Lecture notes: harmonic analysis

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Preface

These notes are intended for a course in harmonic analysis on $\mathbf{R}^n$ which was offered to graduate students at the University of Kentucky in Spring of 2001. The background for this course is a course in real analysis which covers measure theory and the basic facts of life related to L^p spaces. The students who were subjected to this course had studied from *Measure and integral* by Wheeden and Zygmund and the book by Folland, *Real analysis: a modern introduction.*

Much of the material in these notes is taken from the books of Stein *Singular integrals and differentiability properties of functions,* and *Harmonic analysis* and the book of Stein and Weiss, *Fourier analysis on Euclidean spaces.*

The exercises serve a number of purposes. They illustrate extensions of the main ideas that I did not have time to carry out in detail. They occasionally state difficult and unsolvable problems. They provide a chance to state simple results that will be needed later. Often, the result stated will be wrong. Please let me know about errprs.

Chapter 1

The Fourier transform on L^1

In this section, we define the Fourier transform and give the basic properties of the Fourier transform of an $L^1(\mathbf{R}^n)$ function. We will use L^1 to be the space of Lebesgue measurable functions with the norm $\|f\|_1 = \int_{\mathbf{R}^n} |f(x)|\, dx$. More generally, $L^p(\mathbf{R}^n)$ denotes the space of Lebesgue measurable functions for which $\|f\|_p = (\int_{\mathbf{R}^n} |f(x)|^p\, dx)^{1/p}$. When $p = \infty$, the space $L^\infty(\mathbf{R}^n)$ is the collection of measurable functions which are bounded, after we neglect a set of measure zero. These spaces of functions are examples of Banach spaces. We recall that a vector space V over $\mathbf{C}$ with a function $\|\cdot\|$ is called a normed vector space if $\|\cdot\| : V \to [0,\infty)$ and satisfies

$$\begin{aligned} \|f+g\| &\leq \|f\| + \|g\|, \qquad f, g \in V \\ \|\lambda f\| &= |\lambda|\|f\|, \qquad f \in v,\ \lambda \in \mathbf{C} \\ \|f\| &= 0, \qquad \text{if and only if } f = 0. \end{aligned}$$

A function $\|\cdot\|$ which satisfies these properties is called a *norm*. If $\|\cdot\|$ is a norm, then $\|f-g\|$ defines a metric. A normed vector space $V, \|\cdot\|$ is called a *Banach space* if V is complete in the metric defined using the norm. Throughout these notes, functions are assumed to be complex valued.

1.1 Definition and symmetry properties

We define the Fourier transform. In this definition, $x \cdot \xi$ is the inner product of two elements of $\mathbf{R}^n$, $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$.

Definition 1.1 *If $f \in L^1(\mathbf{R}^n)$, then the* Fourier transform *of f, $\hat{f}$, is a function defined* *on* $\mathbf{R}^n$ *and is given by*

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

The Fourier transform is a continuous map from L^1 to the bounded continuous functions on $\mathbf{R}^n$.

Proposition 1.2 *If* $f \in L^1(\mathbf{R}^n)$*, then* $\hat{f}$ *is continuous and*

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

Proof. If $\xi_j \to \xi$, then $e^{-ix\cdot\xi_j} \to e^{-ix\cdot\xi}$. Hence by the Lebesgue dominated convergence theorem, $\hat{f}(\xi_j) \to \hat{f}(\xi)$. ∎

The inequality in the conclusion of Proposition 1.2 is equivalent to the continuity of the map $f \to \hat{f}$. This is an application of the conclusion of the following exercise.

Exercise 1.3 *A linear map* $T : V \to W$ *between normed vector spaces is continuous if and only if there exists a constant* C *so that*

$$\|Tf\|_W \leq C\|f\|_V.$$

In the following proposition, we use $A^{-t} = (A^{-1})^t$ for the transpose of the inverse of an $n \times n$ matrix, A.

Exercise 1.4 *Show that if* A *is an* $n \times n$ *invertible matrix, then* $(A^{-1})^t = (A^t)^{-1}$.

Exercise 1.5 *Show that* A *is an* $n \times n$ *matrix, then* $Ax \cdot y = x \cdot A^t y$.

Proposition 1.6 *If* A *is an* $n \times n$ *invertible matrix, then*

$$\widehat{f \circ A} = |\det A| \hat{f} \circ A^{-t}.$$

Proof. If we make the change of variables, $y = Ax$ in the integral defining $\widehat{f \circ A}$, then we obtain

$$\int_{\mathbf{R}^n} f \circ A(x)\, dx = |\det A|^{-1} \int_{\mathbf{R}^n} f(y) e^{-iA^{-1}y\cdot\xi}\, dy = |\det A|^{-1} \int_{\mathbf{R}^n} f(y) e^{-iy\cdot A^{-t}\xi}\, dy.$$

∎

A simple application of this theorem is that if we set $f_\epsilon(x) = \epsilon^{-n} f(x/\epsilon)$, then

$$\hat{f}_\epsilon(\xi) = \hat{f}(\epsilon\xi). \tag{1.7}$$

Recall that an orthogonal matrix is an $n \times n$-matrix with real entries which satisfies $O^t O = I_n$ where I_n is the $n \times n$ identity matrix. Such matrices are clearly invertible since $O^{-1} = O^t$. The group of all such matrices is usually denoted by $O(n)$.

Corollary 1.8 *If $f \in L^1(\mathbf{R}^n)$ and O is an orthogonal matrix, then $\hat{f} \circ O = \widehat{f \circ O}$.*

Exercise 1.9 *If $x \in \mathbf{R}^n$, show that there is an orthogonal matrix O so that $Ox = (|x|, 0, \ldots, 0)$.*

Exercise 1.10 *Show that an $n \times n$ matrix on $\mathbf{R}^n$ is orthogonal if and only if $Ox \cdot Ox = x \cdot x$ for all $x \in \mathbf{R}^n$.*

We say that function f defined on $\mathbf{R}^n$ is *radial* if there is a function F on $[0, \infty)$ so that $f(x) = F(|x|)$. Equivalently, a function is radial if and only if $f(Ox) = f(x)$ for all orthogonal matrices O.

Corollary 1.11 *Suppose that f is in L^1 and f is radial, then $\hat{f}$ is radial.*

Proof. We fix ξ in $\mathbf{R}^n$ and choose O so that $O\xi = (|\xi|, 0, \ldots, 0)$. Since $f \circ O = f$, we have that $\hat{f}(\xi) = \widehat{f \circ O}(\xi) = \hat{f}(O\xi) = \hat{f}(|\xi|, 0, \ldots, 0)$. ∎

The main applications of the Fourier transform depend on the fact that it turns operations that commute with translations into multiplication operations. That is, it diagonalizes operations which commute with translations. The first glimpse we will see of this is that the operation of translation by h (which surely commutes with translations) corresponds to multiplying the Fourier transform by $e^{ih\cdot\xi}$. We will use τ_h to denote translation by h, $\tau_h f(x) = f(x + h)$.

Exercise 1.12 *If f is a nice function on $\mathbf{R}^n$, show that*

$$\frac{\partial}{\partial x_j} \tau_h f = \tau_h \frac{\partial}{\partial x_j} f.$$

Proposition 1.13 *If f is in $L^1(\mathbf{R}^n)$, then*

$$\widehat{\tau_h f}(\xi) = e^{ih\cdot\xi} \hat{f}(\xi).$$

Also,

$$(e^{ix\cdot h} f)\hat{} = \tau_{-h}(\hat{f}). \tag{1.14}$$

Proof. We change variables $y = x + h$ in the integral

$$\widehat{\tau_h f}(\xi) = \int f(x+h)e^{-ix\cdot\xi}\,dx = \int f(y)e^{-i(y-h)\cdot\xi}\,dy = e^{ih\cdot\xi}\hat{f}(\xi).$$

The proof of the second identity is just as easy and is left as an exercise. ∎

Example 1.15 *If $I = \{x : |x_j| < 1\}$, then the Fourier transform of $f = \chi_I$ is easily computed,*

$$\hat{f}(\xi) = \prod_{j=1}^{n} \int_{-1}^{1} e^{ix_j\xi_j}\,dx_j = \prod_{j=1}^{n} \frac{2\sin\xi_j}{\xi_j}.$$

In the next exercise, we will need to write integrals in polar coordinates. For our purposes, this means that we have a Borel measure σ on the *sphere*, $\mathbf{S}^{n-1} = \{x' \in \mathbf{R}^n : |x'| = 1\}$ so that

$$\int_{\mathbf{R}^n} f(x)\,dx = \int_0^\infty \int_{\mathbf{S}^{n-1}} f(rx')\,d\sigma(x')r^{n-1}\,dr.$$

Exercise 1.16 *If $B_r(x) = \{y : |x - y| < r\}$ and $f = \chi_{B_1(0)}$, compute the Fourier transform $\hat{f}$.*

Hints: 1. Since f is radial, it suffices to compute $\hat{f}$ at $(0, \ldots, r)$ for $r > 0$. 2. Write the integral over the ball as an iterated integral where we integrate with respect to $x' = (x_1, \ldots, x_{n-1})$ and then with respect to x_n. 3. You will need to know the volume of a ball, see exercise 1.29 below. 4. At the moment, we should only complete the computation in 3 dimensions (or odd dimensions, if you are ambitious). In even dimensions, the answer cannot be expressed in terms of elementary functions. See Chapter 13 for the answer in even dimensions. The is

$$\hat{f}(\xi) = \frac{\omega_{n-2}}{n-1} \int_{-1}^{1} e^{-it|\xi|}(1-t^2)^{(n-1)/2}\,dt.$$

Theorem 1.17 *(Riemann-Lebesgue) If f is in $L^1(\mathbf{R}^n)$, then*

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

Proof. We let $X \subset L^1(\mathbf{R}^n)$ be the collection of functions f for which $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$. It is easy to see that X is a vector space. Thanks to Proposition 1.2, X is closed in the L^1-norm. According to Example 1.15, Proposition 1.13 and Proposition 1.6 the characteristic function of every rectangle is in X. Since finite linear combinations of characteristic functions of rectangles are dense in L^1, $X = L^1(\mathbf{R}^n)$. ∎

Combining the Riemann-Lebesgue theorem and the first proposition above, we can show that the image of L^1 under the Fourier transform is contained in $C_0(\mathbf{R}^n)$, the continuous functions on $\mathbf{R}^n$ which vanish at infinity. This containment is strict. We will see that the Fourier transform of the surface measure on the sphere $\mathbf{S}^{n-1}$ is in $C_0(\mathbf{R}^n)$. It is a difficult and unsolved problem to describe the image of L^1 under the Fourier transform.

One of our goals is to relate the properties of f to those of $\hat{f}$. There are two general principles which we will illustrate below. These principles are: *If f is smooth, then $\hat{f}$ decays at infinity* and *If f decays at infinity, then $\hat{f}$ is smooth.* We have already seen some weak illustrations of these principles. Proposition 1.2 asserts that if f is in L^1, which requires decay at infinity, then $\hat{f}$ is continuous. The Riemann-Lebesgue lemma tells us that if f is in L^1, and thus is smoother than the distributions to be discussed below, then $\hat{f}$ has limit 0 at infinity. The propositions below give further illustrations of these principles.

Proposition 1.18 *If f and $x_j f$ are in L^1, then $\hat{f}$ is differentiable and the derivative is given by*

$$i\frac{\partial}{\partial \xi_j}\hat{f} = \widehat{x_j f}.$$

Furthermore, we have

$$\|\frac{\partial f}{\partial \xi_j}\|_\infty \leq \|x_j f\|_1.$$

Proof. Let $h \in \mathbf{R}$ and suppose that e_j is the unit vector parallel to the x_j-axis. Using the mean-value theorem from calculus, one obtains that

$$\left|\frac{e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}}{h}\right| \leq |x_j|.$$

Our hypothesis that $x_j f$ is in L^1 allows to use the dominated convergence theorem to bring the limit inside the integral to compute the partial derivative

$$\frac{\partial \hat{f}(\xi)}{\partial \xi_j} = \lim_{h\to 0}\int \frac{e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}}{h} f(x)\,dx = \int (-ix_j)e^{-ix\cdot\xi} f(x)\,dx.$$

The estimate follows immediately from the formula for the derivative. ∎

Note that the notation in the previous proposition is not ideal since the variable x_j appears multiplying f, but not as the argument for f. One can resolve this problem by decreeing that the symbol x_j stands for the multiplication operator $f \to x_j f$ and the j component of x.

For the next proposition, we need an additional definition. We say f has a partial derivative with respect to x_j in the L^p sense if f is in L^p and there exists a function $\partial f/\partial x_j$ so that

$$\lim_{h\to 0} \| \frac{1}{h}(\tau_{he_j} f - f) - \frac{\partial f}{\partial x_j}\|_p = 0.$$

Proposition 1.19 *If f is differentiable with respect to x_j in the L^1-sense, then*

$$i\xi_j \hat{f} = \widehat{\frac{\partial f}{\partial x_j}}.$$

Furthermore, we have

$$\|\xi_j \hat{f}\|_\infty \leq \left\| \frac{\partial f}{\partial x_j} \right\|_1 .$$

Proof. Let $h > 0$ and let e_j be a unit vector in the direction of the x_j - axis. Since the difference quotient converges in L^1, we have

$$\int_{\mathbf{R}^n} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_j}(x)\, dx = \lim_{h\to 0} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \frac{f(x+he_j) - f(x)}{h}\, dx.$$

In the last integral, we can "difference-by-parts" to move the difference quotient over to the exponential. More precisely, we can make a change of variables $y = x + he_j$ to obtain

$$\int_{\mathbf{R}^n} \frac{e^{-i(x-he_j)\cdot\xi} - e^{-ix\cdot\xi}}{h} f(x)\, dx.$$

Since the difference quotient of the exponential converges pointwise and boundedly (in x) to $i\xi_j e^{-ix\cdot\xi}$, we can use the dominated convergence theorem to obtain $\widehat{\partial f}/\partial x_j = i\xi_j \hat{f}$. ∎

Finally, our last result on translation invariant operators involves convolution. Recall that if f and g are measurable functions on $\mathbf{R}^n$, then the convolution is defined by

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\, dy$$

provided the integral on the right is defined for a.e. x.

Some of the basic properties of convolutions are given in the following exercises. The solutions can be found in most real analysis texts.

Exercise 1.20 *If f is in L^1 and g are in L^p, with $1 \le p \le \infty$, show that $f * g(x)$ is defined a.e. and*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Exercise 1.21 *Show that the convolution is commutative. If $f * g(x)$ is given by a convergent integral, then*

$$f * g(x) = g * f(x).$$

If f, g and h are in L^1, show that convolution is associative

$$f * (g * h) = (f * g) * h.$$

Hint: Change variables.

Exercise 1.22 *The map $f \to f * g$ commutes with translations:*

$$\tau_h(f * g) = (\tau_h f) * g.$$

Exercise 1.23 *(Young's convolution inequality) If the exponents p, q and s satisfy $1/s = 1/p + 1/q - 1$, then*

$$\|f * g\|_s \le \|f\|_p \|g\|_q.$$

Proposition 1.24 *If f and g are in L^1, then*

$$(f * g)\hat{} = \hat{f}\hat{g}.$$

We calculate a very important Fourier transform. The function W in the next proposition gives (a multiple of) the Gaussian probability distribution.

Proposition 1.25 *Let $W(x)$ be defined by $W(x) = \exp(-|x|^2/4)$. Then*

$$\hat{W}(\xi) = (\sqrt{4\pi})^n \exp(-|\xi|^2).$$

Proof. We use Fubini's theorem to write $\hat{W}$ as a product of one-dimensional integrals

$$\int_{\mathbf{R}^n} e^{-|x|^2/4} e^{-ix\cdot\xi}\, dx = \prod_{j=1}^{n} \int_{\mathbf{R}} e^{-x_j^2/4} e^{-ix_j\xi_j}\, dx_j.$$

To evaluate the one-dimensional integral, we use complex analysis which makes everything trivial. We complete the square in the exponent for the first equality and then use Cauchy's integral theorem to shift the contour of integration in the complex plane. This gives

$$\int_{\mathbf{R}} e^{-x^2/4} e^{-ix\xi}\, dx = e^{-|\xi|^2} \int_{\mathbf{R}} e^{-(\frac{x}{2}+i\xi)^2}\, dx = e^{-|\xi|^2} \int_{\mathbf{R}} e^{-|x|^2/4}\, dx = \sqrt{4\pi} e^{-|\xi|^2}.$$

∎

Exercise 1.26 *Carefully justify the shift of contour in the previous proof.*

Exercise 1.27 *Establish the formula*

$$\int_{\mathbf{R}^n} e^{-\pi|x|^2}\,dx = 1$$

which was used above. a) First consider $n = 2$ and write the integral over $\mathbf{R}^2$ in polar coordinates.

b) Deduce the general case from this special case.

In the next exercise, we use the Γ function, defined for $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}.$$

Exercise 1.28 *Use the result of the previous exercise and polar coordinates to compute ω_{n-1}, the $n-1$-dimensional measure of the unit sphere in $\mathbf{R}^n$ and show that*

$$\omega_{n-1} = \sigma(\mathbf{S}^{n-1}) = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}.$$

For the next exercise, we introduce our notation for the *Lebesgue measure* of a set E, $m(E)$

Exercise 1.29 *Use the result of the previous exercise and polar coordinates to find the volume of the unit ball in $\mathbf{R}^n$. Show that*

$$m(B_1(0)) = \omega_{n-1}/n.$$

1.2 The Fourier inversion theorem

In this section, we show how to recover an L^1-function from the Fourier transform. A consequence of this result is that we are able to conclude that the Fourier transform is injective. The proof we give depends on the Lebesgue differentiation theorem. We will discuss the Lebesgue differentiation theorem in the chapter on maximal functions, Chapter 4.

We begin with a simple lemma.

Lemma 1.30 *If f and g are in $L^1(\mathbf{R}^n)$, then*

$$\int_{\mathbf{R}^n} \hat{f}(x)g(x)\,dx = \int_{\mathbf{R}^n} f(x)\hat{g}(x)\,dx.$$

Proof. We consider the integral of $f(x)g(y)e^{-iy\cdot x}$ on $\mathbf{R}^{2n}$. We use Fubini's theorem to write this as an iterated integral. If we compute the integral with respect to x first, we obtain the integral on the left-hand side of the conclusion of this lemma. If we compute the integral with respect to y first, we obtain the right-hand side. ∎

We are now ready to show how to recover a function in L^1 from its Fourier transform.

Theorem 1.31 *(Fourier inversion theorem) If f is in $L^1(\mathbf{R}^n)$ and we define f_t for $t > 0$ by*

$$f_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-t|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi,$$

then

$$\lim_{t\to 0^+} \|f_t - f\|_1 = 0$$

and

$$\lim_{t\to 0^+} f_t(x) = f(x), \qquad a.e.\ x.$$

Proof. We consider the function $g(x) = e^{-t|x|^2+iy\cdot x}$. By Proposition 1.25, (1.7) and (1.14), we have that

$$\hat{g}(x) = (2\pi)^n (4\pi t)^{-n/2} \exp(-|y-x|^2/4t).$$

Thus applying Lemma 1.30 above, we obtain that

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} e^{-t|\xi|^2}\, d\xi = \int_{\mathbf{R}^n} f(x)(4\pi t)^{-n/2} \exp(-\frac{|y-x|^2}{4t})\, dx.$$

Thus, $f_t(x)$ is the convolution of f with the Gaussian and it is known that $f_t \to f$ in L^1. That f_t converges to f pointwise is a standard consequence of the Lebesgue differentiation theorem. A proof will be given below. ∎

It is convenient to have a notation for the inverse operation to the Fourier transform. The most common notation is $\check{f}$. Many properties of the inverse Fourier transform follow easily from the properties of the Fourier transform and the inversion. The following simple formulas illustrate the close connection:

$$\check{f}(x) = \frac{1}{(2\pi)^n} \hat{f}(-x) \tag{1.32}$$

$$\check{f}(x) = \frac{1}{(2\pi)^n} \bar{\hat{\bar{f}}}(x). \tag{1.33}$$

If $\hat{f}$ is in L^1, then the limit in t in the Fourier inversion theorem can be brought inside the integral (by the dominated convergence theorem) and we have

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi.$$

Exercise 1.34 *Prove the formulae (1.32) and (1.33) above.*

Chapter 2

Tempered distributions

In this chapter, we introduce the Schwartz space. This is a space of well-behaved functions on which the Fourier transform is invertible. One of the main interests of this space is that other interesting operations such as differentiation are also continuous on this space. Then, we are able to extend differentiation and the Fourier transform to act on the dual space. This dual space is called the space of tempered distributions. The word tempered means that in a certain sense, the distributions do not grow to rapidly at infinity. Distributions have a certain local regularity–on a compact set, they only involve finitely many derivatives. Given the connection between the local regularity of a function and the growth of its Fourier transform, it seems likely that any space on which the Fourier transform acts should have some restriction on the growth at infinity.

2.1 Test functions and tempered distributions

The main notational complication of this chapter is the use of multi-indices. A multi-index is an n-tuple of non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$. For a multi-index α, we let

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

We also use this notation for partial derivatives,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}.$$

Several other related notations are

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \quad \text{and} \quad \alpha! = \alpha_1! \ldots \alpha_n!.$$

Note that the definition of the *length* of α, $|\alpha|$, appears to conflict with the standard notation for the Euclidean norm. This inconsistency is firmly embedded in analysis and I will not try to change it.

Below are a few exercises which illustrate the use of this notation.

Exercise 2.1 *The multi-nomial theorem.*

$$(x_1 + \ldots + x_n)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} x^\alpha.$$

Exercise 2.2 *Show that*

$$(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma.$$

Exercise 2.3 *The Leibniz rule. If f and g have continuous derivatives of order up to k on $\mathbf{R}^n$ and α is a multi-index of length k, then*

$$\frac{\partial^\alpha (fg)}{\partial x^\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta f}{\partial x^\beta} \frac{\partial^\gamma g}{\partial x^\gamma}. \tag{2.4}$$

Exercise 2.5 *Show for each multi-index α,*

$$\frac{\partial^\alpha}{\partial x^\alpha} x^\alpha = \alpha!.$$

More generally, show that

$$\frac{\partial^\beta}{\partial x^\beta} x^\alpha = \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta}.$$

The right-hand side in this last equation is defined to be zero if any component of $\alpha - \beta$ is negative.

To define the Schwartz space, we define a family of norms on the collection of $C^\infty(\mathbf{R}^n)$ functions which vanish at ∞. For each pair of multi-indices α and β, we let

$$\rho_{\alpha\beta}(f) = \sup_{x\in\mathbf{R}^n} |x^\alpha \frac{\partial^\beta f}{\partial x^\beta}(x)|.$$

We say that a function f is in the *Schwartz space* on $\mathbf{R}^n$ if $\rho_{\alpha\beta}(f) < \infty$ for all α and β. This space is denoted by $\mathcal{S}(\mathbf{R}^n)$. Recall that a norm was defined in Chapter 1. If a function $\rho : V \to [0, \infty)$ satisfies $\rho(f+g) \leq \rho(f) + \rho(g)$ for all f and g in V and $\rho(\lambda f) = |\lambda|\rho(f)$, then ρ is called a *semi-norm* on the vector space V.

The Schwartz space is given a topology using the norms $\rho_{\alpha\beta}$ in the following way. Let ρ_j be some arbitrary ordering of the norms $\rho_{\alpha\beta}$. Let $\bar{\rho}_j = min(\rho_j, 1)$. Then define

$$\rho(f-g) = \sum_{j=1}^{\infty} 2^{-j} \bar{\rho}_j(f-g).$$

Lemma 2.6 *The function ρ is a metric on $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}$ is complete in this metric. The vector operations $(f, g) \to f + g$ and $(\lambda, f) \to \lambda f$ are continuous.*

Exercise 2.7 *Prove the assertions in the previous Lemma.*

Note that our definition of the metric involves an arbitrary ordering of the norms $\rho_{\alpha\beta}$. Readers who are obsessed with details might worry that different choices of the order might lead to different topologies on $\mathcal{S}(\mathbf{R}^n)$. The following proposition guarantees that this does not happen.

Proposition 2.8 *A set $\mathcal{O}$ is open in $\mathcal{S}(\mathbf{R}^n)$ if and only if for each $f \in \mathcal{O}$, there are finitely many semi-norms $\rho_{\alpha_i\beta_i}$ and $\epsilon_i > 0$, $i = 1, \dots, N$ so that*

$$\cap_{i=1}^{N} \{g : \rho_{\alpha_i\beta_i}(f-g) < \epsilon_i\} \subset \mathcal{O}.$$

We will not use this proposition, thus the proof is left as an exercise. It is closely related to Proposition 2.11.

Exercise 2.9 *Prove Proposition 2.8*

Exercise 2.10 *The Schwartz space is an example of a Fréchet space. A Fréchet space is a vector space X whose topology is given by a countable family of semi-norms $\{\rho_j\}$ using a metric $\rho(f-g)$ defined by $\rho(f-g) = \sum 2^{-j}\bar{\rho}_j(f-g)$. The space X is Fréchet if the resulting topology is Hausdorff and if X is complete in the metric ρ. Show that $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space. Hint: If one of the semi-norms is a norm, then it is easy to see the resulting topology is Hausdorff. In our case, each semi-norm is a norm.*

Proposition 2.11 *A linear map T from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ is continuous if and only if for each semi-norm $\rho_{\alpha\beta}$, there exists a finite collection of semi-norms $\{\rho_{\alpha_i\beta_i} : i = 1 \dots, N\}$ and a constant C so that*

$$\rho_{\alpha\beta}(Tf) \leq C \sum_{i=1}^{n} \rho_{\alpha_i\beta_i}(f).$$

A map u from $\mathcal{S}(\mathbf{R}^n)$ to a normed vector space V is continuous if and only if there exists a finite collection of semi-norms $\{\rho_{\alpha_i\beta_i} : i = 1\ldots, N\}$ and a constant C so that

$$\|u(f)\|_V \leq C \sum_{i=1}^{n} \rho_{\alpha_i\beta_i}(f).$$

Proof. We first suppose that $T : \mathcal{S} \to \mathcal{S}$ is continuous. Let the given semi-norm $\rho_{\alpha\beta} = \rho_N$ under the ordering used to define the metric. Then T is continuous at 0 and hence given $\epsilon = 2^{-N-1}$, there exists $\delta > 0$ so that if $\rho(f) < \delta$, then $\rho(Tf) < 2^{-N-1}$. We may choose M so that $\sum_{j=M+1}^{\infty} 2^{-j} < \delta/2$. Given f, we set

$$\tilde{f} = \frac{\delta}{2} \frac{f}{\sum_{j=1}^{M} 2^{-j}\rho_j(f)}.$$

The function $\tilde{f}$ satisfies $\rho(\tilde{f}) < \delta$ and thus $\rho(T\tilde{f}) < 2^{-N-1}$. This implies that $\rho_N(T\tilde{f}) < 1/2$. Thus, by the homogeneity of ρ_N, we obtain

$$\rho_N(Tf) \leq \frac{1}{\delta} \sum_{j=1}^{M} \rho_j(f).$$

Now suppose that the second condition of our theorem holds and we verify that the standard $\epsilon - \delta$ formulation of continuity holds. Since the map T is linear, it suffices to prove that T is continuous at 0. Let $\epsilon > 0$ and then choose N so that $2^{-N} < \epsilon/2$. For each $j = 1, \ldots, N$, there exists C_j and N_j so that

$$\rho_j(Tf) \leq C_j \sum_{k=1}^{N_j} \rho_k(f).$$

If we set $N_0 = \max(N_1, \ldots, N_j)$, and $C_0 = \max(C_1, \ldots, C_N)$, then we have

$$\begin{aligned} \rho(Tf) &\leq \sum_{j=1}^{N} 2^{-j}\rho_j(Tf) + \frac{\epsilon}{2} \\ &\leq C_0 \sum_{j=1}^{N} \left(2^{-j} \sum_{k=1}^{N_0} \rho_k(f) \right) + \frac{\epsilon}{2}. \end{aligned} \tag{2.12}$$

Now we define δ by $\delta = 2^{-N_0} \min(1, \epsilon/(2N_0C_0))$. If we have $\rho(f) < \delta$, then we have $\bar{\rho}_k(f) < 1$ and $\rho_k(f) < \epsilon/(2N_0C_0)$ for $k = 1, \ldots, N$. Hence, we have $\rho_k(f) < \epsilon/(2N_0C_0)$ for $k = 1, \ldots, N_0$. Substituting this into the inequality (2.12) above gives that $\rho(Tf) < \epsilon$.

The proof of the second part is simpler. ∎

Finally, it would be a bit embarrassing if the space $\mathcal{S}(\mathbf{R}^n)$ turned out to contain only the zero function. The following exercise guarantees that this is not the case.

Exercise 2.13 *a) Let*

$$\phi(t) = \begin{cases} \exp(-1/t), & t > 0 \\ 0, \quad t \le 0. \end{cases}$$

Show that $\phi(t)$ is in $C^\infty(\mathbf{R})$. That is, ϕ has derivatives of all orders on the real line. Hint: Show by induction that $\phi^{(k)}(t) = P_{2k}(1/t)e^{-1/t}$ for $t > 0$ where P_{2k} is a polynomial of order $2k$.

b) Show that $\phi(-1/(1-|x|^2))$ is in $\mathcal{S}(\mathbf{R}^n)$. Hint: This is immediate from the chain rule and part a).

Lemma 2.14 *If $1 \le p < \infty$, then $\mathcal{S}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$.*

Proof. If we let $f_\epsilon(x) = \phi_\epsilon * f(x)$ where $\eta(x) = c\phi(x)$ with ϕ the function in the previous exercise and c is chosen so that $\int \eta = 1$. Hence, $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$ will also have integral 1. It is known from real analysis that if $f \in L^p(\mathbf{R}^n)$, then

$$\lim_{\epsilon\to 0^+} \|\eta_\epsilon * f(x) - f(x)\|_p = 0, \qquad 1 \le p < \infty$$

and that this fails when $p = \infty$ and f is not continuous.

Finally, since $\phi(0) = 1$,

$$f_{\epsilon_1,\epsilon_2}(x) = \phi(\epsilon_2 x)(\eta_{\epsilon_1} * f(x),$$

we can choose ϵ_1 and then ϵ_2 small so that when f is in L^p, $p < \infty$, then $\|f - f_{\epsilon_1,\epsilon_2}\|_p$ is as small as we like. Since $f_{\epsilon_1,\epsilon_2}$ is in $\mathcal{S}(\mathbf{R}^n)$, we have proven the density of $\mathcal{S}(\mathbf{R}^n)$ in L^p. ∎

We define the space of *tempered distributions,* $\mathcal{S}'(\mathbf{R}^n)$ as the dual of $\mathcal{S}(\mathbf{R}^n)$. If V is a topological vector space, then the *dual* is the vector space of continuous linear functionals on V. We give some examples of tempered distributions.

Example 2.15 *Each $f \in \mathcal{S}$ gives a tempered distribution by the formula*

$$g \to u_f(g) = \int_{\mathbf{R}^n} f(x)g(x)\,dx.$$

Example 2.16 *If f is in $L^p(\mathbf{R}^n)$ for some p, $1 \le p \le \infty$, then we may define a tempered distribution u_f by*

$$u_f(g) = \int_{\mathbf{R}^n} f(x)g(x)\,dx$$

To see this, note that if N is an integer, then $(1+|x|^2)^{N/2}|f(x)|$ is bounded by a linear combination of the norms, $\rho_{\alpha 0}$ for $|\alpha| \le N$. Thus, for $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$(\int |f(x)|^p\,dx)^{1/p} \le C \sum_{\alpha \le N} \rho_{\alpha 0}(f)(\int_{\mathbf{R}^n} (1+|x|^2)^{-pN}\,dx)^{1/p}. \tag{2.17}$$

If $pN > n/2$, then the integral on the right-hand side of this inequality if finite. Thus, we have an estimate for the $\|f\|_p$ norm. As a consequence, if f is in L^p, then we have $|u_f(g)| \le \|f\|_p \|g\|_{p'}$ by Hölder's inequality. Now the inequality (2.17) applied to g and Proposition 2.11 imply that u_f is continuous.

Exercise 2.18 *Show that the map $f \to u_f$ from $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$ is injective.*

Example 2.19 *The delta function δ is the tempered distribution given by*

$$\delta(f) = f(0).$$

Example 2.20 *More generally, if μ is any finite Borel measure on $\mathbf{R}^n$, we have a distribution u_μ defined by*

$$u_\mu(f) = \int f\,d\mu.$$

This is a tempered distribution because

$$|u_\mu(f)| \le |\mu|(\mathbf{R}^n)\rho_{00}(f).$$

Example 2.21 *Any polynomial P (or any measurable function of polynomial growth) gives a tempered distribution by*

$$u_P(f) = \int P(x)f(x)\,dx.$$

Example 2.22 *For each multi-index α, a distribution is given by*

$$\delta^{(\alpha)}(f) = \frac{\partial^\alpha f(0)}{\partial x^\alpha}.$$

2.2 Operations on tempered distributions

If T is a continuous linear map on $\mathcal{S}(\mathbf{R}^n)$ and u is a tempered distribution, then $f \to u(Tf)$ is also a distribution. The map $u \to u \circ T$ is called the transpose of T and is sometimes written as $T^t u = u \circ T$. This construction is an important part of extending familiar operations on functions to tempered distributions. Our first example considers the map

$$f \to \frac{\partial^\alpha f}{\partial x^\alpha}$$

which is clearly continuous on the Schwartz space. Thus if u is a distribution, then we can define a new distribution by

$$v(f) = u(\frac{\partial^\alpha f}{\partial x^\alpha}).$$

If we have a distribution u which is given by a Schwartz function f, we can integrate by parts and show that

$$(-1)^\alpha u_f(\frac{\partial^\alpha g}{\partial x^\alpha}) = u_{\partial^\alpha f/\partial x^\alpha}(g).$$

Thus we will define the *derivative of a distribution* u by the formula

$$\frac{\partial^\alpha u}{\partial x^\alpha}(g) = (-1)^{|\alpha|} u(\frac{\partial^\alpha g}{\partial x^\alpha}).$$

This extends the definition of derivative from smooth functions to distributions. When we say extend the definition of an operation T from functions to distributions, this means that we have

$$Tu_f = u_{Tf}$$

whenever f is a Schwartz function. Given a map T, we can make this extension if we can find a (formal) transpose of T, T^t, that satisfies

$$\int_{\mathbf{R}^n} Tf g\, dx = \int_{\mathbf{R}^n} f T^t g\, dx$$

for all $f, g \in \mathcal{S}(\mathbf{R}^n)$. Then if T^t is continuous on $\mathcal{S}(\mathbf{R}^n)$, we can define T on $\mathcal{S}'(\mathbf{R}^n)$ by $Tu(f) = u(T^t f)$.

