

Estimates for a family of multi-linear forms

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Abstract

We consider a special class of the multi-linear forms studied by Brascamp and Lieb. For these forms, we are able to characterize the L^p spaces for which the form is bounded. We use this characterization to study a non-linear map that arises in scattering theory.

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1 Introduction

In this note, we consider a family of multi-linear forms involving fractional integration and establish estimates for these forms on products of L^p -spaces. Using these estimates, we are able to give a proof of continuity of a scattering map in two dimensions. This scattering map may be found in work of Fokas [11], as well as later work of several authors including Fokas and Ablowitz [12], Beals and Coifman [2], and Sung [18, 19, 20]. These authors were interested in a two-dimensional scattering theory that served to transform solutions of one of the Davey-Stewartson equations, a nonlinear evolution equation in two space dimensions, into solutions of a linear system. The map reappeared in work of Brown and Uhlmann [9] on the inverse conductivity problem. In the inverse conductivity problem, we are interested in recovering a conductivity coefficient from the Dirichlet to Neumann map. As part of this recovery, it is interesting to know something about the continuity properties of the scattering map. This was one motivation for the work of Brown [8]. This work of Brown shows that the scattering map is continuous in a neighborhood 0 in L^2 . In this article, we provide a new proof of some of the results of Brown and give a description of the set of L^p spaces where certain multi-linear forms are bounded. This description appeared earlier in work of Barthe [1, p. 348].

To describe our main result in more detail, for $n = 0, 1, 2, \dots$ we consider the multi-linear form

$$\begin{aligned} \Lambda_n(t, q_0, q_1, \dots, q_{2n}) \\ = \int_{\mathbf{C}^{2n+1}} \frac{t(\sum_{k=0}^{2n} (-1)^k x_k) \prod_{k=0}^{2n} q_k(x_k)}{(x_0 - x_1)(\bar{x}_1 - \bar{x}_2) \dots (\bar{x}_{2n-1} - \bar{x}_{2n})} dx_0 dx_1 \dots dx_{2n}. \end{aligned} \quad (1.1)$$

In this expression, we are using x_j to stand for a complex variable and dx_j denotes Lebesgue measure on the complex plane. Our goal is to show that there is constant c so that

$$\Lambda_n(t, q_0, q_1, \dots, q_{2n}) \leq c^n \|t\|_{1/2} \prod_{j=0}^{2n} \|q_j\|_{1/2}. \quad (1.2)$$

Here and throughout this paper, we will denote the L^p norm of a function f by $\|f\|_{1/p}$ with the convention that $1/\infty$ is 0. Thus, we provide a new proof of the main estimate in the article [8], but without the precise dependence of the constant. As in Brown's work [8], this leads to the continuity of the scattering map on L^2 . The method is perhaps a bit more flexible and we are able to give an extension of these results when some of the functions come from L^p spaces for $p \neq 2$. We use this to obtain an analogue of the Hausdorff-Young inequality for the scattering map.

We briefly describe the results of this paper. Most of the results of this paper first appeared in the Ph.D. dissertation of the author Nie [15]. The first part of our paper considers general multi-linear forms

$$\Lambda(a_1, a_2, \dots, a_m) = \int_{\mathbf{R}^{k\ell}} \prod_{j=1}^m a_j(f_j \cdot x) dx, \quad a_j \in L^1(\mathbf{R}^\ell) \cap L^\infty(\mathbf{R}^\ell) \quad (1.3)$$

where $x \in \mathbf{R}^{k\ell}$, $x = (x_1, \dots, x_k)$ and each $x_i \in \mathbf{R}^\ell$, $f_j \in \mathbf{R}^k$ and $M = \{f_1, f_2, \dots, f_m\}$ is a collection of vectors in \mathbf{R}^k . We define $f_j \cdot x = \sum_{i=1}^k f_{ji} x_i$. We will arrive at the form Λ_n by setting some of the functions a_j to be $1/x$ which lies in the Lorentz space $L^{2,\infty}(\mathbf{C})$. Thus, we will be interested in estimates in Lorentz spaces. We consider the set Ω_Λ which is defined to be the set of $\theta = (\theta_1, \dots, \theta_m)$ for which we have the inequality

