The following problems will be due on 12 November 2007.

1. Let $\Sigma$ be the $\sigma$-algebra on the real line which is generated by the intervals $[k, k + 1)$, for $k \in \mathbb{Z}$. Is the function given by $f(x) = x$ measurable for this $\sigma$-algebra?

2. (Wheeden and Zygmund, p. 191, #13) Let $\phi$ be a real-valued additive set function on $(\mathcal{S}, \Sigma)$. Let $P_1$ and $P_2$ be two sets as in the Hahn-decomposition for $\phi$. Show that the symmetric difference, $P_1 \Delta P_2$, is a null set. We say that $E$ is a null set for an additive set function $\phi$, if we have $\phi(A) = 0$ for all measurable subsets $A$ of $E$.

3. (Extra credit) Let $f$ be in $L^2(\mathbb{R})$ and for $(x, y)$ in $\mathbb{R}^2$ with $y > 0$, define the Poisson integral of $f$ by

$$u(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} f(t) \frac{y}{y^2 + (x - t)^2} \, dt.$$ 

Provide a careful proof that $D_y u(x, y)$ exists.

November 1, 2007