Exercise 2.23 *Show that if α and β are multi-indices and u is a tempered distribution, then*

$$\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial x^\beta} u = \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\alpha}{\partial x^\alpha} u.$$

Hint: A standard result of vector calculus tells us when partial derivatives of functions commute.

Exercise 2.24 *Let $H(t)$ be the Heaviside function on the real line. Thus $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t < 0$. Find the distributional derivative of H. That is find $H'(\phi)$ for ϕ in $\mathcal{S}$.*

We give some additional examples of extending operations from functions to distributions. If P is a polynomial, then we $f \to Pf$ defines a continuous map on the Schwartz space. We can define multiplication of a distribution by a polynomial by $Pu(f) = u(Pf)$.

Exercise 2.25 *Show that this definition extends to ordinary product of functions in the sense that if f is a Schwartz function,*

$$u_{Pf} = Pu_f.$$

Exercise 2.26 *Show that if f and g are in $\mathcal{S}(\mathbf{R}^n)$, then fg is in $\mathcal{S}(\mathbf{R}^n)$ and that the map*

$$f \to fg$$

is continuous.

Exercise 2.27 *Show that $1/x$ defines a distribution on $\mathbf{R}$ by*

$$u(f) = \lim_{\epsilon \to 0^+} \int_{\{x:|x|>\epsilon\}} f(x) \frac{1}{x}\, dx.$$

This way of giving a value to an integral which is not defined as an absolutely convergent integral is called the principal value *of the integral. Hint: The function $1/x$ is odd, thus if we consider $\int_{\{\epsilon<|x|<1\}} f(x)/x\, dx$, we can subtract a constant from f without changing the value of the integral.*

Exercise 2.28 *Show that we cannot in general define the product between two distributions in such a way that the product is associative. (Vague?)*

Next we consider the convolution of a distribution and a test function. If f and g are in the Schwartz class, we have by Fubini's theorem that

$$\int_{\mathbf{R}^n} f * g(x) h(x)\, dx = \int_{\mathbf{R}^n} f(y) \int_{\mathbf{R}^n} h(x) \tilde{g}(y - x)\, dx\, dy.$$

The *reflection* of g, $\tilde{g}$ is defined by $\tilde{g}(x) = g(-x)$. Thus, we can define the convolution of a tempered distribution u and a test function g, $g * u$ by

$$g * u(f) = u(f * \tilde{g}).$$

This will be a tempered distribution thanks to the following.

Exercise 2.29 *Show that if f and g are in $\mathcal{S}(\mathbf{R}^n)$, then $f * g \in \mathcal{S}(\mathbf{R}^n)$. Furthermore, show that $f \to f * g$ is continuous on $\mathcal{S}$. Hint: The simplest way to do this is to use the Fourier transform to convert the problem into a problem about pointwise products.*

2.3 The Fourier transform

Proposition 2.30 *The Fourier transform is a continuous linear map from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ with a continuous inverse, $f \to \check{f}$.*

Proof. We use the criterion of Proposition 2.11. If we consider the expression in a semi-norm, we have

$$\xi^\alpha \frac{\partial^\beta}{\partial \xi^\beta}\hat{f}(\xi) = (\frac{\partial^\alpha}{\partial x^\alpha} x^\beta f)\hat{}$$

where we have used Propositions 1.18 and 1.19. By the Leibniz rule in (2.4), we have

$$(\frac{\partial^\alpha}{\partial x^\alpha} x^\beta f)\hat{} = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!}((\frac{\partial^\gamma}{\partial x^\gamma} x^\beta)\frac{\partial^\delta}{\partial x^\delta} f)\hat{}.$$

Hence, using the observation of (2.17) and Proposition 1.18, we have that

$$\rho_{\alpha\beta}(\hat{f}) \le C \sum_{\lambda \le \beta, |\gamma| \le |\alpha|+n+1} \rho_{\gamma\lambda}(f).$$

Given (1.33) or (1.32) the continuity of $\check{f}$ is immediate from the continuity of $\hat{f}$ and thus it is clear that $\check{f}$ lies in the Schwartz space and hence L^1 for $f \in \mathcal{S}(\mathbf{R}^)$. Then, we can use (1.33) and then the Fourier inversion theorem to show

$$\hat{\check{f}} = \hat{\hat{\hat{\hat{f}}}} = \check{\hat{\bar{f}}} = f.$$

∎

Next, recall the identity

$$\int \hat{f}(x)g(x)\,dx = \int f(x)\hat{g}(x)\,dx$$

of Lemma 1.30 which holds if f and g are Schwartz functions. Using this identity, it is clear that we want to define the Fourier transform of a tempered distribution by

$$\hat{u}(g) = u(\hat{g}).$$

Then the above identity implies that if u_f is a distribution given by a Schwartz function, or an L^1 function, then

$$u_{\hat{f}}(g) = \hat{u}_f(g).$$

Thus, we have defined a map which extends the Fourier transform.

In a similar way, we can define $\check{u}$ for a tempered distribution u by $\check{u}(f) = u(\check{f})$.

Theorem 2.31 *The Fourier transform is an invertible linear map on $\mathcal{S}'(\mathbf{R}^n)$.*

Proof. We know that $f \to \check{f}$ is the inverse of the map $f \to \hat{f}$ on $\mathcal{S}(\mathbf{R}^n)$. Thus, it is easy to see that $u \to \check{u}$ is an inverse to $u \to \hat{u}$ on $\mathcal{S}'(\mathbf{R}^n)$. ∎

Exercise 2.32 *Show that if f is in $\mathcal{S}$, then f has a derivative in the L^1-sense.*

Exercise 2.33 *Show from the definitions that if u is a tempered distribution, then*

$$(\frac{\partial^\alpha}{\partial x^\alpha} u)\hat{} = (i\xi)^\alpha \hat{u}$$

and that

$$((-ix)^\alpha u)\hat{} = (\frac{\partial^\alpha \hat{u}}{\partial \xi^\alpha}).$$

2.4 More distributions

In addition to the tempered distributions discussed above, there are two more common families of distributions. The (ordinary) distributions $\mathcal{D}'(\mathbf{R}^n)$ and the distributions of compact support, $\mathcal{E}'(\mathbf{R}^n)$. The $\mathcal{D}'$ is defined as the dual of $\mathcal{D}(\mathbf{R}^n)$, the set of functions which are infinitely differentiable and have compact support on $\mathbf{R}^n$. The space $\mathcal{E}'$ is the dual of $\mathcal{E}(\mathbf{R}^n)$, the set of functions which are infinitely differentiable on $\mathbf{R}^n$.

Since we have the containments,

$$\mathcal{D}(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n) \subset \mathcal{E}(\mathbf{R}^n),$$

we obtain the containments

$$\mathcal{E}'(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n).$$

To see this, observe that (for example) each tempered distribution defines an ordinary distribution by restricting the domain of u from $\mathcal{S}$ to $\mathcal{D}$.

The space $\mathcal{D}'(\mathbf{R}^n)$ is important because it can also be defined on open subsets of $\mathbf{R}^n$ or on manifolds. The space $\mathcal{E}'$ is interesting because the Fourier transform of such a distribution will extend holomorphically to $\mathbf{C}^n$. The book of Laurent Schwartz [12, 11] is still a good introduction to the subject of distributions.

Chapter 3

The Fourier transform on L^2.

In this section, we prove that the Fourier transform acts on L^2 and that $f \to (2\pi)^{-n/2}\hat{f}$ is an isometry on this space. Each L^2 function gives a tempered distribution and thus its Fourier transform is defined. Thus, our main accomplishment in is to prove the Plancherel identity which asserts that $f \to (2\pi)^{-n/2}\hat{f}$ is an isometry.

3.1 Plancherel's theorem

Proposition 3.1 *If f and g are in the Schwartz space, then we have*

$$\int_{\mathbf{R}^n} f(x)\bar{g}(x)\,dx = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)\,d\xi.$$

Proof. According to the Fourier inversion theorem in Chapter 1,

$$\bar{g} = \frac{1}{(2\pi)^n}\hat{\bar{\hat{g}}}.$$

Thus, we can use the identity (1.30) of Chapter 1 to conclude the Plancherel identity for Schwartz functions. ∎

Theorem 3.2 *(Plancherel) If f is in L^2, then $\hat{f}$ is in L^2 and we have*

$$\int |f(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\hat{f}(\xi)|^2\,d\xi.$$

Furthermore, the map $f \to \hat{f}$ is invertible.

Proof. If f is in L^2, then we may approximate f by Schwartz functions f_i. Applying the previous proposition with $f = g = f_i - f_j$ we conclude that the sequence $\hat{f}_i$ is Cauchy in L^2. Since this L^2 is complete, the sequence f_i has a limit, F. Since $f_i \to f$ in L^2 we also have that f_i converges to f as tempered distributions. To see this, we use the definition of the Fourier transform, and then that f_i converges in L^2 to obtain that

$$u_{\hat{f}}(g) = \int f\hat{g}\,dx = \lim_{i\to\infty}\int f_i\hat{g}\,dx = \int \hat{f}_i g\,dx = \int Fg\,dx.$$

Thus $\hat{f} = F$. The identity holds for f and $\hat{f}$ since it holds for each f_i.

We know that f has an inverse on $\mathcal{S}$, $f \to \check{f}$. The Plancherel identity tells us this inverse extends continuously to all of L^2. It is easy to see that this extension is still an inverse on L^2. ∎

Recall that a *Hilbert space* $\mathcal{H}$ is a complete normed vector space where the norm comes from an inner product. An *inner product* is a map $\langle,\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbf{C}$ which satisfies

$$\begin{aligned}
\langle x,y\rangle &= \overline{\langle y,x\rangle}, && \text{if } x,y\in\mathcal{H}\\
\langle \lambda x,y\rangle &= \lambda\langle x,y\rangle, && x,y\in\mathcal{H}, \lambda\in\mathbf{C}\\
\langle x,x\rangle &\geq 0, && x\in\mathcal{H}\\
\langle x,x\rangle &= 0, && \text{if and only if } x=0
\end{aligned}$$

Exercise 3.3 *Show that the Plancherel identity holds if f takes values in finite dimensional Hilbert space. Hint: Use a basis.*

Exercise 3.4 *Show by example that the Plancherel identity does not always hold if f does not take values in a Hilbert space. Hint: The characteristic function of $(0,1)\subset\mathbf{R}$ should provide an example. Norm the complex numbers by the ∞-norm, $\|z\| = \max(\operatorname{Re} z, \operatorname{Im} z)$.*

Exercise 3.5 *(Heisenberg inequality.) If f is a Schwartz function, show that we have the inequality:*

$$n\int_{\mathbf{R}^n} |f(x)|^2\,dx \leq 2\|xf\|_2\|\nabla f\|_2.$$

Hint: Write

$$\int_{\mathbf{R}^n} n|f(x)|^2\,dx = \int_{\mathbf{R}^n} (\operatorname{div} x)|f(x)|^2\,dx$$

and integrate by parts. Recall that the gradient *operator* ∇ *and the* divergence *operator,* div *are defined by*

$$\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \text{ and } \operatorname{div}(f_1, \ldots, f_n) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}.$$

This inequality has something to do with the Heisenberg uncertainty principle in quantum mechanics. The function $|f(x)|^2$ *is a probability density and thus has integral 1. The expression* xf *is related to the position of the particle represented by* f *and the expression* ∇f *is related to the momentum. The inequality gives a lower bound on the product of position and momentum.*

If we use Plancherel's theorem and Proposition 1.19, we obtain

$$\int_{\mathbf{R}^n} |\nabla f|^2 \, dx = (2\pi)^{-n} \int_{\mathbf{R}^n} |\xi \hat{f}(\xi)|^2 \, d\xi.$$

If we use this to replace $\|\nabla f\|_2$ *in the above inequality, we obtain a quantitative version of the statement "We cannot have* f *and* $\hat{f}$ *concentrated near the origin."*

3.2 Multiplier operators

If $m(\xi)$ is a tempered distribution, then m defines a *multiplier operator* T_m by

$$(T_m f)\hat{} = m(\xi)\hat{f}.$$

The function m is called the *symbol* of the operator. It is clear that T_m is maps $\mathcal{S}$ to $\mathcal{S}'$. Note that we cannot determine if this map is continuous, since we have not given the topology on $\mathcal{S}'(\mathbf{R}^n)$.

Our main interest is when m is a locally integrable function. Such a function will be a tempered distribution if there are constants C and N so that

$$\int_{B_R(0)} |m(\xi)| \, d\xi \leq C(1 + R^N), \text{for all } R > 0.$$

Exercise 3.6 *Is this condition necessary for a positive function to give a tempered distribution?*

There is a simple, but extremely useful condition for showing that a multiplier operator is bounded on L^2.

Theorem 3.7 *Suppose T_m is a multiplier operator given by a measurable function m. The operator T_m is bounded on L^2 if and only if m is in L^∞. Furthermore, $\|T_m\| = \|m\|_\infty$.*

Proof. If m is in L^∞, then Plancherel's theorem implies the inequality

$$\|T_m f\|_2 \leq \|m\|_\infty \|f\|_2.$$

Now consider $E_t = \{\xi : |m(\xi)| > t\}$ and suppose this set has positive measure. If we choose $F_t \subset E_t$ with $0 < m(F_t) < \infty$, then we have

$$\|T_m(\chi_{F_t})\| \geq t\|\chi_{F_t}\|_2.$$

Hence, $\|T_m\| \geq \|m\|_\infty$. ∎

Exercise 3.8 *(Open.) Find a necessary and sufficient condition for T_m to be bounded on L^p.*

Example 3.9 *If s is a real number, then we can define J_s, the Bessel potential operator of order s by*

$$(J_s f)\hat{} = (1 + |\xi|^2)^{-s/2} \hat{f}.$$

If $s \geq 0$, then Theorem 3.7 implies that $J_s f$ lies in L^2 when f is L^2. Furthermore, if α is multi-index of length $|\alpha| \leq s$, then for some finite constant C we have

$$\left\| \frac{\partial^\alpha}{\partial x^\alpha} J_s f \right\|_{L^2} \leq C\|f\|_2.$$

The operator $f \to \frac{\partial^\alpha}{\partial x^\alpha} J_s f$ is a multiplier operator with symbol $(i\xi)^\alpha/(1 + |\xi|^2)^{s/2}$, which is bounded.

3.3 Sobolev spaces

The Example 3.9 motivates the following definition of the Sobolev space L^2_s. Sobolev spaces are so useful that each mathematician has his or her own notation for them. Some of the more common ones are H^s, $W^{s,2}$ and $B^{2,s}_2$.

Definition 3.10 *The* Sobolev space $L^2_s(\mathbf{R}^n)$ *is the image of $L^2(\mathbf{R}^n)$ under the map J_s. The norm is given by*

$$\|J_s f\|_{2,s} = \|f\|_2$$

or, since $J_s \circ J_{-s}$ is the identity, we have

$$\|f\|_{2,s} = \|J_{-s} f\|_2.$$

Note that if $s \geq 0$, then $L^2_s \subset L^2$ as observed in Example 3.9. For $s = 0$, we have $L^2_0 = L^2$. For $s < 0$, the elements of the Sobolev space are tempered distributions, which are not, in general, given by functions.

The following propositions are easy consequences of the definition and the Plancherel theorem, via Theorem 3.7.

Proposition 3.11 *If $s \geq 0$ is an integer, then a function f is in the Sobolev space L^2_s if and only if f and all its derivatives of order up to s are in L^2.*

Proof. If f is in the Sobolev space L^2_s, then $f = J_s \circ J_{-s} f$. Using the observation of Example 3.9 that

$$f \to \frac{\partial^\alpha}{\partial x^\alpha} J_s f$$

is bounded on L^2, we conclude that

$$\|\frac{\partial^\alpha}{\partial x^\alpha} f\|_2 = \|\frac{\partial^\alpha}{\partial x^\alpha} J_s \circ J_{-s} f\|_2 \leq C \|J_{-s} f\|_2 = \|f\|_{2,s}.$$

If f has all derivatives of order up to s in L^2, then we have that there is a finite constant C so that

$$(1 + |\xi|^2)^{s/2} |\hat{f}(\xi)| \leq C(1 + \sum_{j=1}^{n} |\xi_j|^s) |\hat{f}(\xi)|.$$

Since each term on the right is in L^2, we have f in the Sobolev space. ∎

The characterization of Sobolev spaces in the above theorem is the more standard definition of Sobolev spaces. It is more convenient to define a Sobolev spaces for s a positive integer as the functions which have (distributional) derivatives of order less or equal s in L^2 because this definition extends easily to give Sobolev spaces on open subsets of $\mathbf{R}^n$ and Sobolev spaces based on L^p. The definition using the Fourier transform provides a nice definition of Sobolev spaces when s is not an integer.

Proposition 3.12 *If $s < 0$ and $-|\alpha| \geq s$, then $\partial^\alpha f / \partial x^\alpha$ is in L^2_s and*

$$\|\frac{\partial^\alpha f}{\partial x^\alpha}\|_{2,s} \leq \|f\|_2.$$

Proof. We have

$$(1 + |\xi|^2)^{s/2} (\frac{\partial^\alpha f}{\partial \xi^\alpha})\hat{}(\xi) = [(i\xi)^\alpha (1 + |\xi|^2)^{s/2})] \hat{f}(\xi)).$$

If $|\alpha| \leq -s$, then the factor in square brackets on the right is a bounded multiplier and hence if f is in L^2, then the left-hand side is in L^2. Now Plancherel's theorem tells us that $\partial^\alpha f / \partial x^\alpha$ is in the Sobolev space L^2_s. ∎

Exercise 3.13 *Show that for all* s *in* $\mathbf{R}$, *the map*

$$f \to \frac{\partial^\alpha f}{\partial x^\alpha}$$

maps $L^2_s \to L^2_{s-|\alpha|}$.

Exercise 3.14 *Show that* L^2_{-s} *is the dual of* L^2_s. *More precisely, show that if* $\lambda : L^2_s \to \mathbf{C}$ *is a continuous linear map, then there is a distribution* $u \in L^2_{-s}$ *so that*

$$\lambda(f) = u(f)$$

for each $f \in \mathcal{S}(\mathbf{R}^n)$. *Hint: This is an easy consequence of the theorem that all continuous linear functionals on the Hilbert space* L^2 *are given by* $f \to \int f\bar{g}$.

Chapter 4

Interpolation of operators

In the section, we will say a few things about the theory of interpolation of operators. For a more detailed treatment, we refer the reader to the book of Stein and Weiss [15] and the book of Bergh and Löfstrom [1].

By interpolation, we mean the following type of result. If T is a linear map which is bounded[1] on X_0 and X_1, then T is bounded on X_t for t between 0 and 1. It should not be terribly clear what we mean by "between" when we are talking about pairs of vector spaces. In the context of L^p space, L^q is between L^p and L^r will mean that q is between p and r.

For these results, we will work on a pair of σ-finite measure spaces $(M, \mathcal{M}, \mu)$ and $(N, \mathcal{N}, \nu)$.

4.1 The Riesz-Thorin theorem

We begin with the Riesz-Thorin convexity theorem.

Theorem 4.1 *Let p_j, q_j, $j = 0, 1$ be exponents in the range $[1, \infty]$ and suppose that $p_0 < p_1$. If T is a linear operator defined (at least) on simple functions in $L^1(M)$ into measurable functions on N that satisfies*

$$\|Tf\|_{q_j} \leq M_j \|f\|_{p_j}.$$

[1] A linear map $T : X \to Y$ is bounded between normed vector spaces X and Y if the inequality $\|Tf\|_Y \leq C\|f\|_X$ holds. The least constant C for which this inequality holds is called the operator norm of T.

If we define p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we will have that T extends to be a bounded operator from L^{p_t} to L^{q_t}:

$$\|Tf\|_{q_t} \le M_t \|f\|_{p_t}.$$

The operator norm, M_t satisfies $M_t \le M_0^{1-t} M_1^t$.

Before giving the proof of the Riesz-Thorin theorem, we look at some applications.

Proposition 4.2 *(Hausdorff-Young inequality) The Fourier transform satisfies for $1 \le p \le 2$*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n(1-\frac{1}{p})} \|f\|_p.$$

Proof. This follows by interpolating between the L^1-L^∞ result of Proposition 1.2 and Plancherel's theorem, Theorem 3.2. ∎

The next result appeared as an exercise when we introduced convolution.

Proposition 4.3 *(Young's convolution inequality) If $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$, $1 \le p, q, r \le \infty$ and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

then

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Proof. We fix p $1 \le p \le \infty$ and then will apply Theorem 4.1 to the map $g \to f * g$. Our endpoints are Hölder's inequality which gives

$$|f * g(x)| \le \|f\|_p \|g\|_{p'}$$

and thus $g \to f*g$ maps $L^{p'}(\mathbf{R}^n)$ to $L^\infty(\mathbf{R}^n)$ and the simpler version of Young's inequality which tells us that if g is in L^1, then

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

Thus $g \to f * g$ also maps L^1 to L^p. Thus, this map also takes L^{q_t} to L^{r_t} where

$$\frac{1}{q_t} = \frac{1-t}{1} + t(1 - \frac{1}{p}) \quad \text{and} \quad \frac{1}{r_t} = \frac{1-t}{p} + \frac{t}{\infty}.$$

If we subtract the definitions of $1/r_t$ and $1/q_t$, then we obtain the relation

$$\frac{1}{r_t} - \frac{1}{q_t} = 1 - \frac{1}{p}.$$

The condition $q \geq 1$ is equivalent with $t \geq 0$ and $r \geq 1$ is equivalent with the condition $t \leq 1$. Thus, we obtain the stated inequality for precisely the exponents p, q and r in the hypothesis. ∎

Exercise 4.4 *The simple version of Young's inequality used in the proof above can be proven directly using Hölder's inequality. A proof can also be given which uses the Riesz-Thorin theorem. To do this, use Tonelli's and then Fubini's theorem to establish the inequality*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

The other endpoint is Hölder's inequality:

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_\infty.$$

Then, apply Theorem 4.1 to the map $g \to f * g$.

Below is a simple, useful result that is a small generalization of the simple version of Young's inequality.

Exercise 4.5 *a) Suppose that $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is measurable and that*

$$\int_{\mathbf{R}^n} |K(x,y)| \, dy \leq M_\infty$$

and

$$\int_{\mathbf{R}^n} |K(x,y)| \, dx \leq M_1.$$

Show that

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y) f(y) \, dy$$

defines a bounded operator T on L^p and

$$\|Tf\|_p \leq M_1^{1/p} M_\infty^{1/p'} \|f\|_p.$$

Hint: Show that M_1 is an upper bound for the operator norm on L^1 and M_∞ is an upper bound for the operator norm on L^∞ and then interpolate with the Riesz-Thorin Theorem, Theorem 4.1.

b) Use the result of part a) to provide a proof of Young's convolution inequality

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

To do this, write $f * g(x) = \int_{\mathbf{R}^n} f(x-y) g(y) \, dy$ and then let $K(x,y) = f(x-y)$.

Our next step is a lemma from complex analysis (which makes everything trivial), that is usually called the three lines theorem. This is one of a family of theorems which state that the maximum modulus theorem continues to hold in unbounded regions, provided we put an extra growth condition at infinity. I believe such results are called Phragmen-Lindelöf theorems, though this may or may not be accurate. This theorem considers analytic functions in the strip $\{z : a \le \operatorname{Re} z \le b\}$.

Lemma 4.6 *(Three lines lemma) If f is analytic in the strip $\{z : a \le \operatorname{Re} z \le b\}$, f is bounded and*

$$M_a = \sup |f(a+it)| \qquad \textit{and} \qquad M_b = \sup |f(b+it)|,$$

then

$$|f(x+iy)| \le M_a^{\frac{b-x}{b-a}} M_b^{\frac{x-a}{b-a}}.$$

Proof. We consider $f_\epsilon(x+iy) = e^{\epsilon(x+iy)^2} f(x+iy) M_a^{\frac{x+iy-b}{b-a}} M_b^{\frac{a-(x+iy)}{b-a}}$ for $\epsilon > 0$. This function satisfies

$$|f_\epsilon(a+iy)| \le e^{\epsilon a^2} \qquad \text{and} \qquad |f_\epsilon(b+iy)| \le e^{\epsilon b^2}.$$

and

$$\lim_{y\to\pm\infty} \sup_{a\le x\le b} |f_\epsilon(x+iy)| = 0.$$

Thus by applying the maximum modulus theorem on sufficiently large rectangles, we can conclude that for each $z \in S$,

$$|f_\epsilon(z)| \le \max(e^{\epsilon a^2}, e^{\epsilon b^2}).$$

Letting $\epsilon \to 0^+$ implies the Lemma. ∎

Exercise 4.7 *If instead of assuming that f is bounded, we assume that*

$$|f(x+iy)| \le e^{M|y|}$$

for some $M > 0$, then the above Lemma holds and with the same proof. Show this. What is the best possible growth condition for which the above proof works? What is the best possible growth condition? See [13].

The proof of the Riesz-Thorin theorem will rely on the following family of simple functions.

Lemma 4.8 *Let p_0, p_1 and p with $p_0 < p < p_1$ be given. Consider $s = \sum \alpha_j a_j \chi_{E_j}$ be a simple function with α_j are complex numbers of length 1, $|\alpha_j| = 1$, $a_j > 0$ and $\{E_j\}$ is a pairwise disjoint collection of measurable sets where each is of finite measure. Suppose $\|s\|_p = 1$. Let*

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}$$

and define

$$s_z = \sum \alpha_j a_j^{p/p_z} \chi_{E_j}.$$

This family satisfies

$$\|s_z\|_{p_{\mathrm{Re}\, z}} = 1, \quad \textit{for } 0 < \mathrm{Re}\, z < 1.$$

Proof. We have that

$$\int |s_z|^{p_{\mathrm{Re}\, z}}\, d\mu = \sum a_j^p \mu(E_j).$$

∎

Exercise 4.9 *State and prove a similar lemma for the family of Sobolev spaces. Show that if $s_0 < s < s_1$ and $u \in L^2_s$ with $\|u\|_{L^2_s} = 1$, then we can find a family of distributions u_z so that*

$$\|u_{\mathrm{Re}\, z}\|_{L^2_{s_{\mathrm{Re}\, z}}} = 1, \qquad s_0 < \mathrm{Re}\, z < x_1.$$

This family will be analytic in the sense that if $f \in \mathcal{S}(\mathbf{R}^n)$, then $u_z(f)$ is analytic.

We are now ready to give the proof of the Riesz-Thorin theorem, Theorem 4.1.

Proof. *(Proof of Riesz-Thorin theorem.)* We are now ready to give the proof of the Riesz-Thorin theorem, Theorem . We fix a $p = p_{t_0}$, $0 < t_0 < 1$ and consider simple functions s on M and s' on N which satisfy $\|s\|_{p_{t_0}} = 1$ and $\|s'\|_{q'_{t_0}} = 1$. We let s_z and s'_z be the families from the previous Lemma where s_z is constructed using p_j, $j = 0, 1$ and s'_z is constructed using the exponents q'_j, $j = 0, 1$.

According to our hypothesis,

$$\phi(z) = \int_N s'_z(x) T s_z(x)\, d\nu(x)$$

is an analytic function of z. Also, using Lemma 4.8 and the assumption on T,

$$\sup_{y \in \mathbf{R}} |\phi(j + iy)| \le M_j, \qquad j = 0, 1.$$

Thus by the three lines theorem, Lemma 4.6, we can conclude that

$$|\int s' T s \, d\mu| \leq M_0^{1-t_0} M_1^{t_0}.$$

Since, s' is an arbitrary simple function with norm 1 in $L^{q'}$, we can conclude that

$$\|Ts\|_{q_{t_0}} \leq M_0^{1-t_0} M_1^{t_0}.$$

Finally, since simple functions are dense in L^{p_t}, we may take a limit to conclude that T can be extended to all of L^p and is bounded. ∎

The next exercise may be used to carry the extension of T from simple functions to all of L^p.

Exercise 4.10 *Suppose $T : A \to Y$ is a map defined on a subset A of a metric space X into a metric space Y. Show that if T is uniformly continuous, then T has a unique continuous extension $\bar{T} : \bar{A} \to Y$ to the closure of A, $\bar{A}$. If in addition, X is a vector space, A is a subspace and T is linear, then the extension is also linear.*

Exercise 4.11 *Show that if T is a linear map (say defined on $\mathcal{S}(\mathbf{R}^n)$) which maps $L^2_{s_j}$ into $L^2_{r_j}$ for $j = 0, 1$, then T maps $L^2_{s_t}$ into $L^2_{r_t}$ for $0 < t < 1$, where $s_t = (1-t)s_0 + ts_1$ and $r_t = (1-t)r_0 + tr_1$.*

4.2 Interpolation for analytic families of operators

The main point of this section is that in the Riesz-Thorin theorem, we might as well let the operator T depend on z. This is a very simple idea. We will see below that often a good deal of cleverness is needed in applying this theorem.

I do not wish to get involved the technicalities of analytic operator valued functions. (And am not even sure if there are any technicalities needed here.) If one examines the above proof, we see that the hypothesis we will need on an operator T_z is that for all sets of finite measure, $E \subset M$ and $F \subset N$, we have that

$$z \to \int_N \chi_E T_z(\chi_F) \, d\nu \tag{4.12}$$

is an analytic function of z. This hypothesis can often be proven by using Morera's theorem which replaces the problem of determining analyticity by the simpler problem of checking if an integral condition holds. The integral condition can often be checked with Fubini's theorem.

Theorem 4.13 *(Stein's interpolation theorem) For z in $S = \{z : 0 \le \operatorname{Re} z \le 1\}$, let T_z be a family of linear operators defined simple functions for which we have that the function in 4.12 is bounded and analytic in S. We assume that for $j = 0, 1$, T_{j+iy} maps $L^{p_j}(M)$ to $L^{q_j}(N)$. Also assume that $1 \le p_0 < p_1 \le \infty$. We let p_t and q_t have the meanings as in the Riesz-Thorin theorem and define*

$$M_t = \sup_{y\in\mathbf{R}} \|T_{t+iy}\|$$

where $\|T_{t+iy}\|$ denotes the norm of T_{t+iy} as an operator from $L^{p_t}(M)$ to $L^{q_t}(N)$. We conclude that T_t maps L^{p_t} to L^{q_t} and we have

$$M_t \le M_0^{1-t} M_1^t.$$

The proof of this theorem is the same as the proof of the Riesz-Thorin theorem.

Exercise 4.14 *(Interpolation with change of measure) Suppose that T is a linear map which which maps $L^{p_j}(d\mu)$ into $L^{q_j}(\omega_j d\nu)$ for $j = 0, 1$. Suppose that ω_0 and ω_1 are two non-negative functions which are integrable on every set of finite measure in N. Show that T maps $L^{p_t}(d\mu)$ into $L^{q_t}(\omega_t)$ for $0 < t < 1$. Here, q_t and p_t are defined as in the Riesz-Thorin theorem and $\omega_t = \omega_0^{1-t}\omega_1^t$.*

Exercise 4.15 *Formulate and prove a similar theorem where both measures μ and ν are allowed to vary.*

4.3 Real methods

In this section, we give a special case of the Marcinkiewicz interpolation theorem. This is a special case because we assume that the exponents $p_j = q_j$ are the same. The full theorem includes the off-diagonal case which is only true when $q \ge p$. To indicate the idea of the proof, suppose that we have a map T which is bounded on L^{p_0} and L^{p_1}. If we take a function f in L^p, with p between $p_0 < p < p_1$, then we may truncate f by setting

$$f_\lambda = \begin{cases} f, & |f| \le \lambda \\ 0, & |f| > \lambda. \end{cases} \tag{4.16}$$

and then $f^\lambda = f - f_\lambda$. Since f^λ is L^{p_0} and f_λ is in L^{p_1}, then we can conclude that $Tf = Tf_\lambda + Tf^\lambda$ is defined. As we shall see, if we are clever, we can do this splitting in such a way to see that not only is Tf defined, but we can also compute the norm of

Tf in L^p. The theorem applies to operators which are more general than bounded linear operators. Instead of requiring the operator T to be bounded, we require the following condition. Let $0 < q \leq \infty$ and $0 < p < \infty$ we say that T is *weak-type* p, q if there exists a constant A so that

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq \left(\frac{A\|f\|_p}{\lambda} \right)^q .$$

If $q = \infty$, then an operator is of *weak-type* p, ∞ if there exists a constant A so that

$$\|Tf\|_\infty \leq A\|f\|_p.$$

We say that a map T is *strong-type* p, q if there is a constant A so that

$$\|Tf\|_q \leq A\|f\|_p.$$

For linear operators, this condition is the same as boundedness. The justification for introducing the new term "strong-type" is that we are not requiring the operator T to be linear.

Exercise 4.17 *Show that if T is of strong-type p, q, then T is of weak-type p, q. Hint: Use Chebyshev's inequality.*

The condition that T is linear is replaced by the condition that T is *sub-linear*. This means that for f and g in the domain of T, then

$$|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

The proof of the main theorem will rely on the following well-known representation of the L^p norm of f.