$$\Lambda(a_1, \dots, a_m) \leq C_\theta \prod_{j=1}^m \|a_j\|_{\theta_j} \quad (1.4)$$

for some constant C_θ . We show that this set Ω_Λ is the matroid polytope (or, more precisely, the basis matroid polytope) for the matroid formed by the set of vectors $M \subset \mathbf{R}^k$. Recall that a set of vectors M and the collection of linearly independent subsets of M form a matroid. We recall that the *matroid polytope* for M , Ω_M , is the convex hull of the vectors $\{\chi_B : B \text{ is a basis for } \mathbf{R}^k\}$. We are using χ_S to denote the indicator function of a set $S \subset M$. Thus the i th

component of χ_S is 1 if $f_i \in S$ and 0 otherwise. The matroid polytope can also be described by a finite set of inequalities and we are able to use this description to help establish our estimates. We refer to the monograph of Oxley [17] or the textbook of Lee [14] for basic facts about matroids. We will use two operations on sets in matroids. For matroids given as a subset of a vector space, we may define the *span* of a set $S \subset M$ as $M \cap V$ where V is the span of S in the vector space. The *rank* of S in the matroid sense can be defined as the dimension of the vector space spanned by S .

The characterization of the set Ω_Λ as a matroid polytope may be found in the work of Barthe [1, p. 348], though Barthe does not use the word matroid polytope.

In passing from estimates in Lebesgue spaces to estimates in Lorentz spaces, we make use of a multi-linear interpolation theorem of S. Janson [13]. The constants in this theorem depend on the distance from the boundary of Ω_M and obtaining the correct dependence on n as n tends to infinity requires additional work. We do not attempt to summarize all of the work related to multi-linear forms, but refer readers to the survey paper of Beckner [3], recent work by Bennett, Carbery, Christ, and Tao [4, 5], Carlen, Lieb and Loss [10] as well as the earlier work of Brascamp and Lieb [7] for related work on multi-linear estimates. Note that our work is much simpler in that we do not make an effort to find the optimal estimate in our inequalities. Using estimates in Lorentz spaces to obtain estimates for fractional integration dates back at least to [16] and Beckner [3] discusses forms involving fractional integration.

Both authors thank Jakayla Robbins for pointing out to us that the set Ω_M is a matroid polytope.

2 Estimates in Lebesgue spaces

In this section, we continue to consider the form (1.3). We begin with the following simple proposition.

Proposition 2.1 *If $B \subset M$ is a basis for \mathbf{R}^k , and $\chi_B = (\theta_1, \dots, \theta_m)$, then we have*

$$\Lambda(a_1, \dots, a_m) \leq |\det \hat{B}|^{-\ell} \prod_{j=1}^m \|a_j\|_{\theta_j}.$$

Here we are using \hat{B} to denote the $k \times k$ matrix whose rows are the elements of B .

Proof. We will make a change of variables in the integral defining Λ . We let $B = \{f_{i_1}, \dots, f_{i_k}\}$ and define $y_j = f_{i_j} \cdot x$. If we make this change of variables in the form Λ , the estimate of the Lemma becomes obvious. To obtain the constant, we observe that the determinant of the map $x \rightarrow y$ on $\mathbf{R}^{k\ell}$ is $|\det \hat{B}|^\ell$. ■

As noted above a set of vectors $M = \{f_1, \dots, f_m\}$ gives a matroid. Since each basis for \mathbf{R}^k contains k elements, the matroid polytope for M , Ω_M lies in the hyperplane given by $\sum_{i=1}^m \theta_i = k$. As a corollary of this definition and the previous theorem, we have the following.

Corollary 2.2 *For $\theta \in \Omega_M$ and Λ as defined in (1.3), we have*

$$\Lambda(a_1, \dots, a_m) \leq C \prod_{i=1}^m \|a_i\|_{\theta_i}.$$

where $C = \max\{|\det \hat{B}|^{-\ell} : B \subset M \text{ is a basis for } \mathbf{R}^k\}$.

Proof. The Corollary follows from Proposition 2.1 and the theorem on complex multi-linear interpolation from the monograph of Bergh and Löfstrom [6, Theorem 4.4.1]. \blacksquare

We observe that the constant in this estimate could be improved. In our application, the determinant will be 1 at every vertex and thus we choose to not dwell on the constant.

