Lecture 8: L’Hôpital’s rule

• Recognize indeterminate forms.

• Compute limits using L’Hôpital’s rule.

Some well-known limits

Recall some familiar limits:

\[
\lim_{{x \to 0}} \frac{\sin(2x)}{x} = 2 \\
\lim_{{x \to 0}} \frac{\ln(1 + x)}{x} = 1 \\
\lim_{{x \to 0}} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}
\]

The first two may be viewed as difference quotients and this allows us to know the limit. For example,

\[
\frac{\sin(2x)}{x} = \frac{\sin(2x) - 0}{x - 0}
\]

and thus with \( f(x) = \sin(2x) \),

\[
\lim_{{x \to 0}} \frac{\sin(2x)}{x} = f'(0) = 2
\]

Indeterminate forms

Each of the limits above can be thought of as

\[
\lim_{{x \to a}} \frac{f(x)}{g(x)}
\]

where

\[
\lim_{{x \to a}} f(x) = \lim_{{x \to a}} g(x) = 0.
\]

This is an indeterminate form of type \( \frac{0}{0} \).

As the above examples show, these limits can have many values depending on the functions \( f \) and \( g \). This is why they are called indeterminate forms.

Exercise. Given a value \( a \), can you choose \( f \) and \( g \) so that the indeterminate form has the value \( a \)?
If in the above limit we have
\[ \lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \infty \]
then this is an indeterminate form of type \( \frac{\infty}{\infty} \). (We say the same if one or both of the limits is \(-\infty\).

**Example.**
\[ \lim_{x \to \infty} \frac{e^x}{x}. \]

**Indeterminate forms of type 0\(\infty\).** If
\[ \lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} g(x) = \infty \]
then
\[ \lim_{x \to a} f(x)g(x) \]
is an indeterminate form. This can be rewritten as 0/0 by considering \(f(x)/(g(x)^{-1})\).

**Exercise.** Can you also rewrite this indeterminate form as \(\infty/\infty\)?

**Indeterminate forms of 1\(\infty\).** By rewriting
\[ f(x)^{g(x)} = e^{g(x)\ln(f(x))} \]
such an indeterminate form can be evaluated by if we understand the indeterminate forms discussed above.

**Example.** Compute
\[ \lim_{x \to 0} (1 + x)^{1/x}. \]

**Solution.** We rewrite this as
\[ \lim_{x \to 0} e^{\frac{1}{x} \ln(1+x)} \]
since we evaluated the limit of the exponent above and the exponential function is continuous, we have
\[ \lim_{x \to 0} (1 + x)^{1/x} = e. \]
The rule, finally.

**Theorem 1** If \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is an indeterminate form of type \( 0/0 \) of \( \infty/\infty \), and we have

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L
\]

then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists and equals \( L \).

We will not give the proof here. If \( a \) is finite, then the rule can be proven by a generalization of the mean value theorem.

**Example.** Try to apply L’Hopital’s rule to

\[
\lim_{x \to 0} \frac{x}{e^x}.
\]

What do we obtain? What is the value of the limit.

**Solution.** L’Hopital gives 1 and the correct value is 0.

**Examples**

**Example.** Use L’Hopital to compute

\[
\lim_{x \to 0} (1 + x)x, \quad \lim_{x \to \infty} \frac{e^x}{x}, \quad \lim_{x \to \infty} xe^{-x}, \quad \lim_{x \to 0} 1 - \cos x x^2
\]

**Example.** Show that

\[
\lim_{x \to \infty} e^x x^n = \infty
\]

for \( n = 1, 2, 3, \ldots \).