Lemma 4.18 *Let $p < \infty$ and f be measurable, then*

$$\|f\|_p^p = p \int_0^\infty \mu(\{x : |f(x)| > \lambda\}) \lambda^p \, \frac{d\lambda}{\lambda}.$$

Proof. It is easy to see that this holds for simple functions. Write a general function as an increasing limit of simple functions. ∎

Our main result is the following theorem:

Theorem 4.19 *Let $0 < p_0 < p_1 \leq \infty$ for $j = 0, 1$ and let T take measurable functions on M to measurable functions on N. Assume also that T is sublinear and the domain of T is closed under taking truncations. If T is of weak-type p_j, p_j for $j = 0, 1$, then T is of strong-type p_t, p_t for $0 < t < 1$ and we have for $p_0 < p < p_1$, that when $p_1 < \infty$*

$$\|Tf\|_p \leq 2(\frac{pA_0^{p_0}}{p - p_0} + \frac{p_1 A_1^{p_1}}{p_1 - p})^{1/p} \|f\|_p.$$

When $p_1 = \infty$, we obtain

$$\|Tf\|^p \leq (1 + A_1)(\frac{A_0^{p_0} p}{p - p_0})^{1/p} \|f\|_p^p.$$

Proof. We first consider the case when $p_1 < \infty$ We fix $p = p_t$ with $0 < t < 1$, choose f in the domain of T and let $\lambda > 0$. We write $f = f_\lambda + f^\lambda$ as in (4.16). Since T is sub-linear and then weak-type p_j, p_j, we have that

$$\begin{aligned} \nu(\{x : |Tf(x)| > 2\lambda\}) &\leq \nu(\{x : |Tf^\lambda(x)| > \lambda\}) + \nu(\{x : |Tf_\lambda(x)| > \lambda\}) \\ &\leq \left(\frac{A_0 \|f^\lambda\|_{p_0}}{\lambda}\right)^{p_0} + \left(\frac{A_1 \|f_\lambda\|_{p_1}}{\lambda}\right)^{p_1}. \end{aligned} \tag{4.20}$$

We use the representation of the L^p-norm in Lemma 4.18, the inequality (4.20) and the change of variables $2\lambda \to \lambda$ to obtain

$$\begin{aligned} 2^{-p}\|Tf\|_p^p &\leq A_0^{p_0} p p_0 \int_0^\infty \int_0^\infty \mu(\{x : |f^\lambda(x)| > \tau\}) \tau^{p_0} \frac{d\tau}{\tau} \lambda^{p-p_0} \frac{d\lambda}{\lambda} \\ &\quad + A_1^{p_1} p p_1 \int_0^\infty \int_0^\lambda \mu(\{x : |f_\lambda(x)| > \tau\}) \tau^{p_1} \frac{d\tau}{\tau} \lambda^{p-p_1} \frac{d\lambda}{\lambda}. \end{aligned} \tag{4.21}$$

Note that the second integral on the right extends only to λ since f_λ satisfies the inequality $|f_\lambda| \leq \lambda$. We consider the second term first. We use that $\mu(\{x : |f_\lambda(x)| > \tau\}) \leq \mu(\{x : |f(x)| > \tau\})$ and thus Tonelli's theorem gives the integral in the second term is bounded by

$$\begin{aligned} pp_1 \int_0^\infty \mu(\{x : |f_\lambda(x)| > \tau\}) \int_\tau^\infty \lambda^{p-p_1} \frac{d\lambda}{\lambda} \frac{d\tau}{\tau} &\leq \frac{pp_1}{(p_1 - p)} \int_0^\infty \mu(\{x : |f(x)| > \tau\}) \tau^p \frac{d\tau}{\tau} \\ &= \frac{p_1}{p - p_1} \|f\|_p^p. \end{aligned} \tag{4.22}$$

We now consider the first term to the right of the inequality sign in (4.21). We observe that when $\tau > \lambda$, $\mu(\{x : |f^\lambda(x)| > \tau\}) = \mu(\{x : |f(x)| > \tau\})$, while when $\tau \leq \lambda$, we have $\mu(\{x : |f^\lambda(x)| > \tau\}) = \mu(\{x : |f(x)| > \lambda\})$. Thus, we have

$$\begin{aligned}
& pp_0 \int_0^\infty \int_0^\lambda \mu(\{x : |f_\lambda(x)| > \tau\}) \tau^{p_0} \frac{d\tau}{\tau} \lambda^{p-p_1} \frac{d\lambda}{\lambda} \\
&= pp_0 \int_0^\infty \int_\lambda^\infty \mu(\{x : |f(x)| > \tau\}) \tau^{p_0} \frac{d\tau}{\tau} \lambda^{p-p_0} \frac{d\lambda}{\lambda} \\
&\qquad + p \int_0^\infty \mu(\{x : |f(x)| > \lambda\}) \lambda^p \frac{d\lambda}{\lambda} \\
&= (\frac{p_0}{p-p_0} + 1) \|f\|_p^p. \qquad (4.23)
\end{aligned}$$

Using the estimates (4.22) and (4.23) in (4.21) gives

$$\|Tf\|_p^p \leq 2^p (\frac{pA_0^{p_0}}{p-p_0} + \frac{p_1 A_1^{p_1}}{p_1 - p}) \|f\|_p^p.$$

Which is what we hoped to prove.

Finally, we consider the case $p_1 = \infty$. Since T is of type ∞, ∞, then we can conclude that, with f_λ as above, $\nu(\{x : |Tf_\lambda(x)| > A_1\lambda\}) = 0$. To see how to use this, we write

$$\begin{aligned}
\nu(\{x : |Tf(x)| > (1 + A_1)\lambda\}) &\leq \nu(\{x : |Tf^\lambda(x)| > \lambda\}) + \nu(\{x : |Tf_\lambda(x)| > A_1\lambda\}) \\
&= \nu(\{x : |Tf^\lambda(x)| > \lambda\}).
\end{aligned}$$

Thus, using Lemma 4.18 that T is of weak-type p_0, p_0, and the calculation in (4.22) we have

$$\begin{aligned}
(1 + A_1)^{-p} \|Tf\|_p^p &= A_0^{p_0} pp_0 \int_0^\infty \int_\lambda^\infty \mu(\{x : |f^\lambda(x)| > \tau\}) \tau^{p_0} \frac{d\tau}{\tau} \lambda^{p-p_0} \frac{d\lambda}{\lambda} \\
&\leq \frac{A_0^{p_0} p}{p - p_0} \|f\|_p^p.
\end{aligned}$$

∎

Chapter 5

The Hardy-Littlewood maximal function

In this chapter, we introduce the Hardy-Littlewood maximal function and prove the Lebesgue differentiation theorem. This is the missing step of the Fourier uniqueness theorem in Chapter 1.

Since the material in this chapter is familiar from real analysis, we will omit some of the details. In this chapter, we will work on $\mathbf{R}^n$ with Lebesgue measure.

5.1 The L^p-inequalities

We let $\chi = n\chi_{B_1(0)}/\omega_{n-1}$ be the characteristic function of the unit ball, normalized so that $\int \chi\, dx = 1$ and then we set $\chi_r(x) = r^{-n}\chi(x/r)$. If f is a measurable function, we define the *Hardy-Littlewood maximal function* by

$$Mf(x) = \sup_{r>0} |f| * \chi_r(x).$$

Here and throughout these notes, we use $m(E)$ to denote the *Lebesgue measure* of a set E.

Note that the Hardy-Littlewood maximal function is defined as the supremum of an uncoutable family of functions. Thus, the sort of person[1] who is compulsive about details might worry that Mf may not be measurable. The following lemma implies the measurability of Mf.

Lemma 5.1 *If f is measurable, then Mf is upper semi-continuous.*

[1]Though not your instructor.

Proof. If $Mf(x) > \lambda$, then we can find a radius r so that

$$\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y)|\,dy > \lambda.$$

Since this inequality is strict, for s slightly larger than r, say $r+\delta > s > r$, we still have

$$\frac{1}{m(B_s(x))}\int_{B_r(x)} |f(y)|\,dy > \lambda.$$

But then by the monotonicity of the integral,

$$Mf(z) > \lambda$$

if $B_s(z) \supset B_r(x)$. That is if $|z-x| < \delta$. We have shown that the set $\{x : Mf(x) > \lambda\}$ is open. ∎

Exercise 5.2 *If $\{f_\alpha : \alpha \in A\}$ is a family of continuous real valued functions on $\mathbf{R}^n$ show that*

$$g(x) = \sup_{\alpha\in A} f_\alpha(x)$$

is upper semi-continuous.

If f is locally integrable, then $\chi_r * f$ is continuous for each $r > 0$ and the previous exercise can be used to establish the upper semi-continuity of Mf. Our previous lemma also applies to functions for which the integral over a ball may be infinite.

Oops. At this point, I am not sure if I have defined local integrability. We say that a function is *locally integrable* if it is in $L^1_{loc}(\mathbf{R}^n)$. We say that a function f is $L^p_{loc}(\mathbf{R}^n)$ if $f \in L^p(K)$ for each compact set K. If one were interested (and we are not), one can define a topology by defining the semi-norms,

$$\rho_n(f) = \|f\|_{L^p(B_n(0))}, \qquad \text{for } n = 1, 2\ldots$$

and then using this countable family of semi-norms, construct a metric as we did in defining the topology on the Schwartz space.

Exercise 5.3 *Show that a sequence converges in the metric for $L^p_{loc}(\mathbf{R}^n)$ if and only if the sequence converges in $L^p(K)$ for each compact set K.*

Exercise 5.4 *Let $f = \chi_{(-1,1)}$ on the real line. Show that $Mf \geq 1/|x|$ if $|x| > 1$. Conclude that Mf is not in L^1.*

Exercise 5.5 *Show that if Mf is in $L^1(\mathbf{R}^n)$, then f is zero.*

The first main fact about the Hardy-Littlewood maximal function is that it is finite almost everywhere, if f is in L^1. This is a consequence of the following theorem.

Theorem 5.6 *If f is measurable and $\lambda > 0$, then there exists a constant $C = C(n)$ so that*

$$m(\{x : |Mf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x)|\, dx.$$

The observant reader will realize that this theorem asserts that the Hardy-Littlewood maximal operator is of weak-type $1,1$. It is easy to see that it is sub-linear and of weak type ∞, ∞ and thus by the diagonal case of the Marcinkiewicz interpolation theorem, Theorem 4.19, we can conclude it is of strong-type p, p.

The proof of this theorem depends on a Lemma which allows us to extract from a collection of balls, a subcollection whose elements are disjoint and whose total measure is relatively large.

Lemma 5.7 *Let $\beta = 1/(2 \cdot 3^n)$. If E is a measurable set of finite measure in $\mathbf{R}^n$ and we have a collection of balls $\mathcal{B} = \{B_\alpha\}_{\alpha \in A}$ so that $E \subset \cup B_\alpha$, then we can find a subcollection of the balls $\{B_1, \ldots, B_N\}$ which are pairwise disjoint and which satisfy*

$$\sum_{j=1}^{N} m(B_j) \geq \beta m(E).$$

Proof. We may find $K \subset E$ which is compact and with $m(K) > m(E)/2$. Since K is compact, there is a finite sub-collection of the balls $\mathcal{B}_1 \subset \mathcal{B}$ which cover E. We let B_1 be the largest ball in $\mathcal{B}_1$ and then we let $\mathcal{B}_2$ be the balls in $\mathcal{B}_1$ which do not intersect B_1. We choose B_2 to be the largest ball in $\mathcal{B}_2$ and continue until $\mathcal{B}_{N+1}$ is empty. The balls $B_1, B_2, \ldots, B_N$ are disjoint by construction. If B is a ball in $\mathcal{B}_1$ then either B is one of the chosen balls, call it B_{j_0} or B was discarded in going from $\mathcal{B}_{j_0}$ to $\mathcal{B}_{j_0+1}$ for some j_0. In either case, B intersects one of the chosen balls, B_{j_0}, and B has radius which is less than or equal to the radius of B_{j_0}. Hence, we know that

$$K \subset \cup_{B \in \mathcal{B}_1} B \subset \cup_{j=1}^{N} 3B_j$$

where if $B_j = B_r(x)$, then $3B_j = B_{3r}(x)$. Taking the measure of the sets K and $\cup 3B_j$, we obtain

$$m(E) \leq 2m(K) \leq 2 \cdot 3^n \sum_{j=1}^{N} m(B_j).$$

∎

Now, we can give the proof of the weak-type 1,1 estimate for Mf in Theorem 5.6.

Proof. (*Proof of Theorem 5.6)* We let $E_\lambda = \{x : Mf(x) > \lambda\}$ and choose a measurable set $E \subset E_\lambda$ which is of finite measure. For each $x \in E_\lambda$, there is a ball B_x so that

$$m(B_x)^{-1} \int_{B_x} |f(x)|\, dx > \lambda \tag{5.8}$$

We apply Lemma 5.7 to the collection of balls $\mathcal{B} \subset \{B_x : x \in E\}$ to find a sub-collection $\{B_1, \ldots, B_N\} \subset \mathcal{B}$ of disjoint balls so that

$$\frac{m(E)}{2 \cdot 3^n} \le \sum_{j=1}^{N} m(B_j) \le \frac{1}{\lambda} \int_{B_j} |f(y)|\, dy \le \frac{\|f\|_1}{\lambda}.$$

The first inequality above is part of Lemma 5.7, the second is (5.8) and the last holds because the balls B_j are disjoint. Since E is an arbitrary, measurable subset of E_λ of finite measure, then we can take the supremum over all such E and conclude E_λ also satisfies

$$m(E_\lambda) \le \frac{2 \cdot 3^n \|f\|_1}{\lambda}.$$

∎

Frequently, in analysis it becomes burdensome to keep track of the exact value of the constant C appearing in the inequality. In the next theorem and throughout these notes, we will give the constant and the parameters it depends on without computing its exact value. In the course of a proof, the value of a constant C may change from one occurrence to the next. Thus, the expression $C = 2C$ is true even if $C \neq 0$!

Theorem 5.9 *If f is measurable and $1 < p \le \infty$, then there exists a constant $C = C(n)$*

$$\|Mf\|_p \le \frac{Cp}{p-1} \|f\|_p.$$

Proof. This follows from the weak-type 1,1 estimate in Theorem 5.6, the elementary inequality that $\|Mf\|_\infty \le \|f\|_\infty$ and Theorem 4.19. The dependence of the constant can be read off from the constant in Theorem 4.19. ∎

5.2 Differentiation theorems

The Hardy-Littlewood maximal function is a gadget which can be used to study the identity operator. At first, this may sound like a silly thing to do–what could be easier to understand than the identity? We will illustrate that the identity operator can be interesting by using the Hardy-Littlewood maximal function to prove the Lebesgue differentiation theorem–the identity operator is a pointwise limit of averages on balls. In fact, we will prove a more general result which was used in the proof of the Fourier inversion theorem of Chapter 1. This theorem amounts to a complicated representation of the identity operator. If this does not convince you that the identity operator is interesting, in a few Chapters, we will introduce approximations of the zero operator, $f \to 0$.

The maximal function is constructed by averaging using balls, however, it is not hard to see that many radially symmetric averaging processes can be estimated using M. The following useful result is lifted from Stein's book [14]. Before stating this proposition, given a function ϕ on $\mathbf{R}^n$, we define the non-increasing radial majorant of ϕ by

$$\phi^*(x) = \sup_{|y|>|x|} |\phi(y)|.$$

Proposition 5.10 *Let ϕ be in L^1 and f in L^p, then*

$$\sup_{r>0} |\phi_r * f(x)| \le \int \phi^*(x)\, dx Mf(x).$$

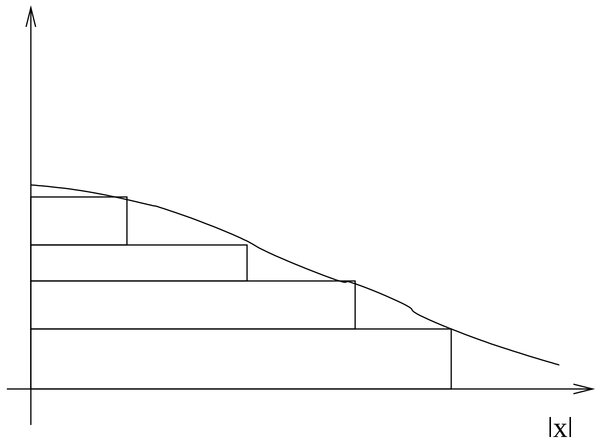
Proof. It suffices to prove the inequality

$$\phi_r * f(x) \le \int \phi(x)\, dx Mf(x)$$

when ϕ is non-negative and radially non-increasing and thus $\phi = \phi^*$ a.e. Also, we may assume $f \ge 0$. We begin with the special case when $\phi(x) = \sum_j a_j \chi_{B_{\rho_j}(0)}(x)$ and then

$$\begin{aligned} \phi_r * f(x) &= r^{-n} \sum_j a_j \frac{m(B_{r\rho_j}(x))}{m(B_{r\rho_j}(x))} \int_{B_{r\rho_j}(x)} f(y)\, dy \\ &\le r^{-n} Mf(x) \sum_j a_j m(B_{r\rho_j}(x)) \\ &= Mf(x) \int \phi. \end{aligned}$$

The remainder of the proof is a picture. We can write a general, non-increasing, radial function as an increasing limit of sums of characteristic functions of balls. The monotone convergence theorem and the special case already treated imply that $\phi_r * f(x) \le Mf(x) \int \phi\, dx$ and the Proposition follows. ∎



Finally, we give the result that is needed in the proof of the Fourier inversion theorem. We begin with a Lemma. Note that this Lemma suffices to prove the Fourier inversion theorem in the class of Schwartz functions. The full differentiation theorem is only needed when f is in L^1.

Lemma 5.11 *If f is continuous and bounded on $\mathbf{R}^n$ and $\phi \in L^1(\mathbf{R}^n)$, then for all x,*

$$\lim_{\epsilon \to 0^+} \phi_\epsilon * f(x) = f(x) \int \phi.$$

Proof. Fix x in $\mathbf{R}^n$ and $\eta > 0$. Recall that $\int \phi_\epsilon$ is independent of ϵ and thus we have

$$\phi_\epsilon * f(x) - f(x) \int \phi(x)\, dx = \int \phi_\epsilon(y)(f(x-y) - f(x))\, dy$$

Since f is continuous at x, there exists $\delta > 0$ so that $|f(x-y) - f(x)| < \eta$ if $|y| < \delta$. In the last integral above, we consider $|y| < \delta$ and $|y| \geq \delta$ separately. We use the continuity of f when $|y|$ is small and the boundedness of $|f|$ for $|y|$ large to obtain:

$$|\phi_\epsilon * f(x) - f(x) \int \phi\, dx| \leq \eta \int_{\{y:|y|<\delta\}} |\phi_\epsilon(y)|\, dy + 2\|f\|_\infty \int_{\{y:|y|>\delta\}} |\phi_\epsilon(y)|\, dy$$

The first term on the right is finite since ϕ is in L^1 and in the second term, a change of variables and the dominated convergence theorem implies we have

$$\lim_{\epsilon \to 0^+} \int_{\{y:|y|>\delta\}} |\phi_\epsilon(y)|\, dy = \lim_{\epsilon \to 0^+} \int_{\{y:|y|>\delta/\epsilon\}} |\phi(y)|\, dy = 0.$$

Thus, we conclude that

$$\limsup_{\epsilon\to 0^+} |\phi_\epsilon * f(x) - f(x)\int \phi(y)\,dy| \leq \eta \int |\phi|\,dy.$$

Since $\eta > 0$ is arbitrary, the conclusion of the lemma follows. ∎

Theorem 5.12 *If ϕ has radial non-increasing majorant in L^1, and f is in L^p for some p, $1 \leq p \leq \infty$, then for a.e. $x \in \mathbf{R}^n$,*

$$\lim_{\epsilon\to 0^+} \phi_\epsilon * f(x) = f(x)\int \phi\,dx.$$

Proof. The proof for $p = 1$, $1 < p < \infty$ and $p = \infty$ are each slightly different.

Let $\theta(f)(x) = \limsup_{\epsilon\to 0^+} |\phi_\epsilon * f(x) - f(x)\int\phi|$. Our goal is to show that $\theta(f) = 0$ a.e. Observe that according to Lemma 5.11, we have if g is continuous and bounded, then

$$\theta(f) = \theta(f-g).$$

Also, according to Proposition 5.10, we have that there is a constant C so that with $I = |\int\phi|$,

$$\theta(f-g)(x) \leq |f(x) - g(x)|I + CM(f-g)(x). \tag{5.13}$$

If f is in L^1 and $\lambda > 0$, we have that for any bounded and continuous g that

$$\begin{aligned} m(\{x : \theta(f)(x) > \lambda\}) &\leq m(\{x : \theta(f-g)(x) > \lambda/2\}) + m(\{x : I|f(x)-g(x)| > \lambda/2\}) \\ &\leq \frac{C}{\lambda}\int_{\mathbf{R}^n} |f(x)-g(x)|\,dx. \end{aligned}$$

The first inequality uses (5.13) and the second uses the weak-type 1,1 property of the maximal function and Tchebishev. Since we can approximate f in the L^1 norm by functions g which are bounded and continuous, we conclude that $m(\{x : \theta(x) > \lambda\}) = 0$. Since this holds for each $\lambda > 0$, then we have that $m(\{x : \theta(x) > 0\}) = 0$

If f is in L^p, $1 < p < \infty$, then we can argue as above and use the that the maximal operator is of strong-type p, p to conclude that for any continous and bounded g,

$$m(\{x : \theta(x) > \lambda\}) \leq \frac{C}{\lambda^p}\int |f(x)-g(x)|^p\,dx.$$

Again, continous and bounded functions are dense in L^p, if $p < \infty$ so we can conclude $\theta(f) = 0$ a.e.

Finally, if $p = \infty$, we claim that for each natural number, n, the set $\{x : \theta(f)(x) > 0 \text{ and } |x| < n\}$ has measure zero. This implies the theorem. To establish the claim, we write $f = \chi_{B_{2n}(0)}f + (1-\chi_{B_{2n}(0)})f = f_1 + f_2$. Since f_1 is in L^p for each p finite, we have $\theta(f_1) = 0$ a.e. and it is easy to see that $\theta(f_2)(x) = 0$ if $|x| < 2n$. Since $\theta(f)(x) \leq \theta(f_1)(x) + \theta(f_2)(x)$, the claim follows. ∎

The standard Lebesgue differentiation therorem is a special case of the result proved above.

Corollary 5.14 *If f is in $L^1_{loc}(\mathbf{R}^n)$, then*

$$f(x) = \lim_{r\to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\, dy.$$

Corollary 5.15 *If f is in $L^1_{loc}(\mathbf{R}^n)$, then there is a measurable set E, with $\mathbf{R}^n \setminus E$ of Lebesgue measure 0 and so that*

$$\lim_{r\to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0, \qquad x \in E.$$

We omit the proof of this last Corollary.

The set E from the previous theorem is called the Lebesgue set of f. It is clear from the definition that the choice of the representative of f may change E by a set of measure zero.

Chapter 6

Singular integrals

In this section, we will introduce a class of symbols for which the multiplier operators introduced in Chapter 3 are also bounded on L^p. The operators we consider are modelled on the Hilbert transform and the Riesz transforms. They were systematically studied by Calderón and Zygmund in the 1950's and are typically called Calderón-Zygmund operators. These operators are (almost) examples of pseudo-differential operators of order zero. The distinction between Calderón Zygmund operators and pseudo-differential operators is the viewpoint from which the operators are studied. If one studies the operator as a convolution operator, which seems to be needed to make estimates in L^p, then one is doing Calderón Zygmund theory. If one is studying the operator as a multiplier, which is more efficient for computing inverses and compositions, then one is studying pseudo-differential operators. One feature of pseudo-differential operators is that there is a general flexible theory for variable coefficient symbols. Our symbols will only depend on the frequency variable ξ.

6.1 Calderón-Zygmund kernels

In this chapter, we will consider linear operators $T : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$. In addition, we assume that T has a kernel $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ which gives the action of T away from the diagonal. The kernel K is a function which is locally integrable on $\mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,y) : x = y\}$. That K gives the action of T away from the diagonal means that that for any two functions f and g in $\mathcal{D}(\mathbf{R}^n)$ and which have disjoint support, we have that

$$Tf(g) = \int_{\mathbf{R}^{2n}} K(x,y)f(y)g(x)\,dx\,dy. \tag{6.1}$$

Note that the left-hand side of this equation denotes the distribution Tf paired with the function g. We say that K is a *Calderón-Zygmund kernel* if there is a constant C_K so that K satisfies the following two estimates:

$$|K(x,y)| \leq \frac{C_K}{|x-y|^n} \tag{6.2}$$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{C_K}{|x-y|^{n+1}} \tag{6.3}$$

Exercise 6.4 *Show that the kernel is uniquely determined by the operator.*

Exercise 6.5 *What is the kernel of the identity operator?*

Exercise 6.6 *Let α be a multi-index. What is the kernel of the operator*

$$T\phi = \frac{\partial^\alpha \phi}{\partial x^\alpha}?$$

Conclude that the operator is not uniquely determined by the kernel.

If an operator T has a Calderón-Zygmund kernel K as described above and T is L^2 bounded, then T is said to be a *Calderon-Zygmund operator.* In this chapter, we will prove two main results. We will show that Calderón-Zygmund operators are also L^p-bounded, $1 < p < \infty$ and we will show that a large class of multipliers operators are Calderón-Zygmund operators.

Since Calderón-Zygmund kernels are locally bounded in the complement of $\{(x,y) : x = y\}$, if f and g are L^2 and have disjoint compact supports, then (6.1) continues to hold. To see this we approximate f and g by smooth functions and note that we can arrange that we only increase the support by a small amount when we approximate.

Exercise 6.7 *Suppose that Ω is a smooth function near the sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$, then show that*

$$K(x,y) = \Omega(\frac{x-y}{|x-y|})\frac{1}{|x-y|^n}$$

is a Calderón-Zygmund kernel.

Exercise 6.8 *If $n \geq 3$ and j, k are in $\{1, \ldots, n\}$, then*

$$\frac{\partial^2}{\partial x_j \partial x_k} \frac{1}{|x-y|^{n-2}}$$

is a Calderón-Zygmund kernel. Of course, this result is also true for $n = 2$, but it is not very interesting.

In two dimensions, show that for and j and k,

$$\frac{\partial^2}{\partial x_j \partial x_k} \log |x - y|$$

is a Calderón-Zygmund kernel.

Theorem 6.9 *If T is a Calderón-Zygmund operator, then for $1 < p < \infty$ there is a constant C so that*

$$\|Tf\|_p \leq C\|f\|_p.$$

The constant $C \leq A \max(p, p')$ where A depends on the dimension n, the constant in the estimates for the Calderón-Zygmund kernel and the bound for T on L^2.

The main step of the proof is to prove a weak-type 1,1 estimate for T and then to interpolate to obtain the range $1 < p < 2$. The range $2 < p < \infty$ follows by applying the first case to the adjoint of T.

Exercise 6.10 *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle , \rangle$ If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear map on a Hilbert space, then the map $x \to \langle Tx, y\rangle$ defines a linear functional on $\mathcal{H}$. Hence, there is a unique element y^* so that $\langle Tx, y\rangle = \langle x, y^*\rangle$.*

a) Show that the map $y \to y^* = T^*y$ is linear and bounded.

b) Suppose now that T is bounded on the Hilbert space L^2, and that, in addition to being bounded on L^2, the map T satisfies $\|Tf\|_p \leq A\|f\|_p$, say for all f in L^2. Show that $\|T^*f\|_{p'} \leq A\|f\|_{p'}$.

Exercise 6.11 *If T is a Calderón-Zygmund operator, then show that T^* is also a Calderón-Zygmund operator and that the kernel of T^* is*

$$K^*(x, y) = \bar{K}(y, x).$$

Exercise 6.12 *If T_m is a multiplier operator with bounded symbol, show that the adjoint is a multiplier operator with symbol $\bar{m}$, $T_m^* = T_{\bar{m}}$.*

Theorem 6.13 *If T is a Calderón-Zygmund operator, f is in $L^2(\mathbf{R}^n)$ and $\lambda > 0$, then*

$$m(\{x : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x)|\, dx.$$

This result depends on the following decomposition lemma for functions. In this Lemma, we will use *cubes* on $\mathbf{R}^n$. By a cube, we mean a set of the form $Q_h(x) = \{y : |x_j - y_j| \leq h/2\}$. We let $\mathcal{D}_0$ be the mesh of cubes with sidelength 1 and whose vertices have integer coordinates. For k an integer, we define $\mathcal{D}_k$ to be the cubes obtained by applying the dilation $x \to 2^k x$ to each cube in $\mathcal{D}_0$. The cubes in $\mathcal{D}_k$ have sidelength 2^k and are obtained by bisecting each of the sides of the cubes in $\mathcal{D}_{k-1}$. Thus, if we take any two cubes Q and Q', in $\mathcal{D} = \cup_k \mathcal{D}_k$, then either one is contained in the other, or the two cubes have disjoint interiors. Also, given a cube Q, we will refer to the 2^n cubes obtained by dividing Q as the children of Q. And of course, if Q is a child of Q', then Q' is a parent of Q. The collection of cubes $\mathcal{D}$ will be called the *dyadic cubes* on $\mathbf{R}^n$.

Lemma 6.14 *(Calderón-Zygmund decomposition) If $f \in L^1(\mathbf{R}^n)$ and $\lambda > 0$, then we can find a family of cubes Q_k with disjoint interiors so that $|f(x)| \leq \lambda$ a.e. in $\mathbf{R}^n \setminus \cup_k Q_k$ and for each cube we have*

$$\lambda < \frac{1}{m(Q_k)} \int_{Q_k} |f(x)|\, dx \leq 2^n \lambda.$$

As a consequence, we can write $f = g + b$ where $|g(x)| \leq 2^n\lambda$ a.e. and $b = \sum b_k$ where each b_k is supported in one of the cubes Q_k, each b_k has mean value zero $\int b_k = 0$ and satisfies $\|b_k\|_1 \leq 2\int_{Q_k} |f|\, dx$. The function g satisfies $\|g\|_1 \leq \|f\|_1$

Proof. Given $f \in L^1$ and $\lambda > 0$, we let $\mathcal{E}$ be the collection of cubes $Q \in \mathcal{D}$ which satisfy the inequality

$$\frac{1}{m(Q)} \int_{Q} |f(x)|\, dx > \lambda. \tag{6.15}$$

Note that because $f \in L^1$, if $m(Q)^{-1}\|f\|_1 \leq \lambda$, then the cube Q will not be in $\mathcal{E}$. That is $\mathcal{E}$ does not contain cubes of arbitrarily large sidelength. Hence, for each cube Q' in $\mathcal{E}$, there is a largest cube Q in $\mathcal{E}$ which contains Q'. We let these maximal cubes form the collection $\{Q_k\}$, which we index in an arbitrary way. If Q'_k is the parent of Q_k, then Q'_k is not in $\mathcal{E}$ and hence the inequality (6.15) fails for Q'_k. This implies that we have

$$\int_{Q_k} |f(x)|\, dx \leq \int_{Q'_k} |f(x)| \leq 2^n m(Q_k)\lambda. \tag{6.16}$$

Hence, the stated conditions on the family of cubes hold.

For each selected cube, Q_k, we define $b_k = f - m(Q)^{-1}\int_{Q_k} f(x)\, dx$ on Q_k and zero elsewhere. We set $b = \sum_k b_k$ and then $g = f - b$. It is clear that $\int b_k = 0$. By the triangle inequality,

$$\int |b_k(x)|\, dx \leq 2\int_{Q_k} |f(x)|\, dx.$$

It is clear that $\|g\|_1 \leq \|f\|_1$. We verify that $|g(x)| \leq 2^n\lambda$ a.e. On each cube Q_k, this follows from the upper bound for the average of $|f|$ on Q_k. For each x in the complement of $\cup_k Q_k$, there is sequence of cubes in $\mathcal{D}$, with arbitrarily small sidelength and which contain x where the inequality (6.15) fails. Thus, the Lebesgue differentiation theorem implies that $|g(x)| \leq \lambda$ a.e. ∎

Our next step in the proof is the following Lemma regarding the kernel.

Lemma 6.17 *If K is a Calderón-Zygmund kernel and x, y are in $\mathbf{R}^n$ with $|x-y| \leq d$, then*

$$\int_{\mathbf{R}^n \setminus B_{2d}(x)} |K(z,x) - K(z,y)|\, dz \leq C.$$

The constant depends only on the dimension and the constant appearing in the definition of the Calderón-Zygmund kernel.

Proof. We apply the mean-value theorem of calculus to conclude that if $y \in \bar{B}_d(x)$ and $z \in \mathbf{R}^n \setminus B_{2d}(x)$, then the kernel estimate (6.3)

$$|K(z,x) - K(z,y)| \leq |x-y| \sup_{y \in B_d(x)} |\nabla_y K(z,y)| \leq 2^{n+1} C_K |x-y||x-z|^{-n-1}. \quad (6.18)$$

The second inequality uses the triangle inequality $|y-z| \geq |x-z| - |y-x|$ and then that $|x-z| - |y-x| \geq |x-z|/2$ if $|x-y| \leq d$ and $|x-z| \geq 2d$. Finally, if we integrate the inequality (6.18) in polar coordinates, we find that

$$\int_{\mathbf{R}^n \setminus B_{2d}(x)} |K(z,x) - K(z,y)|\, dz \leq d C_K 2^{n+1} \omega_{n-1} \int_{2d}^{\infty} r^{-n-1} r^{n-1}\, dr = C_K 2^n \omega_{n-1}.$$

This is the desired conclusion. ∎

Now, we give the proof of the weak-type 1,1 estimate in Theorem 6.13.

Proof of Theorem 6.13. We may assume that f is in $L^1 \cap L^2$. We let $\lambda > 0$. We apply the Calderón-Zygmund decomposition, Lemma 6.14 at λ to write $f = g + b$. We have

$$\{x : |Tf(x)| > \lambda\} \subset \{x : |Tg(x)| > \lambda/2\} \cup \{x : |Tb(x)| > \lambda/2\}.$$

Using Tchebisheff's inequality and that T is L^2-bounded, and then that $|g(x)| \leq C\lambda$ we obtain

$$m(\{x : |Tg(x)| > \lambda/2\}) \leq \frac{C}{\lambda^2} \int_{\mathbf{R}^n} |g(x)|^2\, dx \leq \frac{C}{\lambda} \int_{\mathbf{R}^n} |g(x)|\, dx$$

Finally, since $\|g\|_1 \leq \|f\|_1$, we have

$$m(\{x : |Tg(x)| > \lambda/2\}) \leq \frac{C}{\lambda}\|f\|_1.$$

Now, we turn to the estimate of Tb. We let $O_\lambda = \cup B_k$ where each ball B_k is chosen to have center x_k, the center of the cube Q_k and the radius of B_k is $\sqrt{n}$ multiplied by the sidelength of Q. Thus, if $y \in Q_k$, then the distance $|x_k - y|$ is at most half the radius of B_k. This will be needed to apply Lemma 6.17. We estimate the measure of O_λ using that

$$m(O_\lambda) \leq C\sum_k m(Q_k) \leq \frac{1}{\lambda}\sum_k \int_{Q_k} |f|\,dx \leq \frac{C}{\lambda}\|f\|_1.$$

Next, we obtain an L^1 estimate for Tb_k. If x is in the complement of Q_k, we know that $Tb_k(x) = \int K(x,y)b_k(y)\,dy = \int(K(x,y) - K(x,x_k))b_k(y)\,dy$ where the second equality uses that b_k has mean value zero. Now, applying Fubini's theorem and Lemma 6.17, we can estimate

$$\begin{aligned}\int_{\mathbf{R}^n\setminus B_k} |Tb_k(x)|\,dx &\leq \int_{Q_k} |b_k(y)| \int_{\mathbf{R}^n\setminus B_k} |K(x,y) - K(x,x_k)|\,dx\,dy \\ &\leq C\int_{Q_k} |b_k(y)|\,dy \leq C\int_{Q_k} |f(y)|\,dy\end{aligned}$$

Thus, if we add on k, we obtain

$$\int_{\mathbf{R}^n\setminus O_\lambda} |Tb(y)|\,dy \leq \sum_k \int_{\mathbf{R}^n\setminus B_k} |Tb_k(y)|\,dy \leq C\|f\|_1 \qquad (6.19)$$

Finally, we estimate

$$\begin{aligned} m(\{x : Tb(x) > \lambda/2\}) &\leq m(O_\lambda) + m(\{x \in \mathbf{R}^n \setminus O_\lambda : |Tb(x)| > \lambda/2\}) \\ &\leq m(O_\lambda) + \frac{C}{\lambda}\|f\|_1.\end{aligned}$$

Where the the last inequality uses Chebishev and our estimate (6.19) for the L^1-norm of Tb in the complement of O_λ. ∎

Exercise 6.20 *Let Q be a cube in $\mathbf{R}^n$ of sidelength $h > 0$, $Q = \{x : 0 \leq x_i \leq h\}$. Compute the diameter of Q. Hint: The answer is probably $h\sqrt{n}$.*

Proof of Theorem 6.9. Since we assume that T is L^2-bounded, the result for $1 < p < 2$, follows immediately from Theorem 6.13 and the Marcinkiewicz interpolation theorem,

Theorem 4.19. The result for $2 < p < \infty$ follows by observing that if T is a Calderón-Zygmund operator, then the adjoint T^* is also a Calderón-Zygmund operator and hence T^* is L^p-bounded, $1 < p < 2$. Then it follows that T is L^p-bounded for $2 < p < \infty$.