The converse of Corollary 2.2 also holds. If estimate (1.4) holds for a finite constant, then the point θ lies in the matroid polytope, Ω_M . This converse is not needed in our argument, but is included for completeness. To establish the converse, we recall Theorem 2.1 in the work of Bennett *et. al.* [4], specialized to the form in (1.3).

Theorem 2.3 [4, Theorem 2.1] *We have the estimate (1.4) for $\theta \in [0, 1]^m$ if and only if we have*

$$\sum_{i=1}^m \theta_i = k$$

and for every subspace $V \subset \mathbf{R}^{k\ell}$, we have

$$\dim(V) \leq \sum_{i=1}^m \theta_i \dim(f_i \cdot V).$$

Corollary 2.4 *If the form in (1.3) satisfies the estimate (1.4) for $\theta \in [0, 1]^m$, then we have $\theta \in \Omega_M$.*

Proof. It is known that the matroid polytope can be described as the set of $\theta \in [0, 1]^m$ which lie in the hyperplane $\{\theta : \sum_i \theta_i = k\}$ and which satisfy the inequalities

$$\sum_{\{i: f_i \in S\}} \theta_i \leq \text{rank}(S) \tag{2.5}$$

for all subsets $S \subset M$. See the textbook of J. Lee [14, p. 67], for example.

Assume the form satisfies the estimate (1.4) for θ . Let $S \subset M$ and we will show the above inequality. Towards this end, we let V be the orthogonal

complement of S , $V = S^\perp$. Let $\mathcal{V} = \{(v_1x, v_2x, \dots, v_mx) : v \in V, x \in \mathbf{R}^\ell\}$ and thus $\mathcal{V} \subset \mathbf{R}^{k\ell}$. From Theorem 2.3 we have

$$\ell \dim(V) = \dim(\mathcal{V}) \leq \ell \sum_{i=1}^m \theta_i \dim(f_i \cdot V) = \ell \sum_{\{i: f_i \cdot V \neq \{0\}\}} \theta_i$$

Using that $k = \sum_{i=1}^m \theta_i = \dim(V) + \dim(V^\perp)$ and we arrive at the inequality,

$$\sum_{\{i: f_i \cdot V = \{0\}\}} \theta_i \leq \dim(V^\perp).$$

We observe that $\dim(V^\perp) = \text{rank}(S)$ and the Corollary follows. ■

We now consider an extension of these estimates to the Lorentz spaces. This relies on an interpolation theorem for multi-linear operators of S. Janson [13]. Janson's theorem is based on the real method of interpolation and thus gives us Lorentz spaces as intermediate spaces.

We develop the notation needed to state Janson's result. For $j = 1, \dots, m$, we let $\bar{A}_j = (A_{j0}, A_{j1})$, $j = 1, \dots, m$ and $\bar{B} = (B_0, B_1)$ be Banach couples and then $A_{i\theta, q} = [A_{j0}, A_{j1}]_{\theta, q}$ will be the real interpolation intermediate spaces. We consider multi-linear operators

$$T : \prod_{j=1}^m A_{j0} \cap A_{j1} \rightarrow B_0 + B_1.$$

We fix real numbers $\alpha_0, \alpha_1, \dots, \alpha_m$ with $\alpha_i \neq 0$ for $i = 1, \dots, m$ and define

$$\Omega = \{(\theta_1, \dots, \theta_m) \in [0, 1]^m : (\alpha_0 + \sum_{i=1}^m \alpha_i \theta_i) \in [0, 1] \text{ and } T : \prod_{j=1}^m A_{j\theta_j, q_j} \rightarrow B_{\theta, q} \text{ for some } q, q_1, \dots, q_m \text{ in } (0, \infty]\}.$$

The main results of Janson are that the set Ω is convex and that in the interior of Ω , T is bounded on real interpolation spaces.

A simple application of Janson's results is the following theorem on multi-linear forms. This result depends on the duality properties of Lorentz spaces which may be obtained, for example, from the general result on duality for real interpolation spaces in Bergh and Löfstrom [6, Theorem 4.7.1]. We will apply the next theorem result not to the forms Λ but to forms that are obtained by fixing some of the arguments of Λ . Thus, we state a result for more general multi-linear forms.

