{2k} = 5.$$

To find k, we may divide these equations and find

$$\frac{P_0 e^{2k}}{P_0 e^k} = \frac{5}{2}.$$

Taking the natural logarithm of both sides, we find $k \ln(e) = \ln(5/2)$. Thus, $y(t) = P_0 e^{t \ln(5/2)}$. Substiuting $t = 1$, we find $P_0 e^{\ln(5/2)} = 2$ or $P_0 = 4/5$. Summarizing,

$$y(t) = \frac{4}{5} e^{t \ln(5/2)} = \frac{4}{5} \left(\frac{5}{2}\right)^t.$$

∎

One of the important features of exponential growth is the existence of a time T during which the population doubles, i.e. that $y(t + T) = 2y(t)$. To see that this doubling property is independent of t, we consider

$$\frac{y(t+T)}{y(t)} = \frac{P_0 e^{k(t+T)}}{P_0 e^{kt}} = e^{kT}.$$

Thus, to find the doubling time we need to solve $e^{kT} = 2$ for T.

Example. If $f(t) = 100e^{0.3t}$, find the doubling time.
Can you find the tripling time?

Solution. We have

$$\frac{f(t+T)}{f(t)} = e^{0.3T}.$$

We solve the equation $e^{0.3T} = 2$ to find $T = \ln(2)/0.3$.

The same argument gives that the tripling time is $\ln(3)/0.3$. ∎

In the case of exponential decay, the corresponding notion is half-life. This is the time T so that $y(t+T) = \frac{1}{2}y(t)$.

1.4 Examples

Example. Suppose that a population grows at a rate of 3% per day. If the initial population is 100, when will the population reach 1000. What is the doubling time?

Solution. Let $y(t)$ denote the population at time t where t is measured in days. If $y(t)$ is the population at time t, we know that $y' = 0.03y$ and $y(0) = 100$. Thus, $y(t) = 100e^{0.03t}$. We are asked to find the time T when $y(T) = 1000$. Thus we want $y(T) = 100e^{0.03T} = 1000$. Thus we need $e^{0.03T} = 10$ or $T = \ln(10)/0.03 \approx 76.753$ days.

To find the doubling time, we solve

$$100e^{0.03(t+T)} = 2 \cdot 100e^{0.03t}$$

for T to find $e^{0.03T} = 2$ or $T = \ln(2)/0.03 \approx 23.1$ days. ∎

The carbon in the atmosphere includes two isotopes C_{14} and C_{12} and the ratio of these isotopes in living plants and animals is roughly the same as in the atmosphere. When the organism dies, the C_{14} starts to decay. If $R(t)$ represents the ratio of C_{14} to C_{12} at a time t years after the organism's death, we find that $R(t) = R_a e^{-kt}$ where R_a is ratio of C_{14} to C_{12} in the atmosphere. The half-life of $C14$ is approximately 5730 years.

Example. Suppose that in a sample of wood, the ratio of $C14$ to C_{12} is 23% of the ratio in the atmosphere. How long ago was the wood in a living tree?

Solution. Let $R(t)$ the ratio of C_{14} to C_{12} at a time t years after the tree dies. As $R(t) = R_a e^{-kt}$, we want to find the time T so that $R(T) = 0.23R_a$.

Before we can do this, we need to find k. We use that the half-life is 5730 years to find k. Since $e^{-k5730} = \frac{1}{2}$, we have that $-k = \ln(1/2)/5730$ of $k = \ln(2)/5730 \approx 1.21 \cdot 10^{-4}$. Thus, if $e^{-kT} = 0.23$, we may solve for T and find that $T = \ln(0.23)/(-k) = \ln(0.23)5730/\ln(2) \approx 12,149$ years. ∎

Finally, we give an example related to temperature.

Example. If we place a hot object whose temperature is $\Theta(t)$ in a room of temperature T, the object's temperature will fall and approach T. Newton's law of cooling tells that the rate of change of the temperature is proportional to the difference between the object's temperature and the room's temperature. This can be expressed using the derivative by

$$\frac{d\Theta}{dt} = -k(\Theta - T)$$

where $k > 0$ is a constant. This example indicates that this law can be used to estimate time of death.

A body is found in a room at 12noon and its temperature is 34° C. One hour later, the temperature is 29° C. The temperature of the room is 25 ° C. Estimate the time of death.

Solution. To answer this question, we need to know that the normal temperature of a human is 37° C.

We take $t = 0$ to be 12 noon and measure time in hours. We let $y(t) = \Theta(t) - T$ and note that Newton's law of cooling tells us that $y' = -ky$. We know that $y(0) = 34 - 25$ and thus $y(t) = 9e^{-kt}$. Since we are given that $y(1) = 4$, we can solve the equation $9e^{-k} = 4$ to obtain that $k = \ln(9/4)$. Finally, to answer the question, we want to find the time when $y(t) = 12$. Thus we need to solve $9e^{-kt} = 12$ for t and obtain that $3/4 = e^{kt}$ or $t = \ln(3/4)/k \approx -0.35476$. This means that the time of death was approximately 21 minutes before 12 noon or 11:39 am. ∎

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