The alert reader might observe that Theorem 4.19 appears to give a bound for the operator norm which grows like $|p-2|^{-1}$ near $p = 2$. This growth is a defect of the proof and is not really there. To see this, one can pick one's favorite pair of exponents, say 4/3 and 4 and interpolate (by either Riesz-Thorin or Marcinkiewicz) between them to see that norm is bounded for p near 2. ∎

6.2 Some multiplier operators

In this section, we study multiplier operators where the symbol m is smooth in the complement of the origin. For each $k \in \mathbf{R}$, we define a class of multipliers which we call symbols of order k. We say m is symbol of order k if for each multi-index α, there exists a constant C_α so that

$$\left|\frac{\partial^\alpha m}{\partial \xi^\alpha}(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|+k}. \tag{6.21}$$

The operator given by a symbol of order k corresponds to a generalization of a differential operator of order k. Strictly speaking, these operators are not pseudo-differential operators because we allow symbols which are singular near the origin. The symbols we study transform nicely under dilations. This makes some of the arguments below more elegant, however the inhomogeneous theory is probably more useful.

Exercise 6.22 *a) If $P(\xi)$ is homogeneous polynomial of degree k, then P is a symbol of order k.*

b) The mutiplier for the Bessel potential operator $(1 + |\xi|^2)^{-s/2}$ is a symbol of order $-s$ for $s \geq 0$. What if $s < 0$?

We begin with a lemma to state some of the basic properties of these symbols.

Lemma 6.23 *a) If m_j is a symbol of order k_j for $j = 1, 2$, then $m_1 m_2$ is a symbol of order $k_1 + k_2$ and each constant for $m_1 m_2$ depends on finitely many of the constants for m_1 and m_2.*

b) If $\eta \in \mathcal{S}(\mathbf{R}^n)$, then η is a symbol of order k for any $k \leq 0$.

c) If m is a symbol of order k, then for all $\epsilon > 0$, $\epsilon^{-k} m(\epsilon\xi)$ is a symbol of order k and the constants are independent of ϵ.

d) If m_j, $j = 1, 2$ are symbols of order k, then $m_1 + m_2$ is a symbol of order k.

Proof. A determined reader armed with the Leibniz rule will find that these results are either easy or false. ∎

Exercise 6.24 *a) Use Lemma 6.23 to show that if* m *is a symbol of order 0 and* $\eta \in \mathcal{S}(\mathbf{R}^n)$ *with* $\eta = 1$ *near the origin, then* $m_\epsilon(\xi) = \eta(\epsilon\xi)(1-\eta(\xi/\epsilon))m(\xi)$ *is a symbol of order* 0.

b) Show that if $\eta(0) = 1$*, then for each* $f \in L^2(\mathbf{R}^n)$ *the multiplier operators given by* m *and* m_ϵ *satisfy*

$$\lim_{\epsilon\to 0^+} \|T_m f - T_{m_\epsilon} f\|_2 = 0.$$

c) Do we have $\lim_{\epsilon\to 0^+} \|T_m - T_{m_\epsilon}\| = 0$*? Here,* $\|T\|$ *denotes the operator norm of* T *as an operator on* L^2.

Exercise 6.25 *Show that if* m *is a symbol of order* 0 *and there is a* $\delta > 0$ *so that* $|m(\xi)| \geq \delta$ *for all* $\xi \neq 0$*, then* m^{-1} *is a symbol of order* 0.

Lemma 6.26 *If* m *is in the Schwartz class and* m *is a symbol of order* $k > -n$*, then there is a constant* C *depending only on finitely many of the constants in (6.21) so that*

$$|\check{m}(x)| \leq C|x|^{-n-k}.$$

Proof. To see this, introduce a cutoff function $\eta_0 \in \mathcal{D}(\mathbf{R}^n)$ and fix $|x|$ so that $\eta_0(\xi) = 1$ if $|\xi| < 1$ and $\eta_0 = 0$ if $|\xi| > 2$ and set $\eta_\infty = 1 - \eta_0$. We write

$$K_j(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \eta_j(\xi|x|) m(\xi)\, d\xi, \quad j = 0, \infty.$$

For $j = 0$, the estimate is quite simple since $\eta_0(\xi|x|) = 0$ if $|\xi| > 2/|x|$. Thus,

$$|K_0(x)| \leq (2\pi)^{-n} \int_{|\xi|<2/|x|} |\xi|^k \, d\xi = C|x|^{-k-n}.$$

For the part near ∞, we need to take advantage of the cancellation that results from integrating the oscillatory exponential against the smooth kernel m. Thus, we write $(ix)^\alpha e^{ix\cdot\xi} = \frac{\partial^\alpha}{\partial\xi^\alpha} e^{ix\cdot\xi}$ and then integrate by parts to obtain

$$(ix)^\alpha K_\infty(x) = \int (\frac{\partial^\alpha}{\partial\xi^\alpha} e^{ix\cdot\xi}) \eta_\infty(\xi|x|) m(\xi)\, d\xi = (-1)^{|\alpha|} \int e^{ix\cdot\xi} \frac{\partial^\alpha}{\partial\xi^\alpha} (\eta_\infty(\xi|x|)m(\xi))\, d\xi.$$

The boundary terms vanish since the integrand is in the Schwartz class. Using the symbol estimates (6.21) and that η_∞ is zero for $|\xi|$ near 0, we have for $k - |\alpha| < -n$, that

$$|(ix)^\alpha K_\infty(x)| \leq C \int_{|\xi|>1/|x|} |\xi|^{k-|\alpha|}\, d\xi = C|x|^{-n-k+|\alpha|}.$$

This implies the desired estimate that $|K_\infty(x)| \leq C|x|^{-n-k}$. ∎

We are now ready to show that the symbols of order 0 give Calderón Zygmund operators.

Theorem 6.27 *If m is a symbol of order 0, then T_m is a Calderón-Zygmund operator.*

Proof. The L^2-boundedness of T_m is clear since m is bounded, see Theorem 3.7. We will show that the kernel of T_m is of the form $K(x-y)$ and that for all multi-indices α there is a constant C_α so that K satisfies

$$|\frac{\partial^\alpha}{\partial x^\alpha} K(x)| \le C|x|^{-n-|\alpha|}.$$

The inverse Fourier transform of m, $\check{m}$, is not, in general, a function. Thus, it is convenient to approximate m by nice symbols. To do this, we let $\eta \in \mathcal{D}(\mathbf{R}^n)$ satisfy $\eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ if $|x| > 2$. We define $m_\epsilon(\xi) = \eta(\epsilon\xi)(1 - \eta(\xi/\epsilon))m(\xi)$. By Lemma 6.23, we see that m_ϵ is a symbol with constants independent of ϵ. Since $m_\epsilon \in \mathcal{S}(\mathbf{R}^n)$, by Lemma 6.26 we have that $K_\epsilon = \check{m}_\epsilon$ satisfies for each multi-index α,

$$|\frac{\partial^\alpha}{\partial x^\alpha} K(x)| \le C|x|^{-n-|\alpha|}. \tag{6.28}$$

This is because the derivative of order α of K_ϵ by Proposition 1.19 is the inverse Fourier transform of $(-i\xi)^\alpha m_\epsilon(\xi)$, a symbol of order $|\alpha|$. Since the constants in the estimates are uniform in ϵ, we can apply the Arzela-Ascoli theorem to prove that there is a sequence $\{\epsilon_j\}$ with $\lim_{j\to\infty} \epsilon_j = 0$ so that K_{ϵ_i} and all of its derivatives converge uniformly on compact subsets of $\mathbf{R}^n \setminus \{0\}$ and of course the limit, which we call K, satisfies the estimates (6.28).

It remains to show that $K(x-y)$ is a kernel for the operator T_m. Let f be in $\mathcal{S}(\mathbf{R}^n)$. By the dominated convergence theorem and the Plancherel theorem, $T_{m_\epsilon} f \to T_m f$ in L^2 as $\epsilon \to 0^+$. By Proposition 1.24, $T_{m_\epsilon} f = K_\epsilon * f$. Finally, if f and g have disjoint support, then

$$\begin{aligned} \int T_m f(x) g(x)\, dx &= \lim_{j\to\infty} \int T_{m_{\epsilon_j}} f(x) g(x)\, dx \\ &= \lim_{j\to\infty} \int K_{\epsilon_j}(x-y) f(y) g(x)\, dx\, dy \\ &= \int K(x-y) f(y) g(x)\, dx\, dy. \end{aligned}$$

The first equality above holds because $T_{m_\epsilon} f$ converges in L^2, the second follows from Proposition 1.24 and the third equality holds because of the locally uniform convergence of K in the complement of the origin. This completes the proof that $K(x-y)$ is a kernel for T_m. ∎

We can now state a corollary which is usually known as the Mikhlin multiplier theorem.

Corollary 6.29 *If m is a symbol of order 0, then the multiplier operator T_m is bounded on L^p for $1 < p < \infty$.*

We conclude with a few exercises.

Exercise 6.30 *If m is infinitely differentiable in $\mathbf{R}^n \setminus \{0\}$ and is homogeneous of degree 0, then m is a symbol of order zero.*

In the next exercise, we introduce the *Laplacian* $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

Exercise 6.31 *Let $1 < p < \infty$, $n \geq 3$. If $f \in \mathcal{S}(\mathbf{R}^n)$, then we can find a tempered distribution u so that $\Delta u = f$ and we have the estimate*

$$\|\frac{\partial^2 u}{\partial x_j \partial x_k}\|_p \leq C\|f\|_p$$

where the constant in the estimate C depends only on p and n. Why is $n = 2$ different? In two dimensions, show that we can construct u if $\hat{f}(0) = 0$. (This construction can be extended to all of the Schwartz class, but it is more delicate when $\hat{f}(0) \neq 0$.)

This exercise gives an estimate for the solution of $\Delta u = f$. This estimate follows immediately from our work so far. We should also prove uniqueness: If u is a solution of $\Delta u = 0$ and u has certain growth properties, then $u = 0$. This is a version of the Liouville theorem. The above inequality is not true for every solution of $\Delta u = f$. For example, on $\mathbf{R}^2$, if $u(x) = e^{x_1 + ix_2}$, then we have $\Delta u = 0$, but the second derivatives are not in any $L^p(\mathbf{R}^2)$.

Exercise 6.32 *Let $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ be the wave operator which acts on functions of $n + 1$ variables, $(x, t) \in \mathbf{R}^n \times \mathbf{R}$. Can we find a solution of $\Box u = f$ and prove estimates like those in Exercise 6.31? Why or why not?*

Exercise 6.33 *Show that if $\lambda \in \mathbf{C}$ is not a negative real number, the operator given by $m(\xi) = (\lambda + |\xi|^2)^{-1}$ is bounded on L^p for $1 < p < \infty$ and that we have the estimate*

$$\|T_m f\|_p \leq C\|f\|_p.$$

Find the dependence of the constant on λ.

Chapter 7

Littlewood-Paley theory

In this chapter, we look at a particular singular integral and see how this can be used to characterize the L^p norm of a function in terms of its Fourier transform. The theory discussed here has its roots in the theory of harmonic functions in the disc or the upper half-plane. The expressions $Q_k f$ considered below, share many properties with the $2^{-k}\nabla u(x', 2^{-k})$ where u is the harmonic function in the upper-half plane $x_n > 0$ whose boundary values are f. Recently, many of these ideas have become part of the theory of wavelets. The operators $Q_k f$ decompose f into pieces which are of frequency approximately 2^k. A wavelet decomposition combines this decomposition in frequency with a spatial decomposition, in so far as this is possible.

7.1 A square function that characterizes L^p

We let ψ be a real-valued function in $\mathcal{D}(\mathbf{R}^n)$ which is supported in $\{\xi : 1/2 < |\xi| < 4\}$ and which satisfies $\sum_{k=-\infty}^{\infty} \psi_k(\xi)^2 = 1$ in $\mathbf{R}^n \setminus \{0\}$ where $\psi_k(\xi) = \psi(\xi/2^k)$ and we will call ψ a *Littlewood-Paley function.* It is not completely obvious that such a function exists.

Lemma 7.1 *A Littlewood-Paley function exists.*

Proof. We take a function $\tilde\psi \in \mathcal{D}(\mathbf{R}^n)$ which is non-negative, supported in $\{\xi : 1/2 < |\xi| < 4\}$ and which is strictly positive on $\{\xi : 1 < |\xi| < 2\}$. We set

$$\psi(\xi) = \tilde\psi(\xi)/(\sum_{k=-\infty}^{\infty} \tilde\psi^2(\xi/2^k))^{0.5}.$$

∎

For f in L^p, say, we can define $Q_k f = \check{\psi}_k * f = (\psi_k \hat{f})\check{}$. We define the *square function* $S(f)$ by

$$S(f)(x) = (\sum_{k=-\infty}^{\infty} |Q_k(f)(x)|^2)^{1/2}.$$

From the Plancherel theorem, Theorem 3.2, it is easy to see that

$$\|f\|_2 = \|S(f)\|_2 \tag{7.2}$$

and of course this depends on the identity $\sum_k \psi_k^2 = 1$. We are interested in this operator because we can characterize the L^p spaces in a similar way.

Theorem 7.3 *Let $1 < p < \infty$. There is a finite nonzero constant $C = C(p, n, \psi)$ so that if f is in L^p, then*

$$C_p^{-1}\|f\|_p \le \|S(f)\|_p \le C_p\|f\|_p.$$

This theorem will be proven by considering a vector-valued singular integral. The kernel we consider will be

$$K(x, y) = (\ldots, 2^{nk}\check{\psi}(2^k(x - y)), \ldots).$$

Lemma 7.4 *If ψ is in $\mathcal{S}(\mathbf{R}^n)$, then the kernel K defined above is a Calderón-Zygmund kernel.*

Proof. We write out the norm of K

$$|K(x, y)|^2 = \sum_{k=-\infty}^{\infty} 2^{2nk}|\check{\psi}((2^k(x - y))|^2.$$

We choose N so that $2^N \le |x - y| < 2^{N+1}$ and split the sum above at $-N$. Recall that $\check{\psi}$ is in $\mathcal{S}(\mathbf{R}^n)$ and decays faster than any polynomial. Near 0, that is for $k \le -N$, we use that $\check{\psi}(x) \le C$. For $k > -N$, we use that $\check{\psi}(x) \le C|x|^{-n-1}$. Thus, we have

$$|K(x, y)|^2 \le C(\sum_{k=-\infty}^{-N} 2^{2nk} + \sum_{k=-N+1}^{\infty} 2^{2nk}(2^{k+N})^{-2(n+1)}) = C2^{-2nN}.$$

Recalling that 2^N is approximately $|x - y|$, we obtain the desired upper-bound for $K(x, y)$. To estimate the gradient, we observe that $\nabla_x K(x, y) = (\ldots, 2^{-(n+1)k}(\nabla\check{\psi})((x - y)/2^k), \ldots)$. This time, we will need a higher power of $|x|$ to make the sum converge. Thus, we use that $|\nabla\check{\psi}(x)| \le C$ near the origin and $|\nabla\check{\psi}(x)| \le C|x|^{-n-2}$. This gives that

$$|\nabla K(x)|^2 \le C(\sum_{k=-\infty}^{-N} 2^{2k(n+1)} + \sum_{k=-N+1}^{\infty} 2^{2k(n+1)}(2^{k+N})^{-2(n+2)}) = C2^{-2N(n+1)}.$$

Recalling that 2^N is approximately $|x - y|$ finishes the proof. ∎

Proof of Theorem 7.3. To establish the right-hand inequality, we fix N and consider the map $f \to (\psi_{-N}\hat{f}, \ldots, \psi_N\hat{f})\check{} = K_N * f$. The kernel K_N is a vector-valued function taking values in the vector space $\mathbf{C}^{2N+1}$. We observe that the conclusion of Lemma 6.17 continues to hold, if we interpret the absolute values as the norm in the Hilbert space $\mathbf{C}^{2N+1}$, with the standard norm, $|(z_{-N}, \ldots, z_N)| = (\sum_{k=-N}^{N} |z_k|^2)^{1/2}$.

As a consequence, we conclude that $K_N * f$ satisfies the L^p estimate of Theorem 6.9 and we have the inequality

$$\|(\sum_{k=-N}^{N} |Q_k f|^2)^{1/2}\|_p \leq \|f\|_p. \tag{7.5}$$

We can use the monotone convergence theorem to let $N \to \infty$ and obtain the right-hand inequality in the Theorem.

To obtain the other inequality, we argue by duality. First, using the polarization identity, we can show that for f, g in L^2,

$$\int_{\mathbf{R}^n} \sum_{k=-\infty}^{\infty} Q_k(f)(x)\overline{Q_k(g)}(x)\,dx = \int_{\mathbf{R}^n} f(x)\bar{g}(x)\,dx. \tag{7.6}$$

Next, we suppose that f is $L^2 \cap L^p$ and use duality to find the L^p norm of f, the identity (7.6), and then Cauchy-Schwarz and Hölder to obtain

$$\|f\|_p = \sup_{\|g\|_{p'}=1} \int_{\mathbf{R}^n} f(x)\bar{g}(x)\,dx = \sup_{\|g\|_{p'}=1} \int_{\mathbf{R}^n} \sum Q_k(f)(x)\overline{Q_k(g)}(x)\,dx \leq \|S(f)\|_p \|S(g)\|_{p'}.$$

Now, if we use the right-hand inequality, (7.5) which we have already proven, we obtain the desired conclusion. Note that we should assume g is in $L^2(\mathbf{R}^n) \cap L^{p'}(\mathbf{R}^n)$ to make use of the identity (7.2).

A straightforward limiting argument helps to remove the restriction that f is in L^2 and obtain the inequality for all f in L^p. ∎

7.2 Variations

In this section, we observe two simple extensions of the result above. These modifications will be needed in a later chapter.

For our next proposition, we consider an operators Q_k which are defined as above, except, that we work only in one variable. Thus, we have a function $\psi \in \mathcal{D}(\mathbf{R})$ and suppose that

$$\sum_{k=-\infty}^{\infty} |\psi(\xi_n/2^k)|^2 = 1.$$

We define the operator $f \to Q_k f = (\psi(\xi_n/2^k)\hat{f}(\xi))\check{}$.

Proposition 7.7 *If $f \in L^p(\mathbf{R}^n)$, then for $1 < p < \infty$, we have*

$$C_p \|f\|_p^p \leq \|(\sum_k |Q_k f|^2)^{1/2}\|_p^p \leq C_p \|f\|_p^p.$$

Proof. If we fix $x' = (x_1, \ldots, x_{n-1})$, then we have that

$$C_p \|f(x', \cdot)\|_{L^p(\mathbf{R})}^p \leq \|(\sum_k |Q_k f(x', \cdot)|^2)^{1/2}\|_p^p \leq C_p \|f(x', \cdot)\|_{L^p(\mathbf{R})}^p.$$

This is the one-dimensional version of Theorem 7.3. If we integrate in the remaining variables, then we obtain the Proposition. ∎

We will need the following Corollary for the one-dimensional operators. Of course the same result holds, with the same proof, for the n-dimensional operator.

Corollary 7.8 *If $2 \leq p < \infty$, then we have*

$$\|f\|_p \leq C(\sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2)^{1/2}.$$

If $1 < p \leq 2$, then we have

$$(\sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2)^{1/2} \leq C\|f\|_p.$$

Proof. To prove the first statement, we apply Minkowski's inequality bring the sum out through an $L^{p/2}$ norm to obtain

$$(\int_{\mathbf{R}^n} (\sum_{k=-\infty}^{\infty} |Q_k f(x)|^2)^{p/2} \, dx)^{2/p} \leq \sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2.$$

The application of Minkowski's inequality requires that $p/2 \geq 1$. If we take the square root of this inequality and use Proposition 7.7, we obtain the first result of the Corollary.

The second result has a similar proof. To see this, we use Minkowski's integral inequality to bring the integral out through the $\ell^{2/p}$ norm to obtain

$$(\sum_{k=-\infty}^{\infty} (\int_{\mathbf{R}^n} |Q_k f(x)|^2 \, dx)^{2/p})^{p/2} \leq \int_{\mathbf{R}^n} (\sum_{k=-\infty}^{\infty} |Q_k f(x)|^2)^{p/2}.$$

Now, we may take the pth root and apply Proposition 7.7 to obtain the second part of our Corollary. ∎

Chapter 8

Fractional integration

In this chapter, we study the fractional integration operator or Riesz potentials. To motivate these operators, we consider the following peculiar formulation of the fundamental theorem of calculus: If f is a nice function, then

$$f(x) = \int_{-\infty}^{x} f'(t)(x-t)^{1-1}\, dt.$$

Thus the map $g \to \int_{-\infty}^{x} g(t)\, dt$ is a left-inverse to differentiation. A family of fractional integrals in one dimension are the operators, defined if $\alpha > 0$ by

$$I^+_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt.$$

Exercise 8.1 *Show that if* $\alpha > 0$*and* $\beta > 0$*, then*

$$I^+_\alpha(I^+_\beta(f)) = I^+_{\alpha+\beta}(f).$$

In this section, we consider a family of similar operators in all dimensions. We will establish the L^p mapping properties of these operators. We also will consider the Fourier transform of the distribution given by the function $|x|^{\alpha-n}$. Using these results, we will obtain the Sobolev inequalities.

We begin by giving an example where these operators arise in nature. This exercise will be much easier to solve if we use the results proved below.

Exercise 8.2 *If* f *is in* $\mathcal{S}(\mathbf{R}^n)$*, then*

$$f(x) = \frac{1}{(2-n)\omega_{n-1}} \int_{\mathbf{R}^n} \Delta f(y)|x-y|^{2-n}\, dy.$$

8.1 The Hardy-Littlewood-Sobolev theorem

The operators we consider in $\mathbf{R}^n$ are the family of *Riesz potentials*

$$I_\alpha(f)(x) = \gamma(\alpha, n) \int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n}$$

for α satisfying $0 < \alpha < n$. The constant, $\gamma(\alpha, n)$ is probably given by

$$\gamma(\alpha, n) = \frac{2^{n-\alpha}\Gamma((n-\alpha)/2)}{(4\pi)^{n/2}\Gamma(\alpha/2)}.$$

Note the condition $\alpha > 0$ is needed in order to guarantee that $|x|^{\alpha-n}$ is locally integrable. Our main goal is to prove the L^p mapping properties of the operator I_α. We first observe that the homogeneity properties of this operator imply that the operator can map L^p to L^q only if $1/p - 1/q = \alpha/n$. By homogeneity properties, we mean: If $r > 0$ and we let $\delta_r f(x) = f(rx)$ be the action of dilations on functions, then we have

$$I_\alpha(\delta_r f) = r^{-\alpha}\delta_r(I_\alpha f). \tag{8.3}$$

This is easily proven by changing variables. This observation is essential in the proof of the following Proposition.

Proposition 8.4 *If the inequality*

$$\|I_\alpha f\|_q \le C\|f\|_p$$

holds for all f *in* $\mathcal{S}(\mathbf{R}^n)$ *and a finite constant* C*, then*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Proof. Observe that we have $\|\delta_r f\|_p = r^{-n/p}\|f\|_p$. This is proven by a change of variables if $0 < p < \infty$ and is obvious if $p = \infty$. (Though we will never refer to the case $p < 1$, there is no reason to restrict ourselves to $p \ge 1$.) Next, if f is in $\mathcal{S}(\mathbf{R}^n)$, then by (8.3)

$$\|I_\alpha(\delta_r f)\|_q = r^{-\alpha}\|\delta_r(I_\alpha f)\|_q = r^{-\alpha-n/q}\|I_\alpha f\|_q.$$

Thus if the hypothesis of our proposition holds, we have that for all Schwartz functions f and all $r > 0$, that

$$r^{-\alpha-n/q}\|I_\alpha f\|_q \le C\|f\|_p r^{-n/p}.$$

If $\|I_\alpha f\|_q \ne 0$ then the truth of the above inequality for all $r > 0$ implies that the exponents on each side of the inequality must be equal. If $f \ne 0$ is non-negative, then $I_\alpha f > 0$ everywhere and hence $\|I_\alpha f\|_q > 0$ and we can conclude the desired relation on the exponents. ∎

Next, we observe that the inequality must fail at the endpoint $p = 1$. This follows by choosing a nice function with $\int \phi = 1$. Then with $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$, we have that as $\epsilon \to 0^+$,

$$I_\alpha(\phi_\epsilon)(x) \to \gamma(\alpha, n)|x|^{\alpha-n}.$$

If the inequality $\|I_\alpha\phi_\epsilon\|_{n/(n-\alpha)} \leq C\|\phi_\epsilon\|_1 = C$ holds uniformly as ϵ, then Fatou's Lemma will imply that $|x|^{\alpha-n}$ lies in $L^{n/(n-\alpha)}$, which is false.

Exercise 8.5 *Show that $I_\alpha : L^p \to L^q$ if and only if $I_\alpha : L^{q'} \to L^{p'}$. Hence, we can conclude that I_α does not map $L^{n/\alpha}$ to L^∞.*

Exercise 8.6 *Can you use dilations, δ_r to show that the inequality*

$$\|f * g\|_r \leq \|f\|_p\|g\|_q$$

can hold only if $1/r = 1/p + 1/q - 1$?

Exercise 8.7 *Show that estimate*

$$\|\nabla f\|_p \leq C\|f\|_q$$

can not hold. That is if we fix p and q, there is no constant C so that the above inequality is true for all f in the Schwartz class. Hint: Let $f(x) = \eta(x)e^{i\lambda x_1}$ where η is a smooth bump.

We now give the positive result. The proof we give is due Lars Hedberg [4]. The result was first considered in one dimension (on the circle) by Hardy and Littlewood. The n-dimensional result was considered by Sobolev.

Theorem 8.8 *(Hardy-Littlewood-Sobolev) If $1/p - 1/q = \alpha/n$ and $1 < p < n/\alpha$, then there exists a constant $C = C(n, \alpha, p)$ so that*

$$\|I_\alpha f\|_q \leq C\|f\|_p.$$

The constant C satisfies $C \leq C(\alpha, n)\min((p-1)^{-(1-\frac{\alpha}{n})}, (\frac{\alpha}{n} - \frac{1}{p})^{-(1-\frac{\alpha}{n})})$.

Proof of Hardy-Littlewood-Sobolev inequality. We may assume that the L^p norm of f satisfies $\|f\|_p = 1$. We consider the integral defining I_α and break the integral into sets where $|x - y| < R$ and $|x - y| > R$:

$$I_\alpha f(x) \leq \gamma(\alpha, n)\left(\int_{B_R(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy + \int_{\mathbf{R}^n\setminus B_R(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy\right) \equiv \gamma(\alpha, n)(I + II).$$

By Proposition 5.10, we can estimate

$$I(x,R) \le Mf(x)\omega_{n-1}\int_0^R r^{\alpha-n}r^{n-1}\,dr = Mf(x)\frac{R^\alpha}{\alpha}\omega_{n-1}$$

where we need that $\alpha > 0$ for the integral to converge. To estimate the second integral, $II(x,R)$, we use Hölder's inequality to obtain

$$\begin{aligned} II(x,R) \le \|f\|_p\omega_{n-1}^{1/p'}\left(\int_{r>R} r^{(\alpha-n)p'+n-1}\,dr\right)^{1/p'} &= \|f\|_p\omega_{n-1}^{1/p'}\left(\frac{R^{(\alpha-n)p'+n}}{(\alpha-n)p'+n)}\right)^{1/p'} \\ &= \|f\|_p\omega_{n-1}^{1/p'}\frac{R^{\alpha-\frac{n}{p}}}{((\alpha-n)p'+n)^{1/p'}} \end{aligned}$$

where we need $\alpha < n/p$ for the integral in r to converge. Using the previous two inequalities and recalling that we have set $\|f\|_p = 1$, we have constants C_1 and C_2 so that

$$|I_\alpha(f)(x)| \le C_1R^\alpha Mf(x) + C_2R^{\alpha-\frac{n}{p}}. \tag{8.9}$$

If we were dedicated analysts, we could obtain the best possible inequality (that this method will give) by differentiating with respect to R and using one-variable calculus to find the minimum value of the right-hand side of (8.9). However, we can obtain an answer which is good enough by choosing $R = Mf(x)^{-p/n}$. We substitute this value of R into (8.9) and obtain

$$|I_\alpha(x)| \le (C_1+C_2)Mf(x)^{1-\alpha n/p}$$

and if we raise this inequality to the power $pn/(n-\alpha p)$ we obtain

$$|I_\alpha f(x)|^{np/(n-\alpha p)} \le (C_1+C_2)^{np/(n-\alpha p)}Mf(x)^p.$$

Now, if we integrate and use the Hardy-Littlewood theorem Theorem 5.9 we obtain the conclusion of this theorem. ∎

Exercise 8.10 *The dependence of the constants on p, α and n is probably not clear from the proof above. Convince yourself that the statement of the above theorem is correct.*

Exercise 8.11 *It may seem as if the obsession with the behavior of the constants in the Hardy-Littlewood-Sobolev theorem, Theorem 8.8, is an indication that your instructor does not have enough to do. This is false. Here is an example to illustrate how a good understanding of the constants can be used to obtain additional information. Suppose* *that f is in $L^{n/\alpha}$ and $f = 0$ outside $B_1(0)$. We know that, in general, $I_\alpha f$ is not in L^∞. The following is a substitute result. Consider the integral*

$$\int_{B_1(0)} \exp([\epsilon|I_\alpha f(x)|]^{n/(n-\alpha)})\,dx = \sum_{k=1}^{\infty} \frac{1}{k!}\epsilon^{nk/(n-k\alpha)} \int_{B_1(0)} |I_\alpha f(x)|^{\frac{kn}{n-\alpha}}\,dx.$$

Since f is in $L^{n/\alpha}$ and f is zero outside a ball of radius 1, we have that f is in $L^p(\mathbf{R}^n)$ for all $p < n/\alpha$. Thus, $I_\alpha f$ is in every L^q-space for $\infty > q > n/(n-\alpha)$. Hence, each term on the right-hand side is finite. Show that in fact, we can sum the series for ϵ small.

Exercise 8.12 *If α is real and $0 < \alpha < n$, show by example that I_α does not map $L^{n/\alpha}$ to L^∞. Hint: Consider a function f with $f(x) = |x|^{-\alpha}(-\log|x|)^{-1}$ if $|x| < 1/2$.*

Next, we compute the Fourier transform of the tempered distribution $\gamma(\alpha,n)|x|^{\alpha-n}$. More precisely, we are considering the Fourier transform of the tempered distribution

$$f \to \gamma(\alpha,n)\int_{\mathbf{R}^n} |x|^{\alpha-n} f(x)\,dx.$$

Theorem 8.13 *If $0 < \operatorname{Re}\alpha < n$, then*

$$\gamma(\alpha,n)(|x|^{\alpha-n})\hat{} = |\xi|^{-\alpha}.$$

Proof. We let $\eta(|\xi|)$ be a standard cutoff function which is 1 for $|\xi| < 1$ and 0 for $|\xi| > 2$. We set $m_\epsilon(\xi) = \eta(|\xi|\epsilon)(1-\eta(|\xi|/\epsilon))|\xi|^{-\alpha}$. The multiplier m_ϵ is a symbol of order $-\alpha$ uniformly in ϵ. Hence, by the result Lemma 6.26 of Chapter 6, we have that $K_\epsilon = \check{m}_\epsilon$ satisfies the estimates

$$|\frac{\partial^\beta}{\partial x^\beta} K_\epsilon(x)| \le C(\alpha,\beta)|x|^{\alpha-n-|\beta|}. \tag{8.14}$$

Hence, applying the Arzela-Ascoli theorem we can extract a sequence $\{\epsilon_i\}$ with $\epsilon_i \to 0$ so that K_{ϵ_i} converges uniformly to some function K on each compact subset of $\mathbf{R}^n \setminus \{0\}$. We choose f in $\mathcal{S}(\mathbf{R}^n)$ and recall the definition of the Fourier transform of a distribution to obtain

$$\begin{aligned}\int_{\mathbf{R}^n} K(x)\hat{f}(x)\,dx &= \lim_{j\to\infty} \int K_{\epsilon_j}(x)\hat{f}(x)\,dx \\ &= \lim_{j\to\infty} \int m_{\epsilon_j}(\xi) f(\xi)\,d\xi \\ &= \int |\xi|^{-\alpha} f(\xi)\,d\xi.\end{aligned}$$

The first equality depends on the uniform estimate for K_ϵ in (8.14) and the locally uniform convergence of the sequence K_{ϵ_j}. Thus, we have that $\hat{K}(\xi) = |\xi|^{-\alpha}$ in the sense of distributions. Note that each m_ϵ is radial. Hence, K_ϵ and thus K is radial. See Chapter 1.