Theorem 2.6 *Let Λ be a multi-linear form which is defined at least on $(L^1(\mathbf{R}^\ell) \cap L^\infty(\mathbf{R}^\ell))^m$ and suppose that*

$$\Lambda(a_1, \dots, a_m) \leq A \prod_{i=1}^m \|a_i\|_{\eta_i}$$

for all η in $B(\theta, \delta) \cap \{\eta : \sum_{i=1}^m \eta_i = K\}$ for some K and $B(\theta, \delta) \subset [0, 1]^m$. Then for (q_1, \dots, q_m) satisfying $\sum_{i=1}^m \frac{1}{q_i} \geq 1$, we conclude that

$$\Lambda(a_1, \dots, a_m) \leq C \prod_{i=1}^m \|a_i\|_{\theta_i, q_i}.$$

The constant C depends on θ , (q_1, \dots, q_m) , m , and δ .

Proof. Since we assume that θ is an interior point of the cube $[0, 1]^m$, we have, in particular, that $0 < \theta_1$. We define an $(m-1)$ -linear operator T by

$$\int_{\mathbf{R}^\ell} T(a_2, \dots, a_m) a_1 dx = \Lambda(a_1, \dots, a_m).$$

Our assumption on Λ implies that we have that

$$T : \prod_{i=2}^m L^{1/\eta_i}(\mathbf{R}^k) \rightarrow L^{1/(1-\eta_1)}(\mathbf{R}^k), \quad \eta \in B_\delta(\theta) \cap \{\eta : \sum_i \eta_i = K\}.$$

Our hypotheses allow us to apply Theorem 2 from the article of S. Janson [13] and gives us that for q, q_2, \dots, q_m in $[1, \infty]$, we have

$$\|T(a_2, \dots, a_m)\|_{1-\theta_1, q} \leq C \prod_{i=2}^m \|a_i\|_{\theta_i, q_i}$$

provided $\sum_{i=2}^m 1/q_i \geq 1/q$. Recalling our definition of the operator T and the extension of Hölder's inequality to the Lorentz spaces, we obtain the estimate of the Theorem. ■

3 Estimates for the form Λ_n

The rest of this paper is devoted to the study of the form Λ_n defined (1.1). We will realize this form as a special case of the form introduced in (1.3) where some of the arguments a_j are taken from Lorentz spaces.

In this section we consider the form (1.3) where the functions a_j live on the complex plane, the number of functions is $4m+2$ and the vectors f_j lie in \mathbf{R}^{2m+1} and are defined by

$$\begin{aligned} f_{2j-1} &= e_j, & j &= 1, \dots, 2m+1 \\ f_{2j} &= e_j - e_{j+1}, & j &= 1, \dots, 2m \\ f_{4m+2} &= e_1 - e_2 + \dots + e_{2m+1}. \end{aligned}$$

The number of elements in our matroid is no longer m , but $4m+2$. The vectors f_j are elements of \mathbf{R}^{2m+1} and hence the parameter k in (1.3) is $2m+1$ and the parameter $\ell = 2$ as we have identified the complex plane \mathbf{C} with \mathbf{R}^2 . We let $M = \{f_j : j = 1, \dots, 4m+2\}$ and then Ω_M will be the matroid polytope for M

as introduced in section 2. We will show that the point $(1/2, \dots, 1/2)$ lies in the interior of the set Ω_M . This implies the desired estimate for the form, but without the stated dependence of the constant. The argument below is needed to show that the constant in (1.2) is of the form c^n .

We introduce a set P_δ which we will show lies in Ω_M . For $\delta > 0$, we let

$$P_\delta = \left\{ \theta \in \mathbf{R}^{4m+2} : \sum_{i=4j+1}^{4j+4} \theta_i = 2, \text{ for } j = 0, \dots, m-1, \right. \\ \left. \theta_{4m+1} + \theta_{4m+2} = 1, |\theta_i - 1/2| \leq \delta, i = 1, \dots, 4m+2 \right\}.$$

Theorem 3.1 *If $\delta \leq 1/10$, then $P_\delta \subset \Omega_M$.*

The proof begins with a few technical lemmata. In the following discussion, we will let $B_k = \{f_{4k+1}, \dots, f_{4k+4}\}$, for $k = 0, \dots, m-1$ denote a block of 4 vectors. In addition, it will be useful to view the set M as an ordered set and for $j \leq k$, let $[f_j, f_k] = \{f_i : j \leq i \leq k\}$ denote an interval in M .

