The following problems will be due on 19 September.

1. Provide a proof of Hölder’s inequality from Jensen’s inequality.
   Thus for \( g \geq 0 \) and \( f \geq 0 \) and \( 1 < q < \infty \) suppose \( q' = q/(q - 1) \). In the special case where \( g(x) > 0 \) for all \( x \) and \( \|g\|_q = 1 \), apply Jensen’s inequality where the weight (the function \( p \)) is \( g^q \) and \( f \) is replaced by \( f/g^{q-1} \). Deduce the general case from this special case.

2. (Wheeden and Zygmund, p. 124) Suppose that \( \phi : \mathbb{R} \to \mathbb{R} \) is continuous and for all real numbers \( x \) and \( y \),
   \[
   \phi\left(\frac{x + y}{2}\right) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y).
   \]
   Prove that \( \phi \) is convex.

3. (Not graded) Does the result of the previous problem hold if \( \phi \) is not assumed to be continuous?

4. (Wheeden and Zygmund, p. 143) Prove the converse of Hölder’s inequality when \( p = 1 \) and when \( p = \infty \). You may assume the functions are real-valued.

5. Show that Minkowski’s inequality may fail if \( p < 1 \).

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