Our next step is to show that the kernel K is homogeneous:

$$K(Rx) = R^{\alpha-n}K(x). \tag{8.15}$$

To see this, observe that writing $K = \lim_{j\to\infty} K_{\epsilon_j}$ again gives that

$$\begin{aligned}
\int_{\mathbf{R}^n} K(Rx)\hat{f}(x)\,dx &= \lim_{j\to\infty} K_{\epsilon_j}(Rx)\hat{f}(x)\,dx \\
&= R^{-n}\lim_{j\to\infty}\int m_{\epsilon_j}(\xi/R)f(\xi)\,d\xi = R^{\alpha-n}\int |\xi|^{\alpha-n}f(\xi)\,d\xi \\
&= R^{\alpha-n}\int K(x)\hat{f}(x)\,dx.
\end{aligned}$$

This equality for all f in $\mathcal{S}(\mathbf{R}^n)$ implies that (8.15) holds. If we combine the homogeneity with the rotational invariance of K observed above, we can conclude that

$$\check{m}(x) = c|x|^{\alpha-n}.$$

It remains to compute the value of c. To do this, we only need to find one function where we can compute the integrals explicitly. We use the friendly gaussian. We consider

$$c\int |x|^{\alpha-n}e^{-|x|^2}\,dx = (4\pi)^{-n/2}\int |\xi|^{-\alpha}e^{-|\xi|^2/4}\,d\xi = 2^{n-\alpha}(4\pi)^{-n/2}\int |\xi|^{-\alpha}e^{-|\xi|^2}\,d\xi. \tag{8.16}$$

Writing the integrals in polar coordinates, substituting $s = r^2$, and then recalling the definition of the Gamma function, we obtain

$$\begin{aligned}
\int_{\mathbf{R}^n} |x|^{-\beta}e^{-|x|^2}\,dx &= \omega_{n-1}\int_0^\infty r^{n-\beta}e^{-r^2}\,\frac{dr}{r} \\
&= \frac{\omega_{n-1}}{2}\int_0^\infty s^{(n-\beta)/2}e^{-s}\,\frac{ds}{s} \\
&= \frac{1}{2}\Gamma(\frac{n-\beta}{2})\omega_{n-1}.
\end{aligned}$$

Using this to evaluate the two integrals in (8.16) and solving for c gives

$$c = \frac{2^{n-\alpha}\Gamma((n-\alpha)/2)}{(4\pi)^{n/2}\Gamma(\alpha/2)}.$$

∎

We give a simple consequence.

Corollary 8.17 *For f in $\mathcal{S}(\mathbf{R}^n)$, we have*

$$I_\alpha(f) = (\hat{f}(\xi)|\xi|^{-\alpha})\check{}.$$

A reader who is not paying much attention, might be tricked into thinking that this is just an application of Proposition 1.24. Though I like to advocate such sloppiness, it is traditional to be a bit more careful. Note that Proposition 1.24 does not apply to the study of $I_\alpha f$ because $I_\alpha f$ is not the convolution of two L^1 functions. A proof could be given based on approximating the multiplier $|\xi|^{-\alpha}$ by nice functions. However, I elect to obtain the result by algebra–that is by using distributions. This result should, perhaps, have appeared in Chapter 2. However, following the modern "just-in-time" approach to knowledge delivery, we have waited until the result was needed before making a proof.

Proposition 8.18 *If u is a tempered distribution and f is a Schwartz function, then*

$$(f * u)\hat{} = \hat{f}\hat{u}.$$

Proof. Recall the definition for convolutions involving distributions that appeared in Chapter 2. By this and the definition of the Fourier transform and inverse Fourier transform, we have

$$(f * u)\check{}(g) = f * u(\hat{g}) = \check{\hat{u}}(\tilde{f} * \hat{g}) = \hat{u}((\tilde{f} * \hat{g})\check{}).$$

Now, we argue as in the proof of Proposition 1.24 and use the Fourier inversion theorem, 1.31 to obtain

$$(\tilde{f} * \hat{g})\check{}(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} f(\xi - \eta)\hat{g}(\eta) e^{ix\cdot((\xi-\eta+\eta))}\, d\xi d\eta = \hat{f}(x)g(x).$$

Thus, we have $(f * u)\check{}(g) = \hat{u}(\hat{f}g) = (\hat{f}\hat{u})(g)$. ∎

Proof of Corollary 8.17. This is immediate from Theorem 8.13 which gives the Fourier transform of the distribution given by $\gamma(\alpha, n)|x|^{\alpha-n}$ and the previous proposition. ∎

8.2 A Sobolev inequality

Next step is to establish an inequality relating the L^q-norm of a function f with the L^p-norm of its derivatives. This result, known as a *Sobolev inequality* is immediate from the Hardy-Littlewood-Sobolev inequality, once we have a representation of f in terms of its gradient.

Lemma 8.19 *If f is a Schwartz function, then*

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbf{R}^n} \frac{\nabla f(y) \cdot (x-y)}{|x-y|^n}\, dy.$$

Proof. We let $z' \in \mathbf{S}^{n-1}$ and then write

$$f(x) = -\int_0^\infty \frac{d}{dt} f(x+tz')\, dt = -\int_0^\infty z' \cdot (\nabla f)(x+tz')\, dt.$$

If we integrate both sides with respect to the variable z', and then change from the polar coordinates t and z' to y which is related to t and z' by $y - x = tz'$, we obtain

$$\omega_{n-1} f(x) = -\int_{\mathbf{S}^{n-1}} \int_0^\infty z' \cdot \nabla f(x+tz') t^{n-1+1-n}\, dt\, dz' = \int_{\mathbf{R}^n} \frac{x-y}{|x-y|} \cdot \nabla f(y) \frac{1}{|x-y|^{n-1}}\, dy.$$

This gives the conclusion. ∎

Theorem 8.20 *If $1 < p < n$ (and thus $n \geq 2$), f is in the Sobolev space L^p_1 and q is defined by $1/q = 1/p - 1/n$, then there is a constant $C = C(p,n)$ so that*

$$\|f\|_q \leq C \|\nabla f\|_p.$$

Proof. According to Lemma 8.19, we have that for nice functions,[1]

$$|f(x)| \leq I_1(|\nabla f|)(x).$$

Thus, the inequality of this theorem follows from the Hardy-Littlewood-Sobolev theorem on fractional integration. Since the Schwartz class is dense in the Sobolev space, a routine limiting argument extends the inequality to functions in the Sobolev space. ∎

[1] This assumes that $\omega_{n-1}^{-1} = \gamma(1,n)$, which I have not checked.

The Sobolev inequality continues to hold when $p = 1$. However, the above argument fails. The standard argument for $p = 1$ is an ingenious argument due to Gagliardo, see Stein [14, pp. 128-130] or the original paper of Gagliardo [3].

Exercise 8.21 *If $p > n$, then the Riesz potential, I_1 produces functions which are in Hölder continuous of order $\gamma = 1 - (n/p)$. If $0 < \gamma < 1$, define the Hölder semi-norm by*

$$\|f\|_{C^\gamma} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma},$$

a) Show that if f is a Schwartz function, then $\|I_1(f)\|_{C^\gamma} \leq C\|f\|_p$ provided $p > n$ and $\gamma = 1 - (n/p)$. b) Generalize to I_α. c) The integral defining $I_1(f)$ is not absolutely convergent for all f in L^p if $p > n$. Show that the differences $I_1 f(x) - I_1 f(y)$ can be expressed as an absolutely convergent integral. Conclude that if $f \in L^p_1$, then $f \in C^\gamma$ for γ and p as above.

Exercise 8.22 *Show by example that the Sobolev inequality, $\|f\|_\infty \leq C\|\nabla f\|_n$ fails if $p = n$ and $n \geq 2$. Hint: For appropriate a, try f with $f(x) = \eta(x)(-\log|x|)^a$ with η a smooth function which is supported in $|x| < 1/2$.*

Exercise 8.23 *Show that there is a constant $C = C(n)$ so that if $g = I_1(f)$, then*

$$\sup_{r>0, x \in \mathbf{R}^n} \frac{1}{m(B_r(x)} \int_{B_r(x)} |g(x) - (g)_{r,x}|\, dx \leq C\|f\|_n.$$

Here, $(f)_{r,x}$ denotes the average of f on the ball $B_r(x)$.

$$(f)_{r,x} = \frac{1}{B_r(x)} \int_{B_r(x)} f(y)\, dy.$$

Exercise 8.24 *Show that in one dimension, the inequality $\|f\|_\infty \leq \|f\|_1$ is trivial for nice f. State precise hypotheses that f must satisfy.*

Chapter 9

Singular multipliers

In this section, we establish estimates for an operator whose symbol is singular. The results we prove in this section are more involved than the simple L^2 multiplier theorem that we proved in Chapter 3. However, roughly speaking what we are doing is taking a singular symbol, smoothing it by convolution and then applying the L^2 multiplier theorem. As we shall see, this approach gives estimates in spaces of functions where we control the rate of increase near infinity. Estimates of this type were proven by Agmon and Hörmander. The details of our presentation are different, but the underlying ideas are the same.

9.1 Estimates for an operator with a singular symbol

For the next several chapters, we will be considering a differential operator,

$$\Delta + 2\zeta \cdot \nabla = e^{-x\cdot\zeta}\Delta e^{x\cdot\zeta}$$

where $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j \zeta_j = 0$.

Exercise 9.1 *Show that $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$ if and only if $\zeta = \xi + i\eta$ where ξ and η are in $\mathbf{R}^n$ and satisfy $|\xi| = |\eta|$ and $\xi \cdot \eta = 0$.*

Exercise 9.2 *a) Show that $\Delta e^{x\cdot\zeta} = 0$ if and only if $\zeta \cdot \zeta = 0$. b) Find conditions on $\tau \in \mathbf{R}$ and $\xi \in \mathbf{R}^n$ so that $e^{\tau t + x\cdot\xi}$ satisfies*

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)e^{t\tau + x\cdot\xi} = 0.$$

The symbol of this operator is

$$-|\xi|^2 + 2i\zeta\cdot\xi = |\operatorname{Im}\zeta|^2 - |\operatorname{Im}\zeta+\xi|^2 + 2i\operatorname{Re}\zeta\cdot\xi.$$

Thus it is clear that this symbol vanishes on a sphere of codimension 2 which lies in the hyperplane $\operatorname{Re}\zeta\cdot\xi = 0$ and which has center $-\operatorname{Im}\zeta$ and radius $|\operatorname{Im}\zeta|$. Near this sphere, the symbol vanishes to first order. This means that the reciprocal of the symbol is locally integrable. In fact, we have the following fundamental estimate.

Lemma 9.3 *If $\eta\in\mathbf{R}^n$ and $r>0$ then there exists a constant C depending only on the dimension n so that*

$$\int_{B_r(\eta)} \left|\frac{1}{-|\xi|^2+2i\zeta\cdot\xi}\right| d\xi \le \frac{Cr^{n-1}}{|\zeta|}.$$

Proof. We first observe that we are trying to prove a dilation invariant estimate, and we can simplify our work by scaling out one parameter. If we make the change of variables, $\xi = |\zeta|x$, we obtain

$$\int_{B_r(\eta)} \left|\frac{1}{-|\xi|^2+2i\zeta\cdot\xi}\right| d\xi = |\zeta|^{n-2}\int_{B_{r/|\zeta|}(\eta/|\zeta|)} \frac{1}{-|x|^2+2\hat\zeta\cdot x}\,dx$$

where $\hat\zeta = \zeta/|\zeta|$. Thus, it suffices to consider the estimate when $|\zeta|=1$ and we assume below that we have $|\zeta|=1$.

We also, may make rotation $\xi = Ox$ so that $O^t\operatorname{Re}\zeta = e_1/\sqrt{2}$, with e_1 the unit vector in the x_1 direction and $O^t\operatorname{Im}\zeta = e_2/\sqrt{2}$. Then, we have that

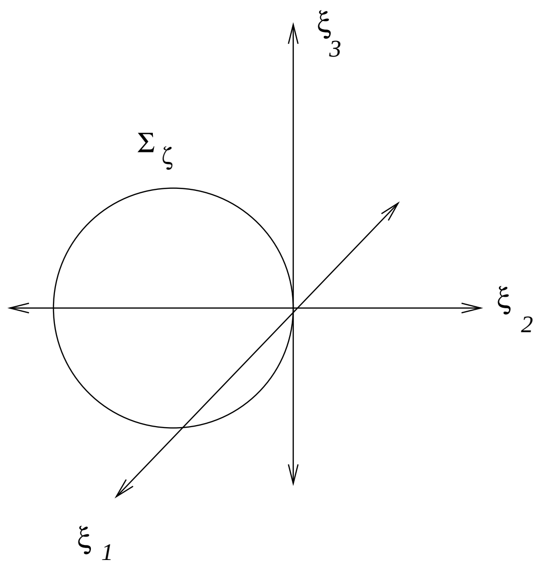
$$\int_{B_r(\eta)} \left|\frac{1}{-|\xi|^2+2i\zeta\cdot\xi}\right| d\xi = \int_{B_r(O^t\eta)} \left|\frac{1}{-|x|^2+2iO^t\zeta\cdot x}\right| dx.$$

Thus, it suffices to prove the Lemma in the special case when $\zeta = (e_1+ie_2)/\sqrt{2}$.

We let $\Sigma_\zeta = \{\xi : -|\xi|^2+2i\zeta\cdot\xi = 0\}$ be the zero set of the symbol.

Case 1. The ball $B_r(\eta)$ satisfies $r<1/100$, $\operatorname{dist}(\eta,\Sigma_\zeta) < 2r$. In this case, we make an additional change of variables. We rotate in the variables $(\xi_2,\dots\xi_n)$ about the center of Σ_ζ, $-e_1/\sqrt{2}$, so that η is within $2r$ units of the origin. We can find a ball B_{3r} of radius $3r$ and centered 0 in Σ_ζ so that $B_r(\eta)\subset B_{3r}$ Now, we use coordinates $x_1 = \operatorname{Re}\zeta\cdot\xi$, $x_2 = |\operatorname{Im}\zeta|^2 - |\operatorname{Im}\zeta+\xi|^2$ and $x_j = \xi_j$ for $j=3,\dots,n$. We leave it as an exercise to compute the Jacobian and show that it is bounded above and below on $B_r(\eta)$. This gives the bound

$$\int_{B_{3r}} \left|\frac{1}{-|\xi|^2+2i\zeta\cdot\xi}\right| d\xi \le C\int_{B_{Cr}(0)} \frac{1}{|x_1+ix_2|}\,dx_1\,dx_2\dots dx_n = Cr^{n-1}.$$



Case 2. We have $B_r(\eta)$ with $\mathrm{dist}(\eta, \Sigma_\zeta) > 2r$. In this case, we have

$$\sup_{\xi \in B_r(\eta)} \frac{1}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \le C/r$$

and the Lemma follows in this case.

Case 3. The ball $B_r(\eta)$ satisfies $r > 1/100$ and $\mathrm{dist}(\eta, \Sigma_\zeta) < 2r$.

In this case, write $B_r(\eta) = B_0 \cup B_\infty$ where $B_0 = B_r(\eta) \cap B_4(0)$ and $B_\infty = B_r(\eta) \setminus B_0$. By case 1 and 2,

$$\int_{B_0} \frac{1}{|-|\xi|^2 + 2i\zeta \cdot \xi|}\, d\xi \le C.$$

Since $B_4(0)$ contains the set Σ_ζ, one can show that

$$\frac{1}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \le C/|\xi|^2$$

on B_∞ and integrating this estimate gives

$$\int_{B_\infty} \frac{1}{|-|\xi|^2 + 2i\zeta \cdot \xi|}\, d\xi \le Cr^{n-2}.$$

Since $r > 1/100$, the estimates on B_0 and B_∞ imply the estimate of the Lemma in this case. ∎

As a consequence of this Lemma, we can define the operator $G_\zeta : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by

$$G_\zeta f = \left(\frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\xi\cdot\zeta}\right)^{\check{}}$$

Lemma 9.4 *The map G_ζ is bounded from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ and we have*

$$(\Delta + 2\zeta\cdot\nabla)G_\zeta f = G_\zeta(\Delta + 2\zeta\cdot\nabla)f = f$$

if $f \in \mathcal{S}(\mathbf{R}^n)$.

Proof. According to the previous lemma, the symbol of G_ζ satisfies the growth condition of Example 2.21 in Chapter 2. Hence $G_\zeta f$ is in $\mathcal{S}'(\mathbf{R}^n)$. The remaining results rely on the Proposition 1.18 of Chapter 1[1]. ∎

It is not enough to know that $G_\zeta f$ is a tempered distribution. We would also like to know that the map G_ζ is bounded between some pair of Banach spaces. This will be useful when we try to construct solutions of perturbations of the operator $\Delta + 2\zeta\cdot\nabla$. The definition of the spaces we will use appears similar to the Besov spaces and the Littlewood-Paley theory in Chapter 7. However, now we are decomposing f rather than $\hat{f}$. To define these spaces, we let

$$B_j = B_{2^j}(0)$$

and then put $R_j = B_j \setminus B_{j-1}$. We let $\dot{M}^{p,s}_q(\mathbf{R}^n)$ denote the space of functions u for which the norm

$$\|u\|_{\dot{M}^{p,s}_q} = \left(\sum_{k=-\infty}^{\infty}\left[2^{ks}\|u\|_{L^p(R_k)}\right]^q\right)^{1/q} < \infty.$$

Also, we let $M^{p,s}_q$ be the space of measurable functions for which the norm

$$\|u\|_{M^{p,s}_q} = \left(\|u\|^q_{L^p(B_0)} + \sum_{k=1}^{\infty}\left[2^{ks}\|u\|_{L^p(R_k)}\right]^q\right)^{1/q}.$$

These definitions are valid for $0 < p \le \infty$, $s \in \mathbf{R}$ and $0 < q < \infty$. We will also need the case when $q = \infty$ and this is defined by replacing the ℓ^q norm of the sequence $2^{ks}\|u\|_{L^p(R_k)}$ by the supremum. Our primary interests are the spaces where $p = 2$, $q = 1$ and $s = 1/2$ and the space where $p = 2$, $q = \infty$ and $s = -1/2$. The following exercises give some practice with these spaces.

[1] There is a sign error in the version of this Proposition handed out in class.

Exercise 9.5 *For which a do we have*

$$(1+|x|^2)^{a/2} \in M^{2,1/2}_{\infty}(\mathbf{R}^n).$$

Exercise 9.6 *Show that if $r \geq q$, then*

$$M^{2,s}_q \subset M^{2,s}_r.$$

Exercise 9.7 *Show that if $s > 0$, then*

$$M^{2,s}_1 \subset \dot{M}^{2,s}_1.$$

Exercise 9.8 *Let T be the multiplication operator*

$$Tf(x) = (1+|x|^2)^{-(1+\epsilon)/2} f(x).$$

Show that if $\epsilon > 0$, then

$$T : M^{2,-1/2}_{\infty} \to M^{2,1/2}_1.$$

Exercise 9.9 *Show that we have the inclusion $M^{2,1/2}_1 \subset L^2(\mathbf{R}^n, d\mu_s)$ where $d\mu_s = (1+|x|^2)^{2s}dx$. This means that we need to establish that for some C depending on n and s, we have the inequality*

$$\|u\|_{L^2(B_0)} + \sum_{k=1}^{\infty} \|u\|_{L^2(R_k)} \leq C(\int_{\mathbf{R}^n} |u(x)|^2(1+|x|^2)^s\, dx)^{1/2}$$

Hint: The integral on $\mathbf{R}^n$ dominates the integral on each ring. On each ring, the weight changes by at most a fixed factor. Thus, it makes sense to replace the weight by its smallest value. This will give an estimate on each ring that can be summed to obtain the $M^{2,1/2}_1$ norm.

The main step of our estimate is the following lemma.

Lemma 9.10 *Let ψ and ψ' are Schwartz functions on $\mathbf{R}^n$ and set $\psi_k(x) = \psi(2^{-k}x)$ and $\psi'_j(x) = \psi'(2^{-j}x)$. We define a kernel $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ by*

$$K(\xi_1, \xi_2) = \int_{\mathbf{R}^n} \frac{\hat{\psi}'_j(\xi_1 - \xi)\hat{\psi}_k(\xi - \xi_2)}{-|\xi|^2 + 2i\zeta \cdot \xi}\, d\xi.$$

Then there is a constant C so that

$$\sup_{\xi_1} \int |K(\xi_1, \xi_2)|\, d\xi_2 \leq \frac{C2^j}{|\zeta|} \tag{9.11}$$

$$\sup_{\xi_2} \int |K(\xi_1, \xi_2)|\, d\xi_1 \leq \frac{C2^k}{|\zeta|}. \tag{9.12}$$

As a consequence, the operator $T_{j,k}$ given by

$$T_{j,k}f(\xi_1) = \int K(\xi_1, \xi_2) f(\xi_2)\, d\xi_2$$

satisfies

$$\|T_{j,k}f\|_p \leq \frac{C}{|\zeta|} 2^{k/p} 2^{j/p'}. \tag{9.13}$$

Proof. Observe that $\hat{\psi}_k(\xi) = 2^{kn}\hat{\psi}(\xi 2^k) = (\hat{\psi})_{2^{-k}}(\xi)$. Thus, $\|\hat{\psi}_k\|_1$ is independent of k. Since $\psi \in \mathcal{S}(\mathbf{R}^n)$, we have that $\|\hat{\psi}\|_1$ is finite. Thus if we use Tonelli's theorem, we have

$$\int_{\mathbf{R}^n} |K(\xi_1, \xi_2)|\, d\xi_2 \leq \|\hat{\psi}\|_1 \int \frac{|\hat{\psi}'_j(\xi_1 - \xi)|}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi.$$

To estimate the integral on the right of this inequality, we break the integral into rings centered at ξ_1 and use that $\hat{\psi}'$ decays rapidly at infinity so that, in particular, we have $\hat{\psi}'(\xi) \leq C\min(1, |\xi|^{-n})$. Then applying Lemma 9.3 gives us

$$\begin{aligned}
\int \frac{|\hat{\psi}'_j(\xi_1 - \xi)|}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi &\leq \|\hat{\psi}\|_\infty 2^{nj} \int_{B_{2^{-j}}(\xi_1)} \frac{1}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi \\
&\quad + \sum_{l=1}^{\infty} C 2^{nj} 2^{-n(j-l)} \int_{B_{2^{-j+l}}(\xi_1)\setminus B_{2^{-j+l-1}}(\xi_1)} \frac{1}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi \\
&\leq \frac{C}{|\zeta|} 2^j \sum_{l=0}^{\infty} 2^{-l}.
\end{aligned}$$

This gives the first estimate (9.11). The second is proven by interchanging the roles of ξ_1 and ξ_2. The estimate (9.11) gives a bound for the operator norm on L^∞. The estimate (9.12) gives a bound for the operator norm on L^1. The bound for the operator norm on L^p follows by the Riesz-Thorin interpolation theorem, Theorem 4.1. See exercise 4.5. ∎

Exercise 9.14 *Show that it suffices to prove the following theorem for $|\zeta| = 1$. That is, show that if the theorem holds when $|\zeta| = 1$, then by rescaling, we can deduce that the result holds for all ζ with $\zeta \cdot \zeta = 0$.*

Exercise 9.15 *The argument given should continue to prove an estimate as long as* $\operatorname{Re}\zeta$ *and* $\operatorname{Im}\zeta$ *are both nonzero. Verify this and show how the constants depend on ζ.*

Theorem 9.16 *The map G_ζ satisfies*

$$\sup_j 2^{-j/2} \|G_\zeta f\|_{L^2(B_j)} \leq \frac{C}{|\zeta|} \|f\|_{\dot{M}_1^{2,1/2}}$$

and

$$\sup_j 2^{-j/2} \|G_\zeta f\|_{L^2(B_j)} \leq \frac{C}{|\zeta|} \|f\|_{M_1^{2,1/2}}.$$

Proof. We first suppose that f is in the Schwartz space. We choose $\psi \geq 0$ as in Chapter 7 so that supp$\psi \subset \{x : 1/2 \leq x \leq 2\}$ and with $\psi_k(x) = \psi(2^{-k}x)$, we have

$$\sum_{k=-\infty}^{\infty} \psi_k^2 = 1, \qquad \text{in } \mathbf{R}^n \setminus \{0\}.$$

We let $\phi = 1$ if $|x| < 1$, $\phi \geq 0$, $\phi \in \mathcal{D}(\mathbf{R}^n)$ and set $\phi_j(x) = \phi(2^{-j}x)$. We decompose f using the ψ_k's to obtain

$$\phi_j G_\zeta f = \sum_{k=-\infty}^{\infty} \phi_j G_\zeta \psi_k^2 f.$$

The Plancherel theorem implies that

$$\|\phi_j G_\zeta(\psi_k^2 f)\|_2^2 = (2\pi)^{-n} \int |T_{j,k} \widehat{\psi_k f}|^2 \, d\xi.$$

Here, the operator $T_{j,k}$ is as in the previous lemma but with ψ replaced by ϕ and ψ' replaced by ψ. Hence, from Lemma 9.10 we can conclude that

$$\|\phi_j G_\zeta \psi_k^2 f\|_2 \leq \frac{C}{|\zeta|} 2^{j/2} 2^{k/2} \sum_{|\ell| \leq 1} \|\psi_{k+\ell} f\|_2. \tag{9.17}$$

Now, using Minkowski's inequality, we have

$$\|G_\zeta f\|_{L^2(B_j)} \leq C \sum_{k=-\infty\infty} \|\phi_j G_\zeta \psi_k^2 f\|_{L^2(\mathbf{R}^n)}. \tag{9.18}$$

The first conclusion of the theorem now follows from (9.17) and (9.18).

The estimate in the inhomogeneous space follows by using Cauchy-Schwarz to show

$$\sum_{k=-\infty}^{0} 2^{k/2}\|f\|_{L^2(R_k)} \leq \left(\sum_{k=-\infty}^{0} \|f\|^2_{L^2(R_k)}\right)^{1/2} \left(\sum_{k=-\infty}^{0} 2^{k/2}\right)^{1/2} = \left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^{1/2} \|f\|_{L^2(B_0)}.$$

Finally, to remove the restriction that f is in the Schwartz space, we observe that the Lemma below tells us that Schwartz functions are dense in $\dot{M}_1^{2,1/2}$ and $M_1^{2,1/2}$. ∎

Lemma 9.19 *We have that $\mathcal{S}(\mathbf{R}^n) \cap \dot{M}_1^{2,1/2}$ is dense in $\dot{M}_1^{2,1/2}$ and $\mathcal{S}(\mathbf{R}^n) \cap M_1^{2,1/2}$ is dense in $\dot{M}_1^{2,1/2}$.*

Proof. To see this, first observe that if we pick f in $\dot{M}_1^{2,1/2}$ and define

$$f_N(x) = \begin{cases} 0, & |x| < 2^{-N} \text{ or } |x| > 2^N \\ f(x), & 2^{-N} \leq |x| \leq 2^N \end{cases}$$

then f_N converges to$f$ in $\dot{M}_1^{2,1/2}$. Next, if we regularize with a standard mollifier, then $f_{N,\epsilon} = f_N * \eta_\epsilon$ converges to f_N in L^2. If we assume that η is supported in the unit ball, then for $\epsilon < 2^{-N-1}$, $f_{N,\epsilon}$ will be supported in the shell $\{x : 2^{-N-1} \leq |x| \leq 2^{N+1}\}$. For such functions, we may use Cauchy-Schwarz to obtain

$$\|f_N - f_{N,\epsilon}\|_{\dot{M}_1^{2,1}} \leq \left(\sum_{k=-N}^{N+1} \|f_{N,\epsilon} - f_N\|^2_{L^2(R_k)}\right)^{1/2} \left(\sum_{k=-N}^{N+1} 2^k\right)^{1/2} = C\|f_N - f_{N,\epsilon}\|_2.$$

Hence, for functions supported in compact subsets of $\mathbf{R}^n \setminus \{0\}$, the L^2 convergence of $f_{N,\epsilon}$ to f_N implies convergence in the space $\dot{M}_1^{2,1/2}$. Approximation in $M_1^{2,1/2}$ is easier since we only need to cut off near infinity. ∎

Exercise 9.20 *Are Schwartz functions dense in $M_\infty^{2,-1/2}$?*

Exercise 9.21 *Use the ideas above to show that*

$$\sup_j 2^{-j/2}\|\nabla G_\zeta f\|_{L^2(B_j)} \leq C\|f\|_{\dot{M}_1^{2,1/2}}.$$

Hint: One only needs to find a replacement for Lemma 9.3.

Exercise 9.22 *Use the ideas above to show that $I_\alpha : \dot{M}_1^{2,\alpha/2} \to \dot{M}_\infty^{2,-\alpha/2}$. Hint: Again, the main step is to find a substitute for Lemma 9.3.*

Finally, we establish uniqueness for the equation $\Delta u + 2\zeta \cdot \nabla u = 0$ in all of $\mathbf{R}^n$. In order to obtain uniqueness, we will need some restriction on the growth of u at infinity.

Theorem 9.23 *If u in L^2_{loc} and satisfies*

$$\lim_{j\to\infty} 2^{-j}\|u\|_{L^2(B_j(0))} = 0$$

and $\Delta u + 2\zeta \cdot \nabla u = 0$, then $u = 0$.

The following is taken from Hörmander [6], see Theorem 7.1.27.

Lemma 9.24 *If u is a tempered distribution which satisfies*

$$\limsup_{R\to\infty} R^{-d}\|u\|_{L^2(B_R(0))} = M < \infty$$

and $\hat{u}$ is supported in a compact surface S of codimension d, then there is a function $u_0 \in L^2(S)$ so that

$$\hat{u}(\phi) = \int_S \phi u_0 d\sigma$$

and $\|u_0\|_{L^2(S)} \leq CM$.

Proof. We choose $\phi \in \mathcal{D}(\mathbf{R}^n)$, supp$\phi \subset B_1(0)$, ϕ even, $\int \phi = 1$ and consider $\hat{u} * \phi_\epsilon$. By Plancherel's theorem, we have that

$$\int |\hat{u} * \phi_{2^{-j}}|^2 \, d\xi = \int |\hat{\phi}(2^{-j}x)u(x)|^2 \, dx \leq C2^{-dj}M^2.$$

To establish this, we break the integral into the integral over the unit ball and integrals over shells. We use that $\hat{\phi}$ is in $\mathcal{S}(\mathbf{R}^n)$ and satisfies $|\hat{\phi}(x)| \leq C\min(1, |x|^{-(d+1)})$. For j large enough so that $2^{-jd}\|u\|_{L^2(B_j)} \leq 2M$, we have

$$\begin{aligned} \int |\hat{\phi}(2^{-j}x)|^2|u(x)|^2 \, dx &\leq \int_{B_j} |u(x)|^2 \, dx + \sum_{k=j}^{\infty} \int_{R_{k+1}} |\hat{\phi}(2^{-j}x)|^2|u(x)|^2 \, dx \\ &\leq C2^{2jd}4M^2 + C2^{2j(d+1)} \sum_{k=j}^{\infty} 2^{-2k}4M^2 = CM^2 2^{dj}. \end{aligned}$$

If we let $S_\epsilon = \{\xi : \operatorname{dist}(\xi, \operatorname{supp} S) < \epsilon\}$ and ψ is in the Schwartz class, then we have

$$\int_S |\psi(x)|^2 \, d\sigma = C_d \lim_{\epsilon\to 0^+} \epsilon^{-d} \int_{S_\epsilon} |\psi(x)|^2 \, dx.$$

Since $\phi_\epsilon * \psi \to \psi$ in $\mathcal{S}$, we have

$$\hat{u}(\psi) = \lim_{j\to\infty} \hat{u}(\phi_{2^{-j}} * \psi).$$

Then using Cauchy-Schwarz and the estimate above for $u * \phi_{2^{-j}}$, we obtain

$$|\hat{u}(\psi * \phi_{2^{-j}})| = |\int_{S_{2^{-j}}} \hat{u} * \phi_{2^{-j}}(x)\psi(x)\,dx| \le CM2^{jd}(\int_{S_{2^{-j}}} |\psi(x)|^2\,dx)^{1/2}.$$

If we let $\epsilon \to 0^+$, we obtain that $|\hat{u}(\psi)| \le CM\|\psi\|_{L^2(S)}$. This inequality implies the existence of u_0. ∎

Now we can present the proof of our uniqueness theorem.

Proof of Theorem 9.23. Since $\Delta u + 2\zeta \cdot \nabla u = 0$, we can conclude that the distribution $\hat{u}$ is supported on the zero set of $-|\xi| + 2i\zeta\cdot\xi$, a sphere of codimension 2. Now the hypothesis on the growth of the L^2 norm and the previous lemma, Lemma 9.24 imply that $\hat{u} = 0$. ∎

Corollary 9.25 *If f is in $M_1^{2,1/2}$, then there is exactly one solution u of*

$$\Delta u + 2\zeta \cdot \nabla u = f$$

which lies in $M_\infty^{2,-1/2}$. This solution satisfies

$$|\zeta|\|u\|_{M_\infty^{2,-1/2}} + \|\nabla u\|_{M_\infty^{2,-1/2}} \le C\|f\|_{M_1^{2,1/2}}.$$

Proof. The existence follows from Theorem 9.16 and exercise 9.21. If u is in $M_\infty^{2,-1/2}$, then we have u is in L^2_{loc} and that

$$\lim_{j\to\infty} 2^{-\alpha j}\|u\|_{L^2(B_{2^j}(0))} = 0$$

if $\alpha > 1/2$. Thus, the uniqueness follows from Theorem 9.23. ∎

9.2 A trace theorem.

The goal of this section is to provide another application of the ideas presented above. The result proven will not be used in this course. Also, this argument will serve to introduce a technical tool that will be needed in Chapter 14.

We begin with a definition of a *Ahlfors condition.* We say that a Borel measure μ in $\mathbf{R}^n$ satisfies an Ahlfors condition if for some constant A it satisfies $\mu(B_r(x)) \leq Ar^{n-1}$. This is a property which is satisfied by surface measure on the boundary of a C^1-domain as well as by surface measure on a graph $\{(x', x_n) : x_n = \phi(x')\}$ provided that the $\|\nabla\phi\|_\infty < \infty$.

Our main result is the following theorem.

Theorem 9.26 *If f is in $\mathcal{S}(\mathbf{R}^n)$ and μ satisfies the Ahlfors condition, then there is a constant C so that*

$$\int_{\mathbf{R}^n} |\hat{u}(\xi)|^2 \, d\mu \leq C\|u\|^2_{\dot{M}_1^{2,1/2}}.$$

This may seem peculiar, but as an application, we observe that this theorem implies a trace theorem for Sobolev spaces.