Lemma 3.2 *If $S = [f_{2k-1}, f_{2k+2j-1}] = [e_k, e_{k+j}]$ is an interval in M , then we have*

$$\text{rank}(S) \geq 1/2 - 3\delta + \sum_{f_i \in S} \theta_i.$$

Proof. The proof proceeds by considering the four cases that arise when k and j are even and odd.

Case 1. Let k be even and j be even.

In this case, $S = \{e_k, e_k - e_{k+1}\} \cup (\cup_{i=k/2}^{(k+j)/2-2} B_i) \cup \{e_{k+j-1}, e_{k+j-1} - e_{k+j}, e_{k+j}\}$. It is clear that the (vector space) span of S is the subspace spanned by $e_k, e_{k+1}, \dots, e_{k+j}$ and thus $\text{rank}(S) = j+1$. We now consider $\sum_{f_i \in S} \theta_i$. Each block contributes 2 to the sum. From the definition of P_δ , we have $\theta_{2k-1} + \theta_{2k} \leq 1 + 2\delta$. Again, from the definition of P_δ , we have that $\theta_{2k+2j-3} + \theta_{2k+2j-2} + \theta_{2k+2j-1} = 2 - \theta_{2(k+j)} \leq 3/2 + \delta$. Thus, we have

$$\sum_{f_i \in S} \theta_i \leq (j+1) - 1/2 + 3\delta.$$

The conclusion of the lemma follows from this upper bound and the observation that $\text{rank}(S) = j+1$.

Case 2. Let k be even and j be odd.

We have $S = \{e_k, e_k - e_{k+1}\} \cup (\cup_{i=k/2}^{(k+j-3)/2} B_i) \cup \{e_{k+j}\}$ and again $\text{rank}(S) = j+1$. We have $(j-1)/2$ blocks in S . If $\theta \in P_\delta$, we may use the upper bound of $1/2 + \delta$ for the θ_i that do not correspond to blocks and obtain that

$$\sum_{f_i \in S} \theta_i \leq (j+1) - 1/2 + 3\delta.$$

Case 3. Let k be odd and j be even.

In this case we have $S = (\cup_{i=(k-1)/2}^{(k+j-3)/2} B_i) \cup \{e_{k+j}\}$. As we have $j/2$ blocks and one extra vector, it is easy to obtain the upper bound

$$\sum_{f_i \in S} \theta_i \leq (j+1) - 1/2 + \delta.$$

As $\text{rank}(S) = j+1$, the estimate of the Lemma follows.

Case 4. Let k be odd and j be odd.

In this case we have

$$S = (\cup_{i=(k-1)/2}^{(k+j)/2-2} B_i) \cup \{e_{k+j-1}, e_{k+j-1} - e_{k+j}, e_{k+j}\}.$$

We have $(j-1)/2$ blocks and three extra vectors, thus we have

$$\sum_{f_i \in S} \theta_i \leq (j+1) - 1/2 + 3\delta.$$

As $\text{rank}(S) = j+1$, the estimate follows again. ■

Lemma 3.3 *Let $S \subset M \setminus \{e_1 - e_2 + e_3 - e_4 + \dots e_{2m+1}\}$ be a dependent set. Suppose that $\text{span}(S) = S$, then we have that S contains a set of the form $\{e_k, e_k - e_{k+1}, e_{k+1}\}$ for some k .*

Proof. Suppose that S contains no set of the form $\{e_k, e_k - e_{k+1}, e_{k+1}\}$. Because $\text{span}(S) = S$, it follows that S contains at most one element from each the sets $\{e_k, e_k - e_{k+1}, e_{k+1}\}$, $k = 1, \dots, 2m$. This contradicts our assumption that S is a dependent set. ■

