1 Lecture: Properties of exponentials

- Compute integrals and derivatives involving exponential functions
- Find solutions of constant coefficient differential equations when the solution is of the form $e^x$.

Recall the “cancellation rules”: $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$.

**Definition.** We define the exponential function $e^x$ to be the inverse function of $\ln x$.

**Example.** What are the domain and range of $e^x$?

**Solution.** The domain of $\ln x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. For $e^x$, these are reversed, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

**Proposition 1**

$$\frac{d}{dx} e^x = e^x \quad \int e^x \, dx = e^x + C$$

**Proof. First solution.** If $g(x)$ is an inverse to $\log x$, then we have $\log g(x) = x$. Differentiating this expression, gives

$$g'(x) \frac{1}{g(x)} = 1$$

and then solving gives $g'(x) = g(x)$.
Second solution. According to the formula for the derivative of an inverse function, we have
\[
f^{-1'}(x) = \frac{1}{f'(f^{-1}(x))}
\]
Here, we have \( f(x) = \log x \), \( f'(x) = 1/x \) and \( f^{-1}(x) = e^x \) so that
\[
f^{-1'}(x) = e^x.
\]
The formula for the integral follows from the formula for anti-derivatives.

Example. Where are the inflection points for \( e^{-x^2} \)?
Find the integrals:
\[
\int x e^{-x^2} \, dx \quad \int e^{-x^2} \, dx.
\]

Proposition 2 \textit{(algebraic properties)} If \( x \) and \( y \) are real numbers, then
\begin{itemize}
  \item \( e^{x+y} = e^x e^y \)
  \item \( e^{x-y} = e^x / e^y \)
  \item \( (e^x)^y = e^{xy} \)
\end{itemize}
Proof. Let \( x = \log a \) and \( y = \log b \). Then, \( e^x e^y = e^{\log a \log b} = ab = e^{\log(ab)} = e^{\log a + \log b} = e^{x+y} \). We have used the properties of log and the cancellation conditions.

Second and third are exercises.

Remark. If one can follow through all the words, one should see that these properties are a restatement of the properties of logarithms.

Example. Show that \( y = \frac{x^e}{e^{x+1}} \) is one-one. Find the inverse function.

Solution. \( y' = e^x/(1+e^x)^2 \). Since \( y' > 0 \) on the interval \( (-\infty, \infty) \), then \( y \) is increasing on this interval.
Solving \( y = e^x/(1 + e^x) \) gives that \( x = \log y(1 - y) \) for \( 0 < y < 1 \).

Example. For what values of \( r \) does \( e^{rx} \) solve
\[
y'' + 2y - 3y = 0, \quad y'' + y = 0.
\]