Corollary 9.27 *If μ satisfies the Ahlfors condition and $s > 1/2$ then we have*

$$\int_{\mathbf{R}^n} |u|^2 \, d\mu \leq C\|u\|^2_{L^2_s(\mathbf{R}^n)}.$$

Proof. First assume that $u \in \mathcal{S}(\mathbf{R}^n)$. Applying the previous theorem to $\check{u}(x) = (2\pi)^{-n}\hat{u}(-x)$ gives that

$$\int |u|^2 \, d\mu(x) \leq C\|\hat{u}\|_{\dot{M}_1^{2,1/2}}.$$

It is elementary (see exercise 9.9), to establish the inequality

$$\|v\|_{M_1^{2,1/2}} \leq C_s \int_{\mathbf{R}^n} |v(x)|^2(1+|x|^2)^s \, dx$$

when $s > 1/2$. Also, from exercise 9.7 or the proof of theorem 9.16, we have

$$\|v\|_{\dot{M}_1^{2,1/2}} \leq C_s\|v\|_{M_1^{2,1/2}}.$$

Combining the two previous inequalities with $v = \hat{u}$ gives the desired conclusion. ∎

Lemma 9.28 *The map $g \to \int \cdot g\,dx$ is an isomorphism from $\dot{M}^{2,-1/2}_\infty$ to the dual space of $M^{2,1/2}_1$, $\dot{M}^{2,1/2}_1{}'$.*

Proof. It is clear by applying Hölder's inequality twice that

$$\int_{\mathbf{R}^n} fg\,dx \le \|f\|_{\dot{M}^{2,1/2}_1} \|g\|_{M^{2,-1/2}_\infty}.$$

Thus, our map takes $\dot{M}^{2,1/2}_\infty$ into the dual of $\dot{M}^{2,1/2}_1$. To see that this map is onto, suppose that $\lambda \in M^{2,1/2'}_1$. Observe $L^2(R_k) \subset \dot{M}^{2,1/2}_1$ in the sense that if $f \in L^2(R_k)$, then the function which is f in R_k and 0 outside R_k lies in $\dot{M}^{2,1/2}_1$. Thus, for such f,

$$\lambda(f) \le \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \|f\|_{\dot{M}^{2,1/2}_\infty} = 2^{k/2} \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \|f\|_{L^2(R_k)}.$$

Since we know the dual of $L^2(R_k)$, we can conclude that there exists g_k with

$$\|g_k\|_{L^2(R_k)} \le 2^{k/2} \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \tag{9.29}$$

so that

$$\lambda(f) = \int_{R_k} fg\,dx \tag{9.30}$$

for $f \in L^2(R_k)$. We set $g = \sum_{k=-\infty}^{\infty} g_k$. Note that there can be no question about the meaning of the infinite sum since for each x at most one summand is not zero. The estimate (9.29) implies $\|g\|_{\dot{M}^{2,-1/2}_\infty} \le \|\lambda\|_{\dot{M}^{2,1/2}_1{}'}$. If f is supported in $\cup_{k=-N}^{N} R_k$, then summing (9.30) implies that

$$\lambda(f) = \int fg\,dx.$$

Finally, such f are dense in $\dot{M}^{2,1/2}_1$, so we conclude $\lambda(f) = \int fg\,dx$ for all f. ∎

We have defined the adjoint of an operator on a Hilbert space earlier. Here, we need a slightly more general notion. If $T : X \to \mathcal{H}$ is a continuous linear map from a normed vector space into a Hilbert space, then $x \to \langle Tx, y\rangle$ is a continuous linear functional of X. Thus, there exists $y* \in X'$ so that $y^*(x) = \langle Tx, y\rangle$. One can show that the map $y \to y^* = T^*y$ is linear and continuous. The map $T^* : \mathcal{H} \to X'$ is the *adjoint* of the map T. There adjoint discussed here is closely related to the transpose of a map introduced when we discussed distributions. For our purposes, the key distinction is that the transpose satisfies $(Tf, g) = (f, T^t g)$ for a bilinear pairing, while the adjoint is satisfies $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for a sesquilinear pairing (this means linear in first variable and conjugate linear in the second variable). The map $T \to T^t$ will be linear, while the map $T \to T^*$ is conjugate linear.

The following lemma is a simple case of what is known to harmonic analysts as the Peter Tomás trick. It was used to prove a restriction theorem for the Fourier transform in [18].

Lemma 9.31 *Let $T : X \to \mathcal{H}$ be a map from a normed vector space X into a Hilbert space $\mathcal{H}$. If $T^*T : X \to X'$, and*

$$\|T^*Tf\|_{X'} \leq A^2\|f\|_X$$

then

$$\|Tf\|_{\mathcal{H}} \leq A\|f\|_X.$$

Proof. We have

$$T^*Tf(f) = \langle Tf, Tf\rangle = \|Tf\|^2_{\mathcal{H}}$$

and since $|T^*Tf(f)| \leq \|T^*Tf\|_{X'}\|f\|_X \leq A^2\|f\|_X$, the lemma follows. ∎

Proof of Theorem 9.26. We consider f in $\dot{M}^{2,1/2}_1$ and let T denote the map $f \to \hat{f}$ as a map into $L^2(\mu)$. The map T^*T is given by

$$T^*Tf(x) = \int_{\mathbf{R}^n} \hat{f}(\xi)e^{-ix\cdot\xi}\, d\mu(\xi).$$

Using the Ahlfors condition on the measure μ one may repeat word for word our proof of Theorem 9.16 to conclude T^*T maps $\dot{M}^{2,1/2}_1 \to \dot{M}^{2,-1/2}_\infty$. Now the two previous Lemmas give that $T : \dot{M}^{2,1/2}_1 \to L^2(\mu)$. ∎

Exercise 9.32 *Prove a similar result for other co-dimensions–even fractional ones. That is suppose that $\mu(B_r(x)) \leq Cr^{n-\alpha}$ for $0 < \alpha < n$. Then show that*

$$\int_{\mathbf{R}^n} |\hat{f}(\xi)|^2\, d\mu(\xi) \leq C\|f\|_{\dot{M}^{2,\alpha/2}_1}.$$

Chapter 10

MA633, the Cliff notes

In this chapter, we introduce a good deal of the machinery of elliptic partial differential equations. This will be needed in the next chapter to introduce the inverse boundary value problem we are going to study.

10.1 Domains in $\mathbf{R}^n$

For $\mathcal{O}$ an open subset of $\mathbf{R}^n$, we let $C^k(\mathcal{O})$ denote the space of functions on $\mathcal{O}$ which have continuous partial derivatives of all orders α with $|\alpha| \leq k$. We let $C^k(\bar{\mathcal{O}})$ be the space of functions for which all derivatives of order up to k extend continuously to the closure $\mathcal{O}$, $\bar{\mathcal{O}}$. Finally, we will let $\mathcal{D}(\mathcal{O})$ to denote the space of functions which are infinitely differentiable and are compactly supported in $\mathcal{O}$.

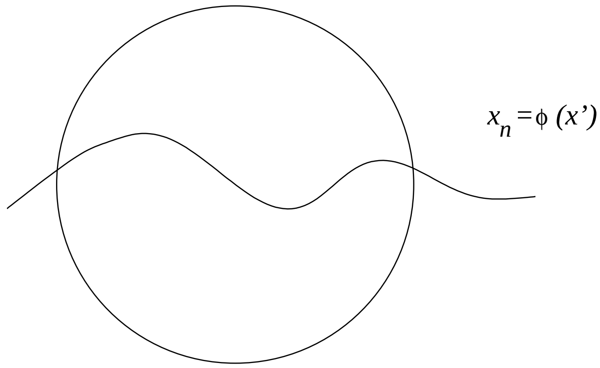
We say that $\Omega \subset \mathbf{R}^n$ is a *domain* if Ω is a bounded connected open set. We say that a domain is of class C^k if for each $x \in \Omega$, there is an $r > 0$, $\phi \in C^k(\mathbf{R}^{n-1})$ and coordinates $(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ (which we assume are a rotation of the standard coordinates) so that

$$\begin{aligned}\partial\Omega \cap B_{2r}(x) &= \{(x', x_n) : x_n = \phi(x')\} \\ \Omega \cap B_{2r}(x) &= \{(x', x_n) : x_n > \phi(x')\}.\end{aligned}$$

Here, $\partial\Omega$ is the boundary of a set. We will need that the map $x \to (x', 2\phi(x') - x_n)$ map $\Omega \cap B_r(x)$ into $\bar{\Omega}^c$. This can always be arranged by decreasing r. We also will assume that $\nabla\phi$ is bounded in all of $\mathbf{R}^{n-1}$.

In these coordinates, we can define *surface measure* $d\sigma$ on the boundary by

$$\int_{B_r(x)\cap\partial\Omega} f(y)\, d\sigma(y) = \int_{B_r(x)\cap\{y: y_n = \phi(y')\}} f(y', \phi(y'))\sqrt{1 + |\nabla\phi(y')|^2}\, dy'.$$



Also, the vector field $\nu(y) = (\nabla\phi(y'), -1)(1 + |\nabla\phi(y')|^2)^{-1/2}$ defines a unit outer normal for $y \in B_r(x) \cap \partial\Omega$.

Since our domain is bounded, the boundary of Ω is a bounded, closed set and hence compact. Thus, we may always find a finite collection of balls, $\{B_r(x_i) : i = 1, \ldots, N\}$ as above which cover $\partial\Omega$.

Many of arguments will proceed more smoothly if we can divide the problem into pieces, choose a convenient coordinate system for each piece and then make our calculations in this coordinate system. To carry out these arguments, we will need partitions of unity. Given a collection of sets, $\{A_\alpha\}$, which are subsets of a topological space X, a partition of unity subordinate to $\{A_\alpha\}$ is a collection of real-valued functions $\{\phi_\alpha\}$ so that $\mathrm{supp}\phi_\alpha \subset A_\alpha$ and so that $\sum_\alpha \phi_\alpha = 1$. Partitions of unity are used to take a problem and divide it into bits that can be more easily solved. For our purposes, the following will be useful.

Lemma 10.1 *If K is a compact subset in $\mathbf{R}^n$ and $\{U_1, \ldots, U_N\}$ is a collection of open sets which cover K, then we can find a collection of functions ϕ_j with each ϕ_j in $\mathcal{D}(U_j)$, $0 \le \phi_j \le 1$ and $\sum_{j=1}^N \phi_j = 1$ on K.*

Proof. By compactness, we can find a finite collection of balls $\{B_k\}_{k=1}^M$ so that each $\bar{B}_k$ lies in some U_j and the balls cover K. If we let $\mathcal{F} = \cup\bar{B}_k$ be the union of the closures of the balls B_k, then the distance between K and $\mathbf{R}^n \setminus \mathcal{F}$ is positive. Hence, we can find finitely many more balls $\{B_{M+1}, \ldots, B_{M+L}\}$ to our collection which cover ∂F and which are contained in $\mathbf{R}^n \setminus K$. We now let $\tilde{\eta}_k$ be the standard bump translated and rescaled to the ball B_k. Thus if $B_k = B_r(x)$, then $\tilde{\eta}_k(y) = \exp(-1/(r^2 - |y-x|^2))$ in B_k and 0 outside B_k. Finally, we put $\tilde{\eta} = \sum_{k=1}^{M+L} \tilde{\eta}_k$ and then $\eta_k = \tilde{\eta}_k/\tilde{\eta}$. Each η_k, $k = 1, \ldots, M$ is smooth since $\tilde{\eta}$ is strictly positive on $\mathcal{O}$. Then we have $\sum_{k=1}^M \eta_k = 1$ on K and we may group to obtain one ϕ_j for each U_j. ∎

The following important result is the Gauss divergence theorem. Recall that for a $\mathbf{C}^n$ valued function $F = (F_1, \ldots, F_n)$, the *divergence* of F, is defined by

$$\mathrm{div} F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.$$

Theorem 10.2 *(Gauss divergence theorem) Let Ω be a C^1 domain and let $F : \Omega \to \mathbf{C}^n$ be in $C^1(\bar{\Omega})$. We have*

$$\int_{\partial\Omega} F(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega} \mathrm{div} F(x)\, dx.$$

The importance of this result may be gauged by the following observation: the theory of weak solutions of elliptic pde (and much of distribution theory) relies on making this result an axiom.

An important Corollary is the following version of Green's identity. In this Corollary and below, we should visualize the gradient of u, ∇u as a column vector so that the product $A\nabla u$ is makes sense as a matrix product.

Corollary 10.3 *If Ω is a C^1-domain, v is in $C^1(\bar{\Omega})$, u is in $C^2(\bar{\Omega})$ and $A(x)$ is an $n \times n$ matrix with $C^1(\bar{\Omega})$ entries, then*

$$\int_{\partial\Omega} v(x) A(x) \nabla u(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega} A(x)\nabla u(x) \cdot \nabla v(x) + v(x) \mathrm{div} A(x) \nabla u(x)\, dx.$$

Proof. Apply the divergence theorem to $vA\nabla u$. ∎

Next, we define *Sobolev spaces* on open subsets of $\mathbf{R}^n$. Our definition is motivated by the result in Proposition 3.11. For k a positive integer, we say that $u \in L^2_k(\Omega)$ if u has weak or distributional derivatives for all α for $|\alpha| \leq k$ and these derivatives, $\partial^\alpha u / \partial x^\alpha$, lie in $L^2(\Omega)$. This means that for all test functions $\phi \in \mathcal{D}(\Omega)$, we have

$$\int_{\Omega} u \frac{\partial^\alpha}{\partial x^\alpha} \phi(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} \phi \frac{\partial^\alpha u}{\partial x^\alpha}(x)\, dx.$$

The weak derivatives of u are defined as we defined the derivatives of a tempered distribution. The differences are that since we are on a bounded open set, our functions are supported there and in this instance we require that the derivative be a distribution given by a function.

It should be clear how to define the norm in this space. In fact, we have that these spaces are Hilbert spaces with the inner product defined by

$$\langle u, v\rangle = \int_\Omega \sum_{|\alpha|\leq k} \frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^\alpha \bar{v}}{\partial x^\alpha}\, dx. \tag{10.4}$$

We let $\|u\|_{L^2_k(\Omega)}$ be the corresponding norm.

Exercise 10.5 *Show that if Ω is a bounded open set, then $C^k(\bar{\Omega}) \subset L^2_k(\Omega)$.*

Example 10.6 *If u is in the Sobolev space $L^2_k(\mathbf{R}^n)$ defined in Chapter 3 and Ω is an open set, then the restriction of u to Ω, call it ru, is in the Sobolev space $L^2_k(\Omega)$. If Ω has reasonable boundary, (C^1 will do) then the restriction map $r : L^2_k(\mathbf{R}^n) \to L^2_k(\Omega)$ is onto. However, this may fail in general.*

Exercise 10.7 *a) Prove the product rule for weak derivatives. If ϕ is in $C^k(\bar{\Omega})$ and all the derivatives of ϕ, $\partial^\alpha\phi/\partial x^\alpha$ with $|\alpha| \leq k$ are bounded, then we have that*

$$\frac{\partial^\alpha \phi u}{\partial x^\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\frac{\partial^\beta \phi}{\partial x^\beta}\frac{\partial^\gamma u}{\partial x^\gamma}.$$

b)If $\phi \in C^k(\bar{\Omega})$, conclude that the map $u \to \phi u$ takes $L^2_k(\Omega)$ to $L^2_k(\Omega)$ and is bounded.
c) If $\phi \in C^1(\bar{\Omega})$, show that the map $u \to \phi u$ maps $L^2_{1,0}(\Omega) \to L^2_{1,0}(\Omega)$.

Lemma 10.8 *If Ω is a C^1 domain and u is in the Sobolev space $L^2_k(\Omega)$, then we may write $u = \sum_{j=0}^N u_j$ where u_0 has support in a fixed (independent of u) compact subset of Ω and each u_j, $j = 1, \ldots, N$ is supported in a ball $B_r(x)$ as in the definition of C^1 domain.*

Proof. We cover the boundary, $\partial\Omega$ by balls $\{B_1, \ldots, B_N\}$ as in the definition of C^1 domain. Then, $K = \Omega \setminus \cup_{k=1}^N B_k$ is a compact set so that the distance from K to $\mathbf{R}^n \setminus \Omega$ is positive, call this distance δ. Thus, we can find an open set $U_0 = \{x : \mathrm{dist}(x, \partial\Omega) > \delta/2\}$ which contains K and is a positive distance from $\partial\Omega$. We use Lemma 10.1 to make a partition of unity $1 = \sum_{j=0}^N \eta_j$ for the open cover of $\bar{\Omega}$ $\{U_0, B_1 \ldots, B_N\}$ and then we decompose $u = \sum_{j=0}^N \eta_j u$. The product rule of exercise 10.7 allows us to conclude that each term $u_j = \eta_j u$ is in $L^2_k(\Omega)$. ∎

Recall that we proved in Chapter 2 that smooth (Schwartz, actually) functions are dense in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$. One step of the argument involved considering the map

$$\eta * u$$

where η is a Schwartz function with $\int \eta = 1$. This approach may appear to break down u is only defined in an open subset of $\mathbf{R}^n$, rather than all of $\mathbf{R}^n$. However, we can make sense of the convolution in most of Ω if we require that the function $\check{\phi}$ have compact support. Thus, we let $\eta \in \mathcal{D}(\mathbf{R}^n)$ be supported in $B_1(0)$ and have $\int \eta = 1$.

Lemma 10.9 *Suppose u is the Sobolev space $L^p_k(\Omega)$, $1 \leq p < \infty$, for $k = 0, 1, 2, \ldots$. Set $\Omega_\epsilon = \Omega \cap \{x : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. If we set $u_\delta = \eta_\delta * u$, then for $|\alpha| \leq k$, we have*

$$\frac{\partial^\alpha}{\partial x^\alpha} u_\delta = (\frac{\partial^\alpha}{\partial x^\alpha} u)_\delta, \qquad \text{for } x \in \Omega_\epsilon \text{ with } \delta < \epsilon.$$

Hence, for each $\epsilon > 0$, we have

$$\lim_{\delta \to 0^+} \|u - u_\delta\|_{L^p_k(\Omega_\epsilon)} = 0.$$

Proof. We assume that u is defined to be zero outside of Ω. The convolution $u * \eta_\delta(x)$ is smooth in all of $\mathbf{R}^n$ and we may differentiate inside the integral and then express the x derivatives as y derivatives to find

$$u * \eta_\delta(x) = \int_\Omega u(y) \frac{\partial^\alpha}{\partial x^\alpha} \eta_\delta(x-y)\, dy = (-1)^{|\alpha|} \int_\Omega u(y) \frac{\partial^\alpha}{\partial y^\alpha} \eta_\delta(x-y)\, dy.$$

If we have $\delta < \epsilon$ and $x \in \Omega_\epsilon$, then $\eta_\delta(x - \cdot)$ will be in the space of test functions $\mathcal{D}(\Omega)$. Thus, we can apply the definition of weak derivative to conclude

$$(-1)^{|\alpha|} \int_\Omega u(y) \frac{\partial^\alpha}{\partial y^\alpha} \eta_\delta(x-y)\, dy = \int_\Omega (\frac{\partial^\alpha}{\partial y^\alpha} u(y)) \eta_\delta(x-y)\, dy.$$

∎

Lemma 10.10 *If Ω is a C^1-domain and $k = 0, 1, 2, \ldots$, then $C^\infty(\bar{\Omega})$ is dense in $L^2_k(\Omega)$.*

Proof. We may use Lemma 10.8 to reduce to the case when u is zero outside $B_r(x) \cap \Omega$ for some ball centered at a boundary point x and $\partial\Omega$ is given as a graph, $\{(x', x_n) : x_n = \phi(x')\}$ in $B_r(x)$. We may translate u to obtain $u_\epsilon(x) = u(x + \epsilon e_n)$. Since u_ϵ has weak derivatives in a neighborhood of Ω, by the local approximation lemma, Lemma 10.9 we may approximate each u_ϵ by functions which are smooth up to the boundary of Ω. ∎

Lemma 10.11 *If Ω and Ω' are bounded open sets and $F : \Omega \to \Omega'$ is $C^1(\bar{\Omega})$ and $F^{-1} : \Omega' \to \Omega$ is also $C^1(\bar{\Omega}')$, then we have $u \in L^2_1(\Omega')$ if and only if $u \in L^2_1(\Omega')$.*

Proof. The result is true for smooth functions by the chain rule and the change of variables formulas of vector calculus. Note that our hypothesis that F is invertible implies that the Jacobian is bounded above and below. The density result of Lemma 10.9 allows us to extend to the Sobolev space. ∎

Lemma 10.12 *If Ω is a C^1-domain, then there exists an extension operator $E : L^2_k(\Omega) \to L^2_k(\mathbf{R}^n)$.*

Proof. We sketch a proof when $k = 1$. We will not use the more general result. The general result requires a more substantial proof. See the book of Stein [14], whose result has the remarkable feature that the extension operator is independent of k.

For the case $k = 1$, we may use a partition of unity and to reduce to the case where u is nonzero outside $B_r(x) \cap \Omega$ and that $\partial\Omega$ is the graph $\{(x', x_n) : x_n = \phi(x')\}$ in $B_r(x)$. By the density result, Lemma 10.10, we may assume that u is smooth up to the boundary. Then we can define Eu by

$$Eu(x) = \begin{cases} u(x), & x_n > \phi(x') \\ u(x', 2\phi(x') - x_n), & x_n < \phi(x') \end{cases}$$

If ψ is test function in $\mathbf{R}^n$, then we can apply the divergence theorem in Ω and in $\mathbf{R}^n \setminus \bar{\Omega}$ to obtain that

$$\begin{aligned} \int_\Omega Eu\frac{\partial \psi}{\partial x_j} + \psi \frac{\partial Eu}{\partial x_j}\, dx &= \int_{\partial\Omega} \psi Eu\nu \cdot e_j \, d\sigma \\ \int_{\mathbf{R}^n\setminus\bar{\Omega}} Eu\frac{\partial \psi}{\partial x_j} + \psi \frac{\partial Eu}{\partial x_j}\, dx &= -\int_{\partial\Omega} \psi Eu\nu \cdot e_j \, d\sigma \end{aligned}$$

In the above expressions, the difference in sign is due to the fact that the normal which points out of Ω is the negative of the normal which points out of $\mathbf{R}^n \setminus \bar{\Omega}$.

Adding these two expressions, we see that Eu has weak derivatives in $\mathbf{R}^n$. These weak derivatives are given by the (ordinary) derivative $\partial Eu/\partial x_j$, which is defined except on $\partial\Omega$. In general, Eu will not have an ordinary derivative on $\partial\Omega$. Using Lemma 10.11, one can see that this extension operator is bounded. The full extension operator is obtained by taking a function u, writing $u = \sum_{j=0}^N \eta_j u$ as in Lemma 10.8 where the support of $\eta_)$ does not meet the boundary. For each η_j which meets the boundary, we apply the local extension operator constructed above and then sum to obtain $Eu = \eta_0 u + \sum_{j=1}^N E(\eta_j u)$. Once we have defined the extension operator on smooth functions in L^2_1, then we can use the density result of Lemma 10.10 to define the extension operator on the full space. ∎

Next, we define an important subspace of $L^2_1(\Omega)$, $L^2_{1,0}(\Omega)$. This space is the closure of $\mathcal{D}(\Omega)$ in the norm of $L^2_1(\Omega)$. The functions in $L^2_{1,0}(\Omega)$ will be defined to be the Sobolev functions which vanish on the boundary. Since a function u in the Sobolev space is initially defined a.e., it should not be clear that we can define the restriction of u to a lower dimensional subset. However, we saw in Chapter 9 that this is possible. We shall present a second proof below. The space $L^2_{1,0}(\Omega)$ will be defined as the space of functions which have zero boundary values.

Remark: Some of you may be familiar with the spaces $L^2_1(\Omega)$ as $H^1(\Omega)$ and $L^2_{1,0}(\Omega)$ as $H^1_0(\Omega)$.

We define the boundary values of a function in $L^2_1(\Omega)$ in the following way. We say that $u = v$ on $\partial\Omega$ if $u - v \in L^2_{1,0}(\Omega)$. Next, we define a space $L^2_{1/2}(\partial\Omega)$ to be the equivalence classes $[u] = u + L^2_{1,0}(\partial\Omega) = \{v : v - u \in L^2_{1,0}(\Omega)\}$. Of course, we need a norm to do analysis. The norm is given by

$$\|u\|_{L^2_{1/2}(\partial\Omega)} = \inf\{\|v\|_{L^2_1(\Omega)} : u - v \in L^2_{1,0}(\Omega)\}. \qquad (10.13)$$

It is easy to see that this is a norm and the resulting space is a Banach space. It is less clear $L^2_{1/2}(\partial\Omega)$ is a Hilbert space. However, if the reader will recall the proof of the projection theorem in Hilbert space one may see that the space on the boundary, $L^2_{1/2}(\partial\Omega)$, can be identified with the orthogonal complement of $L^2_{1,0}(\Omega)$ in $L^2_1(\Omega)$ and thus inherits an inner product from $L^2_1(\Omega)$.

This way of defining functions on the boundary should be unsatisfyingly abstract to the analysts in the audience. The following result gives a concrete realization of the space.

Proposition 10.14 *Let Ω be a C^1-domain. The map*

$$r : C^1(\bar{\Omega}) \to L^2(\partial\Omega)$$

which takes ϕ to the restriction of ϕ to the boundary, $r\phi$ satisfies

$$\|ru\|_{L^2(\partial\Omega)} \leq C\|u\|_{L^2_1(\Omega)}$$

and as a consequence extends continuously to $L^2_1(\Omega)$. Since $r(L^2_{1,0}(\Omega)) = 0$, the map r is well-defined on equivalence classes in $L^2_{1/2}(\partial\Omega)$ and gives a continuous injection $r : L^2_{1/2}(\partial\Omega) \to L^2(\partial\Omega)$.

Exercise 10.15 *Prove the above proposition.*

Exercise 10.16 *If Ω is a C^1 domain, let H be a space of functions f on $\partial\Omega$ defined as follows. We say that $f \in H$ if for each ball $B_r(x)$ as in the definition of C^1 domains and each $\eta \in \mathcal{D}(B_r(x))$, we have $(\eta f)(y', \phi(y'))$ is in the space $L^2_{1/2}(\mathbf{R}^{n-1})$ defined in Chapter 3. In the above, ϕ is the function whose graph describes the boundary of Ω near x. A norm in the space H may be defined by fixing a covering of the boundary by balls as in the definition of C^1-domains, and then a partition of unity subordinate to this collection of balls, $\sum \eta_k$ and finally taking the sum*

$$\sum_k \|\eta_k f_{L^2_{1/2}(\mathbf{R}^{n-1})}$$

.

Show that $H = L^2_{1/2}(\partial\Omega)$.

Hint: We do not have the tools to solve this problem. Thus this exercise is an excuse to indicate the connection without providing proofs.

Lemma 10.17 *If Ω is a C^1 domain and $u \in C^1(\bar{\Omega})$, then there is a constant C so that*

$$\int_{\partial\Omega} |u(x)|^2\, d\sigma(x) \le C \int_\Omega |u(x)|^2 + |\nabla u(x)|^2\, dx.$$

Proof. According to the definition of a C^1-domain, we can find a finite collection of balls $\{B_j : j = 1, \dots, N\}$ and in each of these balls, a unit vector, α_j, which satisfies $\alpha_j \cdot \nu \ge \delta > 0$ for some constant δ. To do this, choose α_j to be $-e_n$ in the coordinate system which is used to describe the boundary near B_j. The lower bound will be $\min_j (1 + \|\nabla\phi_j\|^2_\infty)^{-1/2}$ where ϕ_j is the function which defines the boundary near B_j. Using a partition of unity $\sum_j \phi_j$ subordinate to the family of balls B_j which is 1 on $\partial\Omega$, we construct a vector field

$$\alpha(x) = \sum_{j=1}^N \phi_j(x)\alpha_j.$$

We have $\alpha(x) \cdot \nu(x) \ge \delta$ since each α_j satisfies this condition and each ϕ_j takes values in $[0, 1]$. Thus, the divergence theorem gives

$$\begin{aligned} \delta \int_{\partial\Omega} |u(x)|^2\, d\sigma(x) &\le \int_{\partial\Omega} |u(x)|^2 \alpha(x) \cdot \nu(x)\, dx \\ &= \int_\Omega |u|^2 (\mathrm{div}\alpha) + 2\,\mathrm{Re}(u(x)\alpha \cdot \nabla \bar{u}(x))\, dx. \end{aligned}$$

Applying the Cauchy-Schwarz inequality proves the inequality of the Lemma. The constant depends on Ω through the vector field α and its derivatives. ∎

Proof of Proposition 10.14. The proposition follows from the lemma. That the map r can be extended from nice functions to all of $L^2_1(\Omega)$ depends on Lemma 10.10 which asserts that nice functions are dense in $L^2_1(\Omega)$. ∎

Exercise 10.18 *Suppose that Ω is a C^1 domain. Show that if $\phi \in C^1(\bar{\Omega})$ and $\phi(x) = 0$ on $\partial\Omega$, then $\phi(x)$ is in the Sobolev space $L^2_{1,0}(\Omega)$.*

Finally, we extend the definition of one of the Sobolev spaces of negative order to domains. We define $L^2_{-1}(\Omega)$ to be the dual of the space $L^2_{1,0}(\Omega)$. As in the case of $\mathbf{R}^n$, the following simple lemma gives examples of elements in this space.

Proposition 10.19 *Assume Ω is an open set of finite measure, and g and $f_1, \ldots, f_n$ are functions in $L^2(\Omega)$. Then*

$$\phi \to \lambda(\phi) = \int_\Omega g(x)\phi(x) + \sum_{j=1}^n f_j(x)\frac{\partial\phi(x)}{\partial x_j}\,dx$$

is in $L^2_{-1}(\Omega)$.

Proof. According to the Cauchy-Schwarz inequality, we have

$$\lambda(\phi) \leq \left(\int_\Omega |u(x)|^2 + |\nabla u(x)|^2\,dx\right)^{1/2}\left(\int_\Omega |g(x)|^2 + |\sum_{j=1}^n f_j(x)^2|\,dx\right)^{1/2}.$$

∎

10.2 The weak Dirichlet problem

In this section, we introduce elliptic operators. We let $A(x)$ be function defined on an open set Ω and we assume that this function takes values in $n \times n$-matrices with real entries. We assume that each entry is Lebesgue measurable and that A satisfies the symmetry condition

$$A^t = A \tag{10.20}$$

and ellipticity condition, for some $\lambda > 0$,

$$\lambda|\xi|^2 \leq A(x)\xi\cdot\xi \leq \lambda^{-1}|\xi|^2, \qquad \xi \in \mathbf{R}^n,\ x \in \Omega. \tag{10.21}$$

We say that u is a local weak solution of the equation $\mathrm{div} A(x)\nabla u = f$ for $f \in L^2_{-1}(\Omega)$ if u is in $L^2_{1,loc}(\Omega)$ and for all test functions $\phi \in \mathcal{D}(\Omega)$, we have

$$-\int A(x)\nabla u(x) \cdot \nabla\phi(x)\, dx = f(\phi).$$

Since the derivatives of u are locally in L^2, we can extend to test functions ϕ which are in $L^2_{1,0}(\Omega)$ and which (have a representative) which vanishes outside a compact subset of Ω. However, let us resist the urge to introduce yet another space.

Statement of the Dirichlet problem. The weak formulation of the Dirichlet problem is the following. Let $g \in L^2_1(\Omega)$ and $f \in L^2_{-1}(\Omega)$, then we say that u is a solution of the Dirichlet problem if the following two conditions hold:

$$u \in L^2_1(\Omega) \tag{10.22}$$

$$u - g \in L^2_{1,0}(\Omega) \tag{10.23}$$

$$-\int_\Omega A(x)\nabla u(x)\nabla\phi(x)\, dx = f(\phi) \qquad \phi \in L^2_{1,0}(\Omega). \tag{10.24}$$

Note that both sides of the equation (10.24) are continuous in ϕ in the topology of $L^2_{1,0}(\Omega)$. Thus, we only need to require that this hold for ϕ in a dense subset of $L^2_{1,0}(\Omega)$.

A more traditional way of writing the Dirichlet problem is, given g and f find u which satisfies

$$\begin{cases} \mathrm{div} A\nabla u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Our condition (10.24) is a restatement of the equation, $\mathrm{div} A\nabla u = f$. The condition (10.23) is a restatement of the boundary condition $u = f$. Finally, the condition (10.22) is needed to show that the solution is unique.

Theorem 10.25 *If Ω is an open set of finite measure and $g \in L^2_1(\Omega)$ and $f \in L^2_{-1}(\Omega)$, then there is exactly one weak solution to the Dirichlet problem, (10.22-10.24). There is a constant $C(\lambda, n, \Omega)$ so that the solution u satisfies*

$$\|u\|_{L^2_{1,0}(\Omega)} \le C(\|g\|_{L^2_1(\Omega)} + \|f\|_{L^2_{-1}(\Omega)}).$$

Proof. *Existence.* If $u \in L^2_{1,0}(\Omega)$ and $n \ge 3$ then Hölder's inequality and then the Sobolev inequality of Theorem 8.20 imply

$$\int_\Omega |u(x)|^2\, dx \le \left(\int_\Omega |u(x)|^{\frac{2n}{n-2}}\, dx\right)^{1-\frac{2}{n}} m(\Omega)^{2/n} \le Cm(\Omega)^{2/n} \int_\Omega |\nabla u(x)|^2\, dx.$$

If $n = 2$, the same result holds, though we need to be a bit more persistent and use Hölder's inequality, the Soboleve inequality and Hölder again to obtain:

$$\begin{aligned}\int_\Omega |u(x)|^2\,dx \le \left(\int_\Omega |u(x)|^4\,dx\right)^{1/2} m(\Omega)^{1/2} &\le \left(\int_\Omega |\nabla u(x)|^{4/3}\,dx\right)^{3/2} m(\Omega)^{1/2}\\ &\le \int_\Omega |\nabla u(x)|^2\,dx\ m(\Omega).\end{aligned}$$

Note that in each case, the application of the Sobolev inequality on $\mathbf{R}^n$ is allowed because $L^2_{1,0}(\Omega)$ may be viewed as a subspace of $L^2_1(\Omega)$ by extending functions on Ω to be zero outside Ω. Thus we have

$$\|u\|_{L^2(\Omega)} \le Cm(\Omega)^{1/n}\|\nabla u\|_{L^2(\Omega)}. \tag{10.26}$$

Next, we observe that the ellipticity condition (10.21) implies that

$$\lambda \int_\Omega |\nabla u(x)|^2 \le \int_\Omega A(x)\nabla u(x)\nabla \bar u(x)\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2\,dx. \tag{10.27}$$

We claim the expression

$$\int_\Omega A(x)\nabla u(x)\nabla \bar v(x)\,dx \tag{10.28}$$

provides an inner product on $L^2_{1,0}(\Omega)$ which induces the same topology as the standard inner product on $L^2_{1,0}(\Omega) \subset L^2_1(\Omega)$ defined in (10.4). To see that the topologies are the same, it suffices to establish the inequalities

$$\int_\Omega |\nabla u(x)|^2 + |u(x)|^2\,dx \le \lambda^{-1}(1 + Cm(\Omega)^{2/n})\int_\Omega A(x)\nabla u(x)\nabla \bar u(x)\,dx$$

and that

$$\int_\Omega A(x)\nabla u(x)\nabla \bar u(x)\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2 + |u(x)|^2\,dx.$$

These both follow from the estimates (10.26) and (10.27). As a consequence, standard Hilbert space theory tells us that any continuous linear functional on $L^2_{1,0}(\Omega)$ can be represented using the inner product defined in (10.28). We apply this to the functional

$$\phi \to -\int_\Omega A\nabla g\nabla\phi\,dx - f(\phi)$$

and conclude that there exists $v \in L^2_{1,0}(\Omega)$ so that

$$\int A(x)\nabla v(x)\nabla\phi(x)\,dx = -\int_\Omega A(x)\nabla g(x)\nabla\phi(x)\,dx - f(\phi), \qquad \phi \in L^2_{1,0}(\Omega). \tag{10.29}$$

Rearranging this expression, we can see that $u = g + v$ is a weak solution to the Dirichlet problem.

