1 Lecture 19: Arc length–graphs

1. Compute lengths of curves which are presented as graphs.

A definition of arc-length

By now, we know how to find the length of a line segment. If the coordinates of the endpoints are \((x_1, y_1)\) and \((x_2, y_2)\), then the theorem of Pythagoras tells us that we can find the length of the line segment joining these points by the formula

\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

We will use this simple fact to give a definition of the length of a curve. To do this, we begin by approximating the curve by an inscribed polygonal path. If the curve is of the form \(\{(x, f(x)) : a \leq x \leq b\}\), then we partition the interval \([a, b]\) with \(P = \{a = x_0 < x_1 < x_2 \ldots < x_n = b\}\). We let \(\Delta x_i = x_i - x_{i-1}\) and then we connect the points \(\{(x_i, f(x_i)) : i = 0 \ldots n\}\) to form a polygonal curve whose length is

\[
S = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sum_{i=1}^{n} \sqrt{1 + f'(t_i)^2} \Delta x_i.
\]

This looks suspiciously like a Riemann sum for an integral and if we let the mesh of the partition tend to zero, we obtain the arc length of this curve is

\[
\int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx.
\]

Some examples

We begin by showing that this definition agrees with what we learned in school for lines and circles.

Example. Find the length of the line \(y = 2x\) for \(1 \leq x \leq 4\).

Example. Find the length of the semi-circle \(y = \sqrt{9 - x^2}, 0 \leq x \leq 3\).

Example. Find the length of the curve \(y = x^{3/2}\) for \(0 \leq x \leq 1\).

Solution.

\[
\int_{0}^{1} \sqrt{1 + 9x/4} \, dx = \frac{8}{27}((13/4)^{3/2} - 1) \approx 1.44
\]

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