Uniqueness. If we have two solutions of the Dirichlet problem u_1 and u_2, then their difference $w = u_1 - u_2$ is a weak solution of the Dirichlet problem with $f = g = 0$. In particular, w is in $L^2_{1,0}(\Omega)$ and we can use $\bar{w}$ as a test function and conclude that

$$\int A(x)\nabla w(x) \cdot \nabla \bar{w}(x)\, dx = 0.$$

Thanks to the inequalities (10.26) and (10.27) we conclude that

$$\int_\Omega |w(x)|^2\, dx = 0.$$

Hence, $u_1 = u_2$.

Stability. Finally, we establish the estimate for the solution. We replace the test function ϕ in (10.29) by $\bar{v}$. Using the Cauchy-Schwarz inequality gives

$$\int_\Omega A\nabla v \cdot \nabla \bar{v}\, dx \leq \lambda^{-1}\|v\|_{L^2_{1,0}(\Omega)}\|\nabla g\|_{L^2_1(\Omega)} + \|f\|_{L^2_{-1}(\Omega)}\|v\|_{L^2_{1,0}(\Omega)}.$$

If we use that the left-hand side of this inequality is equivalent with the norm in $L^2_{1,0}(\Omega)$, cancel the common factor, we obtain that

$$\|v\|_{L^2_{1,0}(\Omega)} \leq C\|g\|_{L^2_{1,0}(\Omega)} + \|f\|_{L^2_{-1}(\Omega)}.$$

We have $u = v + g$ and the triangle inequality gives

$$\|u\|_{L^2_1(\Omega)} \leq \|g\|_{L^2_1(\Omega)} + \|v\|_{L^2_{1,0}(\Omega)}$$

so combining the last two inequalities implies the estimate of the theorem. ∎

Exercise 10.30 *(Dirichlet's principle.) Let $g \in L^2_1(\Omega)$ and suppose that $f = 0$ in the weak formulation of the Dirichlet problem.*

a) Show that the expression

$$I(u) = \int_\Omega A(x)\nabla u(x) \cdot \nabla \bar{u}(x)\, dx$$

attains a minimum value on the set $g + L^2_{1,0}(\Omega) = \{g + v : v \in L^2_{1,0}(\Omega)\}$. Hint: Use the foil method. This is a general fact in Hilbert space.

b) If u is a minimizer for I, then u is a weak solution of the Dirichlet problem, $\mathrm{div} A\nabla u = 0$ *and $u = g$ on the boundary.*

c) Can you extend this approach to solve the general Dirichlet problem $\mathrm{div} A\nabla u = f$ *in Ω and $u = g$ on the boundary?*

Chapter 11

Inverse Problems: Boundary identifiability

11.1 The Dirichlet to Neumann map

In this section, we introduce the Dirichlet to Neumann map. Recall the space $L^2_{1/2}(\partial\Omega)$ which was introduced in Chapter 10. We let Ω be a bounded open set, A a matrix which satisfies the ellipticity condition and given f in $L^2_{1/2}(\partial\Omega)$, we let $u = u_f$ be the weak solution of the Dirichlet problem

$$\begin{cases} \mathrm{div} A\nabla u = 0, & \text{on } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

Given $u \in L^2_1(\Omega)$ we can define a continuous linear functional on $L^2_1(\Omega)$ by

$$\phi \to \int_\Omega A(x)\nabla u(x)\nabla\phi(x)\,dx.$$

If we recall the Green's identity (10.3), we see that if u and A are smooth, then

$$\int_{\partial\Omega} A(x)\nabla u(x)\cdot\nu(x)\phi(x)\,d\sigma(x) = \int_\Omega A(x)\nabla u(x)\nabla\phi(x) + \phi(x)\mathrm{div}A(x)\nabla u(x)\,dx.$$

Thus, if u solves the equation $\mathrm{div}A\nabla u = 0$, then it reasonable to define $A\nabla u\cdot\nu$ as a linear functional on $L^2_{1/2}(\partial\Omega)$ by

$$A\nabla u\cdot\nu(\phi) = \int_\Omega A(x)\nabla u(x)\cdot\nabla\phi(x)\,dx. \tag{11.2}$$

We will show that this map is defined on $L^2_{1/2}(\partial\Omega)$ The expression $A\nabla u \cdot \nu$ is called the *conormal derivative* of u at the boundary. Note that it is something of a miracle that we can make sense of this expression at the boundary. To appreciate that this is surprising, observe that we are not asserting that the full gradient of u is defined at the boundary, only the particular component $A\nabla u \cdot \nu$. The gradient of u may only be in $L^2(\Omega)$ and thus there is no reason to expect that any expression involving ∇u could make sense on the boundary, a set of measure zero.

A potential problem is that this definition may depend on the representative of ϕ which is used to define the right-hand side of (11.2). Fortunately, this is not the case.

Lemma 11.3 *If $u \in L^2_1(\Omega)$ and u is a weak solution of* $\mathrm{div} A\nabla u = 0$*, then the value of $A\nabla u \cdot \nu(\phi)$ is independent of the extension of ϕ from $\partial\Omega$ to Ω.*

The linear functional defined in (11.2) is a continuous linear functional on $L^2_{1/2}(\partial\Omega)$.

Proof. To establish that $A\nabla u \cdot \nu$ is well defined, we will use that u is a solution of $\mathrm{div} A\nabla u = 0$. We choose ϕ_1, ϕ_2 in $L^2_1(\Omega)$ and suppose $\phi_1 - \phi_2 \in L^2_{1,0}(\Omega)$. According to the definition of weak solution,

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla(\phi_1(x) - \phi_2(x))\, dx = 0.$$

To establish the continuity, we need to choose a representative of ϕ which is close to the infinum in the definition of the $L^2_{1/2}$-norm (see (10.13)). Thus we need $\|\phi\|_{L^2_1(\Omega)} \leq 2\|r\phi\|_{L^2_{1/2}(\partial\Omega)}$. Here, $r\phi$ denotes the restriction of ϕ to the boundary. With this choice of ϕ and Cauchy-Schwarz we have

$$|A\nabla u \cdot \nu(\phi)| \leq C\|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}.$$

This inequality implies the continuity. ∎

We will define $L^2_{-1/2}(\partial\Omega)$ as the dual of the space $L^2_{1/2}(\partial\Omega)$. Now, we are ready to define the *Dirichlet to Neumann map.* This is a map

$$\Lambda_A : L^2_{1/2}(\Omega) \to L^2_{-1/2}(\partial\Omega)$$

defined by

$$\Lambda_A f = A\nabla u \cdot \nu$$

where u is the solution of the Dirichlet problem with boundary data f.

The traditional goal in pde is to consider the direct problem. For example, given the coefficient matrix A, show that we can solve the Dirichlet problem. If we were more persistent, we could establish additional properties of the solution. For example, we could show that the map $A \to \Lambda_A$ is continuous, on the set of strictly positive definite matrix valued functions on Ω.

However, that would be the easy way out. The more interesting and difficult problem is the inverse problem. Given the map Λ_A, can we recover the coefficient matrix, A. That is given some information about the solutions to a pde, can we recover the equation. The answer to the problem, as stated, is no, of course not.

Exercise 11.4 *Let Ω be a bounded domain and let $F : \Omega \to \Omega$ be a $C^1(\bar{\Omega})$ diffeomorphism that fixes a neighborhood of the boundary. Show that if A gives an elliptic operator* $\mathrm{div} A\nabla$ *on Ω, then there is an operator* $\mathrm{div} B\nabla$ *so that*

$$\mathrm{div} A\nabla u = 0 \iff \mathrm{div} B\nabla u \circ F = 0.$$

As a consequence, it is clear that the maps $\Lambda_A = \Lambda_B$. Hint: See Lemma 11.10 below for the answer.

Exercise 11.5 *Show that the only obstruction to uniqueness is the change of variables described in the previous problem.*

Remark: This has been solved in two dimensions, by John Sylvester [16]. In three dimensions and above, this problem is open.

Exercise 11.6 *Prove that the map $A \to \Lambda_A$ is continuous on the set of strictly positive definite and bounded matrix-valued functions. That is show that*

$$\|\Lambda_A - \Lambda_B\|_{\mathcal{L}(L^2_{1/2}, L^2_{-1/2})} \leq C_\lambda \|A - B\|_\infty.$$

Here, $\|\cdot\|_{\mathcal{L}(L^2_{1/2}, L^2_{-1/2})}$ denotes the norm on linear operators from $L^2_{1/2}$ to $L^2_{-1/2}$.

a) As a first step, show that if we let u_A and u_B satisfy $\mathrm{div} A\nabla u_A = \mathrm{div} B\nabla u_B = 0$ *in an open set Ω and $u_A = u_B = f$ on $\partial\Omega$, then we have*

$$\int_\Omega |\nabla u_A - \nabla u_B|^2 \, dx \leq C\|f\|_{L^2_{1/2}(\partial\Omega)} \|A - B\|^2_\infty.$$

Hint: We have $\mathrm{div} B\nabla u_A = \mathrm{div}(B - A)\nabla u_A$ *since u_A is a solution.*

b) Conclude the estimate above on the Dirichlet to Neumann maps.

However, there is a restricted version of the inverse problem which can be solved. In the remainder of these notes, we will concentrate on elliptic operators when the matrix A is of the form $A(x) = \gamma(x)I$ where I is the $n \times n$ identity matrix and $\gamma(x)$ is a scalar function which satisfies

$$\lambda \leq \gamma(x) \leq \lambda^{-1} \tag{11.7}$$

for some constant $\lambda > 0$. We change notation a bit and let Λ_γ be the Dirichlet to Neumann map for the operator $\mathrm{div}\gamma\nabla$. Then the *inverse conductivity problem* can be formulated as the following question:

Is the map $\gamma \to \Lambda_\gamma$ injective?

We will answer this question with a yes, if the dimension $n \geq 3$ and we have some reasonable smoothness assumptions on the domain and γ. This is a theorem of J. Sylvester and G. Uhlmann [17]. Closely related work was done by Henkin and R. Novikov at about the same time [5, 9]. One can also ask for a more or less explicit construcion of the inverse map. A construction is given in in the work of Novikov and the work of A. Nachman [7] for three dimensions and [8] in two dimensions. This last paper also gives the first proof of injectivity in two dimensions. My favorite contribution to this story is in [2]. But this is not the the place for a complete history.

We take a moment to explain the appearance of the word conductivity in the above. For this discussion, we will assume that function u and γ are smooth. The problem we are considering is a mathematical model for the problem of determining the conductivity γ by making measurements of current and voltage at the boundary. To try and explain this, we suppose that u represents the voltage potential in Ω and then ∇u is the electric field. The electric field is what makes electrons flow and thus we assume that the current is proportional to the electric field, $J = \gamma\nabla u$ where the conductivity γ is a constant of proportionality. Since we assume that charge is conserved, for each subregion $B \subset \Omega$, the net flow of electrons or current through B must be zero. Thus,

$$0 = \int_{\partial B} \gamma\nabla u(x) \cdot \nu(x)\, d\sigma(x).$$

The divergence theorem gives that

$$0 = \int_{\partial B} \gamma(x)\nabla u(x) \cdot \nu(x)\, d\sigma(x) = \int_B \mathrm{div}\gamma(x)\nabla u(x)\, dx.$$

Finally, since the integral on the right vanishes, say, for each ball $B \subset \Omega$, we can conclude that $\mathrm{div}\gamma\nabla u = 0$ in Ω.

11.2 Identifiability

Our solution of the inverse conductivity problem has two steps. The first is to show that the Dirichlet to Neumann map determines γ on the boundary. The second step is to use the knowledge of γ on the boundary to relate the inverse conductivity problem to a problem in all of $\mathbf{R}^n$ which turns out to be a type of scattering problem. We will use the results of Chapter 9 to study this problem in $\mathbf{R}^n$.

Theorem 11.8 *Suppose that $\partial\Omega$ is C^1. If γ is in $C^0(\bar\Omega)$ and satisfies (11.7), then for each $x \in \partial\Omega$, there exists a sequence of functions u_N so that*

$$\gamma(x) = \lim_{N\to\infty} \Lambda_\gamma u_N(\bar u_N).$$

Theorem 11.9 *Suppose Ω and γ are as in the previous theorem and also $\partial\Omega$ is C^2 and γ is in $C^1(\bar\Omega)$. If e is a constant vector and u_N as in the previous theorem, then we have*

$$\nabla\gamma(x)\cdot e = \lim_{N\to\infty}\int_{\partial\Omega}\left(\gamma(x)|\nabla u_N(x)|^2 e\cdot\nu(x) - 2\,\mathrm{Re}\,\gamma(x)\frac{\partial u_N}{\partial\nu}(x)e\cdot\nabla\bar u(x)\right)d\sigma.$$

The construction of the solutions u_N proceeds in two steps. The first step is to write down an explicit function which is an approximate solution and show that the conclusion of our Theorem holds for this function. The second step is to show that we really do have an approximate solution. This is not deep, but requires a certain amount of persistence. I say that the result is not deep because it relies only on estimates which are a byproduct of our existence theory in Theorem 10.25.

In the construction of the solution, it will be convenient to change coordinates so that in the new coordinates, the boundary is flat. The following lemma keeps track of how the operator $\mathrm{div}\gamma\nabla$ transforms under a change of variables.

Lemma 11.10 *Let A be an elliptic matrix and $F:\Omega'\to\Omega$ be a $C^1(\bar\Omega)$-diffeomorphism, $F:\Omega'\to\Omega$. Then have that* $\mathrm{div}A\nabla u = 0$ *if and only if* $\mathrm{div}B\nabla u\circ F$ *where*

$$B(y) = |\det DF(y)|DF^{-1}(F(y))^t A(F(y))DF^{-1}(F(y)).$$

Proof. The proof of this lemma indicates one of the advantages of the weak formulation of the equation. Since the weak formulation only involves one derivative, we only need to use the chain rule once.

We use the chain rule to compute

$$\nabla u(x) = \nabla(u(F(F^{-1}(x))) = DF^{-1}(x)\nabla(u\circ F)(F^{-1}(x)).$$

This is valid for Sobolev functions also by approximation (see Lemma 10.11). We insert this expression for the gradient and make a change of variables $x = F(y)$ to obtain

$$\begin{aligned}
&\int_\Omega A(x)\nabla u(x)\cdot\nabla\phi(x)\,dx \\
&= \int_{\Omega'} A(F(y))DF^{-1}(F(y))\nabla(u\circ F(y))\cdot DF^{-1}(F(y))\nabla(\phi\circ F(y))|\det DF(y)|\,dy \\
&= \int_{\Omega'} |\det DF(y)|DF^{-1}(F(y))^t A(F(y))DF^{-1}(F(y))\nabla(u\circ F(y))\cdot\nabla(\phi\circ F(y))\,dy.
\end{aligned}$$

This last integral is the weak formulation of the equation $\mathrm{div}B\nabla u = 0$ with the test function $\phi\circ F$. To finish the proof, one must convince oneself that the map $\phi\to\phi\circ F$ is an isomorphism[1] from $L^2_{1,0}(\Omega)$ to $L^2_{1,0}(\Omega')$. ∎

Exercise 11.11 *Figure out how to index the matrix DF^{-1} so that in the application of the chain rule in the previous Lemma, the product $DF^{-1}\nabla(u\circ F)$ is matrix multiplication. Assume that the gradient is a column vector.*

Solution The chain rule reads

$$\frac{\partial}{\partial x_i}u\circ G = \frac{\partial G_j}{\partial x_i}\frac{\partial u}{\partial x_j}\circ G.$$

Thus, we want

$$(DG)_{ij} = \frac{\partial G_j}{\partial x_i}.$$

In the rest of this chapter, we fix a point x on the boundary and choose coordinates so that x is the origin. Thus, we suppose that we are trying to find the value of γ and $\nabla\gamma$ at 0. We assume that $\partial\Omega$ is C^1 near 0 and thus we have a ball $B_r(0)$ so that $B_{2r}(0)\cap\partial\Omega = \{(x',x_n) : x_n = \phi(x')\}\cap B_{2r}(0)$. We let $x = F(y',y_n) = (y',\phi(y')+y_n)$. Note that we assume that the function ϕ is defined in all of $\mathbf{R}^{n-1}$ and thus, the map F is invertible on all of $\mathbf{R}^n$. In the coordinates, (y',y_n), the operator $\mathrm{div}\gamma\nabla$ takes the form

$$\mathrm{div}A\nabla u = 0$$

with $A(y) = \gamma(y)B(y)$. (Strictly speaking, this is $\gamma(F(y))$. However, to simplify the notation, we will use $\gamma(z)$ to represent the value of γ at the point corresponding to z in the current coordinate system. This is a fairly common convention. To carry it out

[1] An *isomorphism* for Banach (or Hilbert spaces) is an invertible linear map with continuous inverse. A map which also preserves the norm is called an *isometry.*

precisely would require yet another chapter that we don't have time for...) The matrix B depends on ϕ and, by the above lemma, takes the form

$$B(y) = \begin{pmatrix} 1_{n-1} & -\nabla\phi(y') \\ -\nabla\phi(y')^t & 1 + |\nabla\phi(y')|^2 \end{pmatrix}.$$

Apparently, we are writing the gradient as a column vector. The domain Ω' has 0 on the boundary and near 0, $\partial\Omega'$ lies in the hyperplane $y_n = 0$ and Ω' lies in the regions $y_n > 0$. We introduce a real-valued cutoff function $\eta(y) = f(y')g(y_n)$ which is normalized so that

$$\int_{\mathbf{R}^{n-1}} f(y')^2 \, dy' = 1 \tag{11.12}$$

and so that $g(y_n) = 1$ if $|y_n| < 1$ and $g(y_n) = 0$ if $|y_n| > 2$. Our next step is to set $\eta_N(x) = N^{(n-1)/4}\eta(N^{1/2}y)$. We choose a vector $\alpha \in \mathbf{R}^n$ and which satisfies

$$B(0)\alpha \cdot e_n = 0 \tag{11.13}$$

$$B(0)\alpha \cdot \alpha = B(0)e_n \cdot e_n. \tag{11.14}$$

We define E_N by

$$E_N(y) = N^{-1/2} \exp(-N(y_n + i\alpha \cdot y))$$

and then we put

$$v_N(y) = \eta_N(y)E_N(y). \tag{11.15}$$

The function v_N is our approximate solution. The main facts that we need to prove about v_N are in Lemma 11.16 and Lemma 11.19 below.

Lemma 11.16 *With v_N and Ω' as above,*

$$\lim_{N\to\infty} \|\mathrm{div}\gamma B\nabla v_N\|_{L^2_{-1}(\Omega')} = 0.$$

To visualize why this might be true, observe that E_N is a solution of the equation with constant coefficients $B(0)$. The cutoff function oscillates less rapidly than E_N (consider the relative size of the gradients) and thus it introduces an error that is negligible for N large and allows us to disregard the fact that E_N is not a solution away from the origin.

Our proof will require yet more lemmas. The function v_N is concentrated near the boundary. In the course of making estimates, we will need to consider integrals pairing v_N and its derivatives against functions which are in $L^2_{1,0}(\Omega)$. To make optimal estimates, we will want to exploit the fact that functions in $L^2_{1,0}(\Omega)$ are small near the boundary.

The next estimate, a version of *Hardy's inequality* makes this precise. If we have not already made this definition, then we define

$$\delta(x) = \inf_{y\in\partial\Omega} |x-y|.$$

The function δ gives the distance from x to the boundary of Ω.

Lemma 11.17 *(Hardy's inequality) a) Let f be a C^1 function on the real line and suppose that $f(0)=0$, then for $1<p\leq\infty$,*

$$\int_0^\infty |\frac{f(t)}{t}|^p\,dt \leq p' \int_0^\infty |f'(t)|^p\,dt.$$

b) If f is in $L^2_{1,0}(\Omega)$, then

$$\int_\Omega \left|\frac{f(x)}{\delta(x)}\right|^2\,dx \leq C\int_\Omega |\nabla f(x)|^2 + |f(x)|^2\,dx.$$

Proof. *a)* We prove the one-dimensional result with $p<\infty$ first. We use the fundamental theorem of calculus to write

$$f(t) = -\int_0^t f'(s)\,ds$$

Now, we confuse the issue by rewriting this as

$$\begin{aligned} -t^{\frac{1}{p}-1}f(t) &= \int (t/s)^{-1/p'}\chi_{(1,\infty)}(t/s)s^{1/p}f(s)\,\frac{ds}{s} \\ &= \int K(t/s)s^{1/p}f'(s)\,\frac{ds}{s} \end{aligned} \tag{11.18}$$

where $K(u) = u^{-1/p'}\chi_{(1,\infty)}(u)$. A computation shows that

$$\int_0^\infty K(t/s)\frac{ds}{s} = \int_0^\infty K(t/s)\,\frac{dt}{t} = p'$$

which will be finite if $p>1$. Thus, by exercise 4.5 we have that $g\to\int K(t/s)g(s)\,ds/s$ maps $L^p(ds/s)$ into itself. Using this in (11.18) gives

$$\left(\int_0^\infty \left|\frac{f(t)}{t}\right|^p\,dt\right)^{1/p} \leq p'\left(\int_0^\infty |f'(t)|^p\,dt\right)^{1/p}.$$

Which is what we wanted to prove. The remaining case $p=\infty$ where the L^p norms must be replaced by L^∞ norms is easy and thus omitted.

b) Since $\mathcal{D}(\Omega)$ is dense in $L^2_{1,0}(\Omega)$, it suffices to consider functions in $\mathcal{D}(\Omega)$. By a partition of unity, as in Lemma 10.8 we can further reduce to a function f which is compactly supported $B_r(x)\cap\Omega$, for some ball centered at x on the boundary, or to a function f which is supported at a fixed distance away from the boundary. In the first case have that $\partial\Omega$ is given by the graph $\{(y', y_n) : y_n = \phi(y')\}$ near x. Applying the one-dimensional result in the y_n variable and then integrating in the remaining variables , we may conclude that

$$\int_{\Omega\cap B_r(x)} \frac{|f(y)|^2}{(y_n-\phi(y'))^2}\,dy \le 4\int_{\Omega\cap B_r(x)} |\frac{\partial u}{\partial y_n}(y)|^2\,dy.$$

This is the desired inequality once we convince ourselves that $(y_n-\phi(y'))/\delta(y)$ is bounded above and below in $B_r(x)\cap\Omega$.

The second case where f is supported strictly away from the boundary is an easy consequence of the Sobolev inequality, Theorem 8.20, because $1/\delta(x)$ is bounded above on each compact subset of Ω. ∎

The following Lemma will be useful in obtaining the properties of the approximate solutions and may serve to explain some of the peculiar normalizations in the definition.

Lemma 11.19 *Let v_N, E_N and η_N be as defined in (11.15). Let β be continuous at 0 then*

$$\lim_{N\to\infty} N\int_{\Omega'} \beta(y)|\eta_N(y)|^2 e^{-2Ny_n}\,dy = \beta(0)/2. \tag{11.20}$$

If $k > -1$ and $\tilde\eta \in \mathcal{D}(\mathbf{R}^n)$, then for N sufficiently large there is a constant C so that

$$|\int_{\Omega'} \delta(y)^k \tilde\eta(N^{1/2}y)e^{-2Ny_n}\,dy| \le CN^{\frac{1-n}{2}-1-k}. \tag{11.21}$$

Proof. To prove the first statement, we observe that by the definition and the normalization of the cutoff function, f, in (11.12) we have that

$$\begin{aligned}\int_{\Omega'} \eta_N(y)^2 e^{-2Ny_n}\,dy &= N^{\frac{n-1}{2}}\int_{\{y:y_n>0\}} f(N^{1/2}y')^2 e^{-2Ny_n}\,dy\\ &\quad +N^{\frac{n-1}{2}}\int_{\{y:y_n>0\}} (g(N^{1/2}y_n)^2-1)f(N^{1/2}y')^2 e^{-2Ny_n}\,dy.\end{aligned}$$

The first integral is $1/(2)$ and the second is bounded by a multiple of $(2N)^{-1}e^{-2N^{1/2}}$. The estimate of the second depends on our assumption that $g(t) = 1$ for $t < 1$. Thus, we have that

$$\lim_{N\to\infty} N\int_{\Omega'} \eta_N(y)^2 e^{-2Ny_n}dy = 1/2.$$

Using this to express the $\frac{1}{2}$ as a limit gives

$$\begin{aligned}|\frac{1}{2}\beta(0) - \lim_{N\to\infty} N\int_{\Omega'}\beta(y)\eta_N(y)^2 e^{-2Ny_n}dy| &\leq \lim_{N\to\infty} N\int_{\Omega'}|\beta(0)-\beta(y)| \\ &\qquad \times\eta_N(y)^2 e^{-2Ny_n}dy \\ &\leq \lim_{N\to\infty}\sup_{\{y:|y|<2^{1/2}N^{-1/2}\}}\frac{1}{2}|\beta(0)-\beta(y)| \\ &\qquad \times N\int_{\Omega'}\eta_N(y)^2 e^{-2Ny_n}dy.\end{aligned}$$

Now the continuity of β implies that this last limit is 0.

The inequalities in the second statement follow easily, by observing that for N sufficiently large, we have $\delta(y) = y_n$ on the support of $\tilde\eta(N^{1/2}y)$. If $\mathrm{supp}\tilde\eta \subset B_R(0)$, then we can estimate our integral by

$$\begin{aligned}\left|\int_{\Omega'}\delta(y)^k\tilde\eta(N^{1/2}y)e^{-2Ny_n}\,dy\right| &\leq \|\tilde\eta\|_\infty\int_{\{y':|y'|<N^{-1/2}R\}}\int_0^\infty y_n^k e^{-2Ny_n}\,dy'dy_n \\ &\leq CN^{\frac{1-n}{2}-1-k}.\end{aligned}$$

∎

We can now give the proof of Lemma 11.22.

Lemma 11.22 *With Ω' and v_N as above, suppose β is a bounded function on Ω' which is continuous at 0, then*

$$\lim_{N\to\infty}\int_{\Omega'}\beta(y)B(y)\nabla v_N(y)\cdot\nabla\bar v_N(y)\,dy = \beta(0)B(0)e_n\cdot e_n.$$

Proof. Using the product rule, expanding the square and that η_N is real valued gives

$$\begin{aligned}\int_{\Omega'}\beta(y)B(y)\nabla v_N(y)\nabla\bar v_N(y)\,dy &= N\int\beta(y)(B(y)\alpha\cdot\alpha + B(y)e_n\cdot e_n)\eta_N(y)^2e^{-2Ny_n}\,dy \\ &\quad -2\int\beta(y)(B(y)\nabla\eta_N(y)\cdot e_n)\eta_N(y)e^{-2Ny_n}\,dy \\ &\quad +N^{-1}\int_{\Omega'}\beta(y)B(y)\nabla\eta_N(y)\cdot\nabla\eta_N(y)e^{-2Ny_n}\,dy \\ &= I + II + III.\end{aligned}$$

By (11.20) of our Lemma 11.19, we have that

$$\lim_{N\to\infty} I = \beta(0)(B(0)e_n\cdot e_n). \tag{11.23}$$

where we have used (11.14) to replace $B(0)\alpha \cdot \alpha$ by $B(0)e_n \cdot e_n$. The integral II can be bounded above by

$$II \le 2N^{\frac{n}{2}} \|\beta B\|_\infty \int_{\Omega'} |(\nabla\eta_1)(N^{1/2}y)\eta_1(N^{1/2}y)| e^{-2Ny_n}\, dy \le CN^{-1/2}. \tag{11.24}$$

Here, we are using the second part of Lemma 11.19, (11.21). The observant reader will note that we have taken the norm of the matrix B in the above estimate. The estimate above holds if matrices are normed with the operator norm–though since we do not care about the exact value of the constant, it does not matter so much how matrices are normed.

Finally, the estimate for III also follows from (11.21) in Lemma 11.19 as follows:

$$III \le N^{\frac{n-1}{2}} \|\beta B\|_\infty \int_{\Omega'} |(\nabla\eta_1)(N^{1/2}y)|^2 e^{-2Ny_n}\, dy \le CN^{-1}. \tag{11.25}$$

The conclusion of the Lemma follows from (11.23–11.25). ∎

Now, we can make precise our assertion that v_N is an approximate solution of the equation $\mathrm{div} A\nabla v = 0$.

Lemma 11.26 *With v_N and Ω' as above,*

$$\lim_{N\to\infty} \|\mathrm{div} A\nabla v_N\|_{L^2_{-1}(\Omega')} = 0.$$

Proof. We compute and use that $\mathrm{div} A(0)\nabla E_N = 0$ to obtain

$$\begin{aligned}
\mathrm{div} A(y)\nabla v_N(y) &= \mathrm{div}(A(y) - A(0))\nabla v_N(y) + \mathrm{div} A(0)\nabla v_N(y) \\
&= \mathrm{div}(A(y) - A(0))\nabla v_N(y) \\
&\qquad +2A(0)\nabla\eta_N(y)\nabla E_N(y) + E_N \mathrm{div} A(0)\nabla\eta_N(y) \\
&= I + II + III.
\end{aligned}$$

In the term I, the divergence must be interpreted as a weak derivative. To estimate the norm in $L^2_{-1}(\Omega)$, we must pair each of I through III with a test function ψ. With I, we use the definition of weak derivative and recall that η_N is supported in a small ball to obtain

$$\begin{aligned}
|I(\psi)| &= \left| \int (A(y) - A(0))\nabla v_N(y) \cdot \nabla\psi(y)\, dy \right| \\
&\le \sup_{|y|<2^{3/2}N^{-1/2}} |A(y) - A(0)| \|\nabla v_N\|_{L^2(\Omega)} \|\nabla\psi\|_{L^2(\Omega)}.
\end{aligned}$$

This last expression goes to zero with N because A is continuous at 0 and according to (11.22) the $L^2(\Omega)$ of the gradient of v_N is bounded as $N \to 0$.

To make estimates for II, we multiply and divide by $\delta(y)$, use the Cauchy-Schwarz inequality, the Hardy inequality, Lemma 11.17, and then (11.21)

$$\begin{aligned}
|II(\psi)| &= |\int_{\Omega'} 2A(0)\nabla\eta_N(y)\nabla E_N(y)\psi(y)\,dy| \\
&\leq \left(\int_{\Omega'} \left|\frac{\psi(y)}{\delta(y)}\right|^2 dy\right)^{1/2} \left(N^{\frac{n+3}{2}} \int_{\Omega'} \delta(y)^2 |(\nabla\eta_1(N^{1/2}y)|^2 e^{-2Ny_n}\,dy\right)^{1/2} \\
&\leq CN^{-1/2}\|\psi\|_{L^2_{1,0}(\Omega')}.
\end{aligned}$$

Finally, we make estimates for the third term

$$\begin{aligned}
|III(\psi)| &= \left|\int E_N(y)\mathrm{div}A(0)\nabla\eta_N(y)\,dy\right| \\
&\leq \left(\int_{\Omega'} \left|\frac{\psi(y)}{\delta(y)}\right|^2 dy\right)^{1/2} \left(N^{\frac{n+1}{2}} \int_{\Omega'} \delta(y)^2 |(\mathrm{div}A(0)\nabla\eta_1)(N^{1/2}y)|^2 e^{-2Ny_n}\,dy\right)^{1/2} \\
&\leq C\|\psi\|_{L^2_{1,0}(\Omega')}N^{-1}.
\end{aligned}$$

∎

Now, it is easy to patch up v_N to make it a solution, rather than an approximate solution.

Lemma 11.27 *With Ω' and B as above, we can find a family of solutions, w_N, of $\mathrm{div}A\nabla w_N = 0$ with $w_N - v_N \in L^2_{1,0}(\Omega')$ so that*

$$\lim_{N\to\infty} \int_{\Omega'} \beta(y)B(y)\nabla w_N(y) \cdot \nabla \bar{w}_N(y)\,dy = \beta(0)B(0)e_n \cdot e_n.$$

Proof. According to Theorem 10.25 we can solve the Dirichlet problem

$$\begin{cases} \mathrm{div}A\nabla\tilde{v}_N = -\mathrm{div}A\nabla v_N, & \text{in } \Omega' \\ \tilde{v}_N = v_N, & \text{on } \partial\Omega \end{cases}$$

The solution $\tilde{v}_N$ will satisfy

$$\lim_{N\to\infty} \|\nabla\tilde{v}_N\|_{L^2(\Omega')} \leq \lim_{N\to\infty} C\|\mathrm{div}A\nabla v_n\|_{L^2_{-1}(\Omega')} = 0 \tag{11.28}$$

by the estimates from the existence theorem, Theorem 10.25 and the estimate of Lemma 11.26.

If we set $w_N = v_N + \tilde{v}_N$, then we have a solution with the correct boundary values and by (11.28) and Lemma 11.22

$$\begin{aligned}\lim_{N\to\infty}\int_{\Omega'}\beta(y)B(y)\nabla w_N(y)\cdot\nabla\bar{w}_N(y)\,dy &= \lim_{N\to\infty}\int_{\Omega'}A\nabla v_N(y)\cdot\nabla\bar{v}_N(y)\,dy \\ &= \beta(0)B(0)e_n\cdot e_n.\end{aligned}$$

∎

We will need another result from partial differential equations–this one will not be proven in this course. This Lemma asserts that solutions of elliptic equations are as smooth as one might expect.

Lemma 11.29 *If A is matrix with $C^1(\bar{\Omega})$ entries and Ω is a domain with C^2-boundary, then the solution of the Dirichlet problem,*

$$\begin{cases}\mathrm{div}A\nabla u = 0 & \text{in } \Omega\\ u = f & \text{on } \partial\Omega\end{cases}$$

will satisfy

$$\|u\|_{L^2_2(\Omega)} \le C\|f\|_{L^2_2(\Omega)}.$$

As mentioned above, this will not be proven. To obtain an idea of why it might be true. Let u be a solution as in the theorem. This, we can differentiate and obtain that $v = \partial u/\partial x_j$ satisfies an equation of the form $\mathrm{div}\gamma\nabla v = \mathrm{div}(\partial\gamma/\partial x_j)\nabla u$. The right-hand side is in L^2_{-1} and hence it is reasonable to expect that v satisfies the energy estimates of Theorem 10.25. This argument cannot be right because it does not explain how the boundary data enters into the estimate. To see the full story, take MA633.

Finally, we can give the proofs of our main theorems.

Proof of Theorem 11.8 and Theorem 11.9. We let $F : \Omega' \to \Omega$ be the diffeomorphism used above and let $u_N = w_N \circ F^{-1}/(1 + |\nabla\phi(0)|^2)$. According to the change of variables lemma, u_N will be a solution of the original equation, $\mathrm{div}\gamma\nabla u_N = 0$ in Ω. Also, the Dirichlet integral is preserved:

$$\int_\Omega \beta(x)|\nabla u_N(x)|^2\,dx = \frac{1}{1+|\nabla\phi(0)|^2}\int_{\Omega'}\beta(y)B(y)\nabla w_N(y)\cdot\nabla\bar{w}_N(y)\,dy.$$

Thus, the recovery of γ at the boundary follows from the result in Ω' of Lemma 11.27 and we have

$$\gamma(0) = \lim_{N\to\infty} \int_\Omega \gamma(x)|\nabla u_N(x)|^2\,dx = \lim_{N\to\infty} \Lambda_\gamma(u_N)(\bar{u}_N).$$

For the proof of the second theorem, we use the same family of solutions and the Rellich identity [10]:

$$\int_{\partial\Omega} \gamma(x)e\cdot\nu(x)|\nabla u_N(x)|^2 - 2\,\mathrm{Re}\,\gamma(x)\frac{\partial u}{\partial\nu}(x)\,e\cdot\nabla\bar{u}(x)\,dx = \int_\Omega e\cdot\nabla\gamma(x)|\nabla u_N(x)|^2\,dx.$$

This is proven by an application of the divergence theorem. The smoothness result in Lemma 11.29 is needed to justify the application of the divergence theorem: we need to know that u_N has two derivatives to carry this out. The full gradient of u_N is determined by the boundary values of u_N and the Dirichlet to Neumann map.

By Lemma 11.22, if $\gamma \in C^1(\bar{\Omega})$, we can take the limit of the right-hand side and obtain that

$$\frac{\partial\gamma}{\partial x_j}(0) = \lim_{N\to\infty} \int_\Omega \frac{\partial\gamma}{\partial x_j}(x)|\nabla u_N(x)|^2\,dx.$$

∎

Corollary 11.30 *If we have a C^2 domain and for two $C^1(\bar{\Omega})$ functions, $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ on the boundary and $\nabla\gamma_2 = \nabla\gamma_1$ on the boundary.*

Proof. The boundary values of the function u_N are independent of γ_j. The expression $\Lambda_\gamma u_N(\bar{u}_N)$ in Theorem 11.8 clearly depends only on u_N and the map Λ_γ. The left-hand side Theorem 11.9 depends only on γ and ∇u_N. Since ∇u_N which can be computed from u_N and the normal derivative of u_N. Hence, we can use Theorem 11.9 to determine $\nabla\gamma$ from the Dirichlet to Neumann map. ∎

Exercise 11.31 *If γ and $\partial\Omega$ are regular enough, can we determine the second order derivatives of γ from the Dirichlet to Neumann map?*

It is known that all derivatives of u are determined by the Dirichlet to Neumann map. I do not know if there is a proof in the style of Theorems 11.8 and 11.9 which tell how to compute second derivatives of γ by looking at some particular expression on the boundary.

Exercise 11.32 *If one examines the above proof, one will observe that there is a bit of slop. We made an arbitrary choice for the vector α and used α in the determination of one function, γ. It is likely that in fact, we can determine $(n-1)$ parameters at the boundary by considering $(n-1)$ linearly independent choices for α. Run with this.*

Chapter 12

Inverse problem: Global uniqueness

The goal of this chapter is to prove the following theorem.

Theorem 12.1 *If Ω is a C^2-domain in $\mathbf{R}^n$, $n \geq 3$, and we have two $C^2(\bar{\Omega})$ conductivities with $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

The proof of this result relies on converting the problem of the uniqueness of γ for the equation $\mathrm{div}\gamma\nabla$ to a similar question about the uniqueness of the potential q for a Schrödinger equation of the form $\Delta - q$ with $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. One reason why this Chapter is so long, is that we spend a great deal of time convincing ourselves that that the uniqueness question for one equation is equivalent with the uniqueness question for the other. Most of this chapter is lifted from the paper of Sylvester and Uhlmann [17]. A few of the details are taken from later works that simplify parts of the argument.

12.1 A Schrödinger equation

Here, we extend our notion of weak solution to equations with a potential or zeroth order term.

We say that v is a weak solution of

$$\begin{cases} \Delta v - qv = 0 & \text{on } \Omega \\ v = f & \text{on } \partial\Omega \end{cases}$$

if $v \in L^2_1(\Omega)$, $v - f \in L^2_{1,0}(\Omega)$ and

$$\int_\Omega \nabla v(x) \cdot \nabla \phi(x) + q(x)v(x)\phi(x)\,dx = 0, \qquad \phi \in L^2_{1,0}(\Omega).$$

If $q \geq 0$ and $q \in L^\infty$, then the quadratic form associated with this equation clearly provides an inner product on $L^2_{1,0}(\Omega)$ and hence we can prove an existence theorem by imitating the arguments from Chapter 10. However, the potentials that we are studying do not satisfy $q \geq 0$, in general. Still, it is possible that the quadratic form is non-negative even without this bound. That is one consequence of the following Lemma. We will use the Lemma below to relate the existence and uniqueness for $\Delta - q$ to $\mathrm{div}\gamma\nabla$.

Lemma 12.2 *Suppose that Ω is C^1, γ is $C^2(\bar{\Omega})$ and that γ is bounded above and below as in (11.7) A function u in $L^2_1(\Omega)$ satisfies*

$$\begin{cases} \mathrm{div}\gamma\nabla u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \end{cases}$$

if and only if $v = \sqrt{\gamma} u$ is a weak solution of

$$\begin{cases} \Delta v - qv = 0, & \text{in } \Omega \\ v = \sqrt{\gamma} f, & \text{on } \partial\Omega \end{cases}$$

Proof. We let $C^1_c(\Omega)$ denote the space of functions in $C^1(\Omega)$ which are compactly supported in Ω. If $\phi \in C^1_c(\Omega)$, then $\sqrt{\gamma}\phi$ is also in $C^1_c(\Omega)$ and hence lies in $L^2_{1,0}(\Omega)$. We consider the quadratic expression in the weak formulation of $\mathrm{div}\gamma\nabla u = 0$ and then use the product rule twice to obtain

$$\begin{aligned} \int_\Omega \gamma(x)\nabla u(x)\nabla\phi(x)\,dx &= \int_\Omega \nabla(\sqrt{\gamma}(x)u(x)) \cdot \sqrt{\gamma}(x)\nabla\phi(x) \\ &\qquad -u(x)(\nabla\sqrt{\gamma}(x)) \cdot (\sqrt{\gamma}(x)\nabla\phi(x))\,dx \\ &= \int_\Omega \nabla(\sqrt{\gamma}(x)u(x)) \cdot \nabla(\sqrt{\gamma}(x)\phi(x)) \\ &\qquad -u(x)(\nabla\sqrt{\gamma}(x)) \cdot (\sqrt{\gamma}(x)\nabla\phi(x)) \\ &\qquad -\nabla(\sqrt{\gamma}(x)u(x) \cdot (\nabla\sqrt{\gamma}(x))\phi(x)\,dx \end{aligned}$$

Now, in the middle term, we use the divergence theorem to move the gradient operator from ϕ to the remaining terms. Since we are not assuming that ϕ vanishes on the boundary, we pick up a term on the boundary:

$$\begin{aligned} \int_\Omega u(x)(\nabla\sqrt{\gamma(x)}) \cdot (\sqrt{\gamma(x)}\nabla\phi(x))\,dx &= -\int_\Omega \frac{\Delta\sqrt{\gamma}(x)}{\sqrt{\gamma}(x)}(\sqrt{\gamma}(x)u(x))(\sqrt{\gamma}(x)\phi(x)) \\ &\qquad +\nabla(\sqrt{\gamma}(x)u(x)) \cdot (\nabla\sqrt{\gamma}(x))\phi(x)\,dx \\ &\qquad +\int_{\partial\Omega} \phi(x)\sqrt{\gamma}(x)\nabla\sqrt{\gamma}(x) \cdot \nu(x)\,d\sigma(x). \end{aligned}$$

We use this to simplify the above expression, note that two terms cancel and we obtain, with $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$, that

$$\begin{aligned} \int_\Omega \gamma(x)\nabla u(x)\cdot\nabla\phi(x)\,dx &= \int_\Omega \nabla(\sqrt{\gamma}(x)u(x))\cdot\nabla(\sqrt{\gamma}(x)\phi(x)) \\ &\quad +q(x)\sqrt{\gamma}(x)u(x)\sqrt{\gamma}(x)\phi(x)\,dx \\ &\quad -\int_{\partial\Omega}\sqrt{\gamma}(x)u(x)\phi(x)\nabla\sqrt{\gamma}(x)\cdot\nu\,d\sigma. \end{aligned} \tag{12.3}$$

Since the map $\phi \to \sqrt{\gamma}\phi$ is invertible on $C^1_c(\Omega)$ we have that

$$\int_\Omega \gamma(x)\nabla u(x)\cdot\nabla\phi(x)\,dx = 0, \quad \text{for all } \phi \in C^1_c(\Omega),$$

if and only if with $v = \sqrt{\gamma}u$

$$\int_\Omega \nabla v(x)\cdot\nabla\phi(x) + q(x)v(x)\phi(x)\,dx = 0, \quad \text{for all } \phi \in C^1_c(\Omega).$$

∎

Corollary 12.4 *With q as above, if $f \in L^2_1(\Omega)$, then there exist a unique weak solution of the Dirichlet problem for $\Delta - q$.*

Proof. According to Lemma 12.2, solutions of the Dirichlet problem for $\Delta v - qv = 0$ with data f are taken to solutions of the Dirichlet problem for $\mathrm{div}\gamma\nabla u = 0$ with data $f/\sqrt{\gamma}$ by the map $v \to v/\sqrt{\gamma}$ and this map is invertible, the existence and uniqueness for $\Delta - q$ follows from the existence and uniqueness in Theorem 10.25. ∎

Exercise 12.5 *We claimed above that if Ω is bounded and $q \geq 0$ is real and in L^∞, then the expression*

$$\int_\Omega \nabla u\cdot\nabla\bar{v} + qu\bar{v}\,dx$$

defines an inner product on $L^2_{1,0}(\Omega)$ which induces the same topology as the standard inner product. Verify this.

Show that this continues to hold for $n \geq 3$, if $q \in L^{n/2}(\Omega)$. What goes wrong if $n = 2$?

The following Lemma asserts that a function $\beta \in C^1(\Omega)$ function defines a multiplier on $L^2_{1/2}(\partial\Omega)$ which depends only on the boundary values of β. This should seem obvious. That we need to prove such obvious statements is the price we pay for our cheap definition of the space $L^2_{1/2}(\Omega)$.

Lemma 12.6 *Let Ω be a C^1-domain. If $\beta_1,\ \beta_2 \in C^1(\bar{\Omega})$, $\beta_1 = \beta_2$ on $\partial\Omega$, then for each $f \in L^2_1(\Omega)$, $(\beta_1 - \beta_2)f \in L^2_{1,0}(\Omega)$. As a consequence, for each $f \in L^2_{1/2}(\partial\Omega)$, $\beta_1 f = \beta_2 f$.*

Proof. First note that the product rule, exercise 10.7 implies that $\beta_j f$ is in $L^2_1(\Omega)$ if f is in $L^2_1(\Omega)$. To see that $(\beta_1 - \beta_2)f$ is in $L^2_{1,0}(\Omega)$, we will establish the following:

Claim. If $\beta \in C^1(\bar{\Omega})$ and $\beta = 0$ on $\partial\Omega$, then the map $f \to \beta f$ maps $L^2_1(\Omega)$ into $L^2_{1,0}(\Omega)$.

To establish the claim, we may use a partition of unity to reduce to a function f which is supported in a ball $B_r(x_0)$ and so that near x, the boundary lies in a graph $\{(x', x_n) : x_n = \phi(x')\}$. We let $\lambda(t)$ be a function which is smooth on all of $\mathbf{R}$, is 0 if $t < 1$ and is 1 if $t > 2$. We let

$$\eta_\epsilon(x) = \lambda((x_n - \phi(x'))/\epsilon).$$

Thus, η_ϵ vanishes on $\partial\Omega \cap B_r(x_0)$. The product $\eta_\epsilon(x)\beta(x)f(x)$ will be compactly supported in Ω, hence we can regularize as in Lemma 10.9 in order to approximate in the $L^2_1(\Omega)$-norm by functions in $\mathcal{D}(\Omega)$ and conclude that $\eta_\epsilon(x)\beta(x)f(x)$ is in $L^2_{1,0}(\Omega)$. Now we show that

$$\lim_{\epsilon \to 0^+} \|\eta_\epsilon \beta f - \beta f\|_{L^2_1(\Omega)} = 0. \tag{12.7}$$

This will imply that βf is in $L^2_{1,0}(\Omega)$, since $L^2_{1,0}(\Omega)$ is (by definition) a closed subspace of $L^2_1(\Omega)$.

We establish (12.7). It is an immediate consequence of the Lebesgue dominated convergence theorem that $\eta_\epsilon \beta f \to \beta f$ in $L^2(\Omega)$ as $\epsilon \to 0^+$. Now, we turn to the derivatives. We compute the derivative

$$\frac{\partial}{\partial x_j}(\eta_\epsilon(x)\beta(x)f(x)) = (\frac{\partial \eta_\epsilon}{\partial x_j}(x)\beta(x)f(x) + \eta_\epsilon(x)\frac{\partial}{\partial x_j}(\beta(x)f(x)).$$

By the dominated convergence theorem, the second term on the right converges in $L^2(\Omega)$ to the derivative of βf. We show the first term on the right goes to zero in L^2. To see this, we apply the mean value theorem of one variable calculus on the line segment joining $(x', \phi(x'))$ to (x', x_n) and use that $\beta(x', \phi(x')) = 0$ to conclude that

$$|\beta(x)| \le 2\epsilon \|\nabla \beta\|_\infty.$$

Using this, and observing that $\nabla \eta_\epsilon$ is supported in a thin strip along the boundary and satisfies $|\nabla \eta_\epsilon| \le C/\epsilon$, we conclude that

$$\int_\Omega |\frac{\partial \eta_\epsilon}{\partial x_j}(x)\beta(x)f(x)|^2\, dx \le C\|\beta\|_\infty \int_{\{B_r(x_0) \cap \{x: 0 < x_n - \phi(x') < 2\epsilon\}} |f(x)|^2\, dx.$$

The last integral goes to zero as $\epsilon \to 0^+$. Thus the claim follows.

It is easy to see that $f \to \beta_j f$ gives a map on $L^2_1(\Omega)$.

Now, if each β_1 and β_2 are as in the theorem and f is a representative of an element of $L^2_{1/2}(\partial\Omega)$, then since $\beta_1 f - \beta_2 f \in L^2_{1,0}(\Omega)$, we can conclude that $\beta_1 f$ and $\beta_2 f$ give the same function $L^2_{1/2}(\partial\Omega)$. ∎

Our next step is to establish a relation between the Dirichlet to Neumann map for q and that for γ.

Lemma 12.8 *If γ is in $C^2(\bar{\Omega})$ and satisfies the ellipticity condition (11.7), and Ω is a C^1-domain, then we have*

$$\Lambda_q(\cdot) - \frac{1}{\sqrt{\gamma}}\nabla\sqrt{\gamma}\cdot\nu = \frac{1}{\sqrt{\gamma}}\Lambda_\gamma(\frac{1}{\sqrt{\gamma}}\cdot).$$

Proof. We fix f in $L^2_{1/2}(\partial\Omega)$ and suppose that u is the solution of the Dirichlet problem for $\mathrm{div}\gamma\nabla$ with boundary data f. According to the identity (12.3) in the proof of Lemma 12.2, we have

$$\begin{aligned}
\Lambda_\gamma(f)(\phi) &= \int_\Omega \gamma(x)\nabla u(x)\cdot\nabla\phi(x)\,dx \\
&= \int_\Omega \nabla(\sqrt{\gamma}(x)u(x))\cdot\nabla(\sqrt{\gamma}(x)\phi(x)) + q(x)\sqrt{\gamma}(x)u(x)\sqrt{\gamma}(x)\phi(x)\,dx \\
&\qquad - \int_{\partial\Omega}\sqrt{\gamma}(x)u(x)\phi(x)\nabla\sqrt{\gamma}(x)\cdot\nu\,d\sigma \\
&= \sqrt{\gamma}\Lambda_q(\sqrt{\gamma}f)(\phi) - \int_{\partial\Omega}\sqrt{\gamma}f\nabla\sqrt{\gamma}\cdot\nu\phi\,d\sigma.
\end{aligned}$$

Making the substitution $f = g/\sqrt{\gamma}$ and dividing by $\sqrt{\gamma}$ gives the desired conclusion. [1] ∎

Remark. A clearer and more direct proof of this lemma can be given if we assume the regularity result of Lemma 11.29. We may choose f which is nice, solve the Dirichlet problem for $\mathrm{div}\gamma\nabla$ with data f to obtain u. We have that $v = \sqrt{\gamma}u$ solves the Schrödinger equation $\Delta v - qv = 0$. Taking the normal derivative we have

$$\Lambda_q(\sqrt{\gamma}f) = \sqrt{\gamma}\frac{\partial u}{\partial\nu} + u\frac{\partial\sqrt{\gamma}}{\partial\nu}.$$

We now consider two conductivities γ_1 and γ_2 and the corresponding potentials $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$.

[1] In the above equation, we are not distinguishing between the multiplication operator that $\sqrt{\gamma}$ gives on $L^2_{1/2}(\partial\Omega)$ and the transpose of this operator to the dual, $L^2_{-1/2}(\partial\Omega)$. Did anyone notice?

Proposition 12.9 *If $\gamma_1,\ \gamma_2 \in C^1(\bar\Omega)$, $\gamma_1 = \gamma_2$ and $\nabla\gamma_1 = \nabla\gamma_2$, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

if and only if

$$\Lambda_{q_1} = \Lambda_{q_2}.$$

Proof. This result follows from Lemma 12.8 and 12.6. ∎

12.2 Exponentially growing solutions

In this section, we consider potential q which are defined in all of $\mathbf{R}^n$ and are bounded and compactly supported. In applications, q will be of the form $\Delta\sqrt{\gamma}/\sqrt{\gamma}$ in Ω and 0 outside Ω. The assumption that q is bounded is needed in this approach. The assumtion that q is compactly supported is too strong. What is needed is that q defines a multiplication operator from $M^{2,-1/2}_\infty \to M^{2,1/2}_1$ and thus there is a constant $M(q)$ so that

$$\|q\phi\|_{M_1^{2,1/2}} \leq M(q)\|\phi\|_{M_\infty^{2,-1/2}}. \tag{12.10}$$

This requires that q decay faster than $(1+|x|^2)^{-1/2}$ at infinity, which is true if q is bounded and compactly supported.

Our goal is to construct solutions of the equation $\Delta v - qv = 0$ which are close to the harmonic functions $e^{x\cdot\zeta}$. Recall that such an exponential will be harmonic if $\zeta\cdot\zeta = 0$. We will succeed if ζ is large.

Theorem 12.11 *Assume $M(q)$ is finite and let $\zeta \in \mathbf{C}^n$ satisfy $\zeta\cdot\zeta = 0$. There exists a constant $C = C(n)$ so that if $|\zeta| > C(n)M(q)$, then we can find a solution of*

$$\Delta v - qv = 0$$

of the form $v(x) = e^{x\cdot\zeta}(1+\psi(x,\zeta))$ which satisfies

$$\|\psi\|_{M_\infty^{2,-1/2}} \leq \frac{CM(q)}{|\zeta|}\|q\|_{M_1^{2,1/2}}.$$

Furthermore, the function ψ is the only function in $M^{2,-1/2}_\infty$ for which v as defined above will satisfy $\Delta v - qv = 0$.

Proof. *Existence.* If we differentiate we see that $\Delta v - qv = 0$ if and only if

$$\Delta\psi + 2\zeta \cdot \nabla\psi - q\psi = q. \tag{12.12}$$

A solution of this equation may be constructed by solving the integral equation

$$\psi - G_\zeta(q\psi) = G_\zeta(q).$$

A solution of the integral equation (12.12) is given by the series

$$\psi = \sum_{j=1}^{\infty}(G_\zeta q)^j(1).$$

To see that the series can be summed, we apply the second estimate of Theorem 9.16 of Chapter 9, and use the estimate for the multiplication operator given by q, (12.10), to obtain

$$\|(G_\zeta q)^j(1)\|_{M^{2,-1/2}_\infty} \leq (\frac{CM(q)}{|\zeta|})^{j-1}\frac{C}{|\zeta|}\|q\|_{M^{2,1/2}_1}.$$

Thus, if $|\zeta|$ is large, this series converges and defines a functions ψ in $M^{2,-1/2}_\infty$. Furthermore, according to exercise 9.21 $\nabla\psi = \nabla G_\zeta(q(1+\psi))$ is in $M^{2,-1/2}_\infty$. Thus, v will be a weak solution of the equation $\Delta v - qv = 0$ in $\mathbf{R}^n$.

Uniqueness. If we have two solutions, ψ_1 and ψ_2 of (12.12) which are in $M^{2,-1/2}_\infty$, then their difference satisfies

$$\Delta(\psi_1 - \psi_2) + 2\zeta \cdot \nabla(\psi_1 - \psi_2) - q(\psi_1 - \psi_2) = 0.$$

According to Theorem 9.23 we have $\psi_1 - \psi_2 = G_\zeta(q(\psi_1 - \psi_2))$. Thus from the estimate in Theorem 9.16 we have

$$\|\psi_2 - \psi_2\|_{M^{2,-1/2}_\infty} \leq \frac{CM(q)}{|\zeta|}\|\psi_1 - \psi_2\|_{M^{2,-1/2}_\infty}.$$

If we have $CM(q)/|\zeta| < 1$, then this inequality will imply that $\|\psi_1 - \psi_2\|_{M^{2,-1/2}_\infty} = 0$. ∎

Lemma 12.13 *Suppose that Ω is C^1 and suppose that each q_j is supported in $\bar{\Omega}$ and that q_j are of the form $\Delta\sqrt{\gamma}_j/\sqrt{\gamma_j}$. If $\Lambda_{q_1} = \Lambda_{q_2}$ and $v_j = (1+\psi_j)e^{x\cdot\zeta}$ are the solutions for $\Delta - q_j$ from Theorem 12.11, then $\psi_1(x,\zeta) = \psi_2(x,\zeta)$ for $x \in \mathbf{R}^n \setminus \Omega$ and all ζ sufficiently large.*

Proof. We use a cut-and-paste argument. Define a new function by

$$\tilde{\psi}_1(x,\zeta) = \begin{cases} \psi_2(x,\zeta), & x \in \mathbf{R}^n \setminus \bar{\Omega} \\ \psi(x,\zeta), & x \in \Omega. \end{cases}$$

Here, $\psi(x,\zeta) = e^{-x\cdot\zeta}v(x) - 1$ where v is the solution of the Dirichlet problem

$$\begin{cases} \Delta v - q_1 v = 0 & x \in \Omega \\ v(x) = e^{x\cdot\zeta}(1+\psi_2(x,\zeta)) & x \in \partial\Omega \end{cases}$$

We claim that $\tilde{v}_1(x,\zeta) = e^{x\cdot\zeta}(1+\tilde{\psi}_1)$ is a solution of $\Delta v - q_1 v = 0$ in all of $\mathbf{R}^n$. This depends on the the hypothesis $\Lambda_{q_1} = \Lambda_{q_2}$. To establish this claim, we let $\phi \in \mathcal{D}(\mathbf{R}^n)$ and consider

$$\int_{\mathbf{R}^n} \nabla\tilde{v}_1 \cdot \nabla\phi + q_1\tilde{v}_1\phi\, dx = \int_{\mathbf{R}^n\setminus\Omega} \nabla v_2 \cdot \nabla\phi\, dx + \int_\Omega \nabla v(x)\cdot\nabla\phi(x) + q_1(x)v(x)\phi(x)\, dx.$$

Since v_2 is a solution of $\Delta v_2 - q_2 v_2 = 0$ in $\mathbf{R}^n$, we have that

$$\int_{\mathbf{R}^n\setminus\Omega} \nabla v_2 \cdot \nabla\phi\, dx = -\int_\Omega \nabla v_2\cdot\nabla\phi + q_2 v_2\phi\, dx = \Lambda_{q_2}(v_2)(\phi).$$

Since $v_2 = v$ on the boundary of Ω and $\Lambda_{q_1} = \Lambda_{q_2}$ we have

$$\Lambda_{q_2}(v_2)(\phi) = \Lambda_{q_1}(v)(\phi) = \int_\Omega \nabla v\cdot\nabla\phi + q_1 v\phi\, dx.$$

Combining these last three equations shows that $\tilde{v}_1$ is a weak solution of $\Delta - q_1$ in $\mathbf{R}^n$. By the uniqueness statement in Theorem 12.11, the function $\tilde{\psi}_1$ defined by $\tilde{\psi}_1 = e^{-x\cdot\zeta}\tilde{v}_1 - 1$ must equal ψ_1. In particular, $\psi_1 = \psi_2$ outside Ω. ∎

Lemma 12.14 *Let q be a potential for which we can solve the Dirichlet problem. The operator Λ_q is symmetric. That is we have $\Lambda_q(\phi)(\psi) = \Lambda_q(\psi)(\phi)$.*

Proof. Let ϕ_1, ϕ_2 be in $L^2_{1/2}(\partial\Omega)$. We solve the Dirichlet problem with data ϕ_j to find a function u_j which solves the Dirichlet problem for $\Delta - q$ with boundary data ϕ_j. Then we have

$$\Lambda(\phi_1)(\phi_2) = \int_\Omega \nabla u_1\cdot\nabla u_2 + q u_1 u_2\, dx.$$

The integral on the right-hand side is symmetric in u_1 and u_2 so we can conclude

$$\Lambda_q(\phi_1)(\phi_2) = \Lambda_q(\phi_2)(\phi_1).$$

∎

Proof of Theorem 12.1. According to Corollary 11.30 and Proposition 12.9, we have that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\Lambda_{q_1} = \Lambda_{q_2}$. We will show that if $\Lambda_{q_1} = \Lambda_{q_2}$, then the Fourier transforms satisfy $\hat{q}_2 = \hat{q}_1$ (where we are assuming that each q_j has been defined to be zero outside Ω). We fix $\xi \in \mathbf{R}^n$ and choose two unit vectors α and β which satisfy $\alpha \cdot \beta = \alpha \cdot \xi = \beta \cdot \xi = 0$. (Here, we use our assumption that $n \geq 3$ in order to find three mutually orthogonal vectors.) Next, for $R > |\xi|/2$, we define ζ_1 and ζ_2 by

$$\zeta_1 = R\alpha + i\beta\sqrt{R^2 - |\xi|^2/4} - i\xi/2 \qquad \text{and} \qquad \zeta_2 = -R\alpha - i\beta\sqrt{R^2 - |\xi|^2/4} - i\xi/2.$$

These vectors satisfy $\zeta_j \cdot \zeta_j = 0$, $\zeta_1 + \zeta_2 = -i\xi$ and $|\zeta_j| = \sqrt{2}R$.

For R large, we let v_j be the solution of $\Delta v_j - q_j v_j = 0$ corresponding to ζ_j as in Theorem 12.11. Since the Dirichlet to Neumann maps are equal and using Lemma 12.14, we have

$$0 = \Lambda_{q_1}(v_1)(v_2) - \Lambda_{q_2}(v_2)(v_1) = \int_\Omega (q_1(x) - q_2(x))e^{-ix\cdot\xi}(1 + \psi_1 + \psi_2 + \psi_1\psi_2)\,dx.$$

Recall, that the ψ_j depend on the parameter R and that Theorem 12.11 implies that the $\psi_j \to 0$ in L^2_{loc} as $R \to \infty$. Thus, we conclude

$$\hat{q}_1 = \hat{q}_2.$$

The Fourier inversion theorem implies $q_1 = q_2$. Finally, the Lemma below tells us that if $q_1 = q_2$ and $\gamma_1 = gamma_2$ on the boundary, $\gamma_1 = \gamma_2$. ∎

Lemma 12.15 *If γ_1 and γ_2 in $C^2(\bar{\Omega})$ and if $\Delta\sqrt{\gamma}_1/\sqrt{\gamma}_1 = \Delta\sqrt{\gamma}_2/\sqrt{\gamma}_2$, then $u = \log(\gamma_1/\gamma_2)$ satisfies the equation*

$$\mathrm{div}\sqrt{\gamma_1\gamma_2}\nabla u = 0.$$

As a consequence, if Ω is C^1, and $\gamma_1 = \gamma_2$ on the boundary, then $\gamma_1 = \gamma_2$.

Proof. Let ϕ be a $\mathcal{D}(\Omega)$ function, say, which is compactly supported in Ω. We multiply our hypothesis, $\Delta\sqrt{\gamma}_1/\sqrt{\gamma}_1 = \Delta\sqrt{\gamma}_2/\sqrt{\gamma}_2$ by ϕ and integrate by parts to obtain

$$0 = \int_\Omega (\frac{\Delta\sqrt{\gamma}_1}{\sqrt{\gamma}_1} - \frac{\Delta\sqrt{\gamma}_2}{\sqrt{\gamma}_2})\phi\,dx = -\int_\Omega \nabla\sqrt{\gamma_1}\cdot\nabla(\frac{1}{\sqrt{\gamma_1}}\phi) - \nabla\sqrt{\gamma_2}\cdot\nabla(\frac{1}{\sqrt{\gamma_2}}\phi)\,dx$$

If we make the substitution $\phi = \sqrt{\gamma_1}\sqrt{\gamma_2}\psi$, then we have

$$\int_\Omega \sqrt{\gamma_1\gamma_2}\nabla(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2})\cdot\nabla\psi\,dx = 0.$$

If $\gamma_1 = \gamma_2$ on the boundary and Ω is C^1, then by Lemma 12.6we have $\log(\gamma_1/\gamma_2)$ is in $L^2_{1,0}(\Omega)$. We can conclude that this function is zero in Ω from the uniqueness assertion of Theorem 10.25. ∎

Exercise 12.16 *Suppose the* Ω *is a* C^1*-domain in* $\mathbf{R}^n$*. Suppose that* u^+ *is a weak solution* $\Delta u^+ = 0$ *in* Ω *and* u^- *is a local weak solution of* $\Delta u^- = 0$ *in* $\mathbf{R}^n \setminus \bar{\Omega}$ *and that* ∇u^- *is in* L^2 *of every bounded set in* $\mathbf{R}^n \setminus \bar{\Omega}$*. Set* $\gamma = 1$ *in* $\mathbf{R}^n \setminus \bar{\Omega}$ *and set* $\gamma = 2$ *in* Ω*. Define* u *by*

$$u(x) = \begin{cases} u^+(x), & x \in \Omega \\ u^-(x), & x \in \mathbf{R}^n \setminus \bar{\Omega} \end{cases}$$

What conditions must $u^\pm$ *satisfy in order for* u *to be a local weak solution of* $\mathrm{div}\gamma\nabla$ *in all of* $\mathbf{R}^n$*. Hint: There are two conditions. The first is needed to make the first derivatives of* u *to be locally in* L^2*. The second is needed to make the function* u *satisfy the weak formulation of the equation.*

Exercise 12.17 *Show that the result of Lemma 12.15 continues to hold if we only require that the coefficients* γ_1 *and* γ_2 *are elliptic and in* $L^2_1(\Omega)$*. In fact, the proof is somewhat simpler because the equation* $\Delta\sqrt{\gamma_1}/\sqrt{\gamma_1} = \Delta\sqrt{\gamma_2}/\sqrt{\gamma_2}$ *and the boundary condition are assumed to hold in a weak formulation. The proof we gave amounts to showing that the ordinary formulation of these conditions imply the weak formulation.*

Exercise 12.18 *(Open) Show that a uniqueness theorem along the lines of Theorem 12.1 holds under the assumption that* γ *is only* $C^1(\bar{\Omega})$*.*

Chapter 13

Bessel functions

Chapter 14

Restriction to the sphere

Chapter 15

The uniform sobolev inequality

In this chapter, we give the proof of a theorem of Kenig, Ruiz and Sogge which can be viewed as giving a generalization of the Sobolev inequality. One version of the Sobolev inequality is that if $1 < p < n/2$, then we have

$$\|u\|_p \leq C(n,p)\|\Delta u\|_p.$$

This can be proven using the result of exercise 8.2 and the Hardy-Littlewood-Sobolev theorem, Theorem 8.8. In our generalization, we will consider more operators, but fewer exponents p. The result is

Theorem 15.1 *Let $L = \Delta + a \cdot \nabla + b$ where $a \in complexes^n$ and $b \in \mathbf{C}$ and let p satisfy $1/p - 1/p' = 2/n$. For each f with $f \in L^p$ and $D^2 f L^p$ we have*

$$\|f\|_{p'} \leq C\|Lf\|_p.$$

Chapter 16

Inverse problems: potentials in $L^{n/2}$

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Bibliography

[1] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, 1976.

[2] Russell M. Brown and Gunther A. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. Partial Differential Equations **22** (1997), no. 5-6, 1009–1027.

[3] Emilio Gagliardo, *Proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. **7** (1958), 102–137.

[4] Lars Inge Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[5] G.M. Henkin and R.G. Novikov, *A multidimensional inverse problem in quantum and acoustic scattering*, Inverse problems **4** (1988), 103–121.

[6] Lars Hörmander, *The analysis of linear partial differential operators. I*, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis.

[7] Adrian I. Nachman, *Reconstructions from boundary measurements*, Annals of Math. **128** (1988), 531–587.

[8] ———, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math. (2) **143** (1996), 71–96.

[9] R.G. Novikov, *Multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Functional anal. and appl. **22** (1988), 263–272.

[10] F. Rellich, *Darstellung der eigenwerte von* $\Delta u + \lambda u$ *durch ein Randintegral*, Math. Z. **46** (1940), 635–646.

[11] L. Schwartz, *Théorie des distributions. Tome I*, Hermann & Cie., Paris, 1950, Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9.

[12] Laurent Schwartz, *Théorie des distributions. Tome II*, Hermann & Cie., Paris, 1951, Actualités Sci. Ind., no. 1122 = Publ. Inst. Math. Univ. Strasbourg **10**.

[13] C. Segovia and R.L. Wheeden, *On the function g_λ^* and the heat equation*, Studia Math. **37** (1970), 57–93.

[14] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton NJ, 1970.

[15] E.M. Stein and Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.

[16] J. Sylvester, *An anisotropic inverse boundary value problem*, Comm. Pure Appl. Math. **43** (1990), 201–232.

[17] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Annals of Math. **125** (1987), 153–169.

[18] Peter A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

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