Lemma 3.4 *Let $S \subset M \setminus \{e_1 - e_2 + \dots + e_{2n+1}\}$. If $\text{span}(S) = S$, then we may write*

$$S = \bigcup_{i=0}^k S_i$$

where the collection $\{S_i\}$ is pairwise disjoint, for each $i = 1, \dots, k$, $S_i = [e_{s_i}, e_{t_i}]$ is an interval and the set S_0 is independent. For this decomposition, we have

$$\sum_{i=0}^k \text{rank}(S_i) = \text{rank}(S).$$

Proof. If S is linearly dependent, then by Lemma 3.3, we may find an index k so that $\{e_k, e_k - e_{k+1}, e_{k+1}\}$ lies in S . Since $\text{span}(S) = S$, if $\{e_k, e_k - e_{k+1}, e_{k+1}\} \subset S$ and e_{k-1} lies in S , then $e_{k-1} - e_k$ also lies in S . Similarly, either e_{k+1} and $e_{k+1} - e_{k+2}$ both lie in S or both do not lie in S . We let S_1 be the maximal interval of the form $[e_s, e_t]$ which contains $\{e_k, e_k - e_{k+1}, e_{k+1}\}$. It is clear that we have $\text{rank}(S) = \text{rank}(S_1) + \text{rank}(S \setminus S_1)$. If $S \setminus S_1$ is dependent, then we repeat the above argument to find a interval S_2 . We continue until $S \setminus (\cup S_i)$ is independent and then name this set S_0 . It is clear that we have the rank of S is the sum of the ranks of the subsets. ■

Proposition 3.5 Suppose that $\text{span}(S) = S$, $e_1 - e_2 + \dots + e_{2m+1} \in S$ and

$$e_1 - e_2 + \dots + e_{2m+1} \in \text{span}(S \setminus \{e_1 - e_2 + \dots + e_{2m+1}\}).$$

If $S \setminus \{e_1 - e_2 + \dots + e_{2m+1}\} = \cup_i S_i$ and each S_i is an interval of the form $[e_s, e_t]$, then $S = B$.

Proof. Since $\text{span}(S) = S$, if $e_k - e_{k+1} \notin S$, then also $e_k \notin S$ or $e_{k+1} \notin S$. If e_j is not in S , then we have that $e_{j-1} - e_j$ and $e_j - e_{j+1}$ are not in S_i for any i . But this implies that no vector in $S \setminus \{e_1 - e_2 + \dots + e_{2m+1}\}$ has a non-zero e_j component and thus $e_1 - e_2 + \dots + e_{2m+1}$ is not in $\text{span}(S \setminus \{e_1 - e_2 + \dots + e_{2m+1}\})$. ■

We are ready to give the proof of our Theorem.

Proof of Theorem 3.1. To show $P_\delta \subset \Omega_M$, we use the characterization of the matroid polytope by the inequalities in (2.5). Note that it suffices to consider these inequalities for sets which satisfy $\text{span}(S) = S$.

We begin by considering sets $S \subset M \setminus \{e_1 - e_2 + \dots + e_{2m+1}\}$. By Lemma 3.4, we may write $S = \cup_{i=0}^k S_i$ where the set S_0 is independent and each S_i is an interval of the form $[e_s, e_t]$. We let L denote the cardinality of S_0 . Using Lemma 3.4 and then Lemma 3.2 for each of the intervals in this decomposition, we obtain

$$\begin{aligned} \text{rank}(S) &= \sum_{i=0}^k \text{rank}(S_i) \\ &\geq \text{rank}(S_0) + k(1/2 - 3\delta) + \sum_{i=1}^k \sum_{f_j \in S_i} \theta_j \\ &\geq L(1/2 - \delta) + k(1/2 - 3\delta) + \sum_{f_j \in S} \theta_j. \end{aligned}$$

In the last inequality, we use that S_0 is independent and each $\theta_j \leq 1/2 + \delta$. From this, it is clear that we have the inequality (2.5) when $\delta \leq 1/6$.

Now we consider the case when $e_1 - e_2 + \dots + e_{2m+1} \in S$ and thus we write

$$S = S' \cup \{e_1 - e_2 + \dots + e_{2m+1}\}.$$

If $\{e_1 - e_2 + \dots + e_{2m+1}\} \notin \text{span}(S')$, then we have

$$\text{rank}(S') \geq \sum_{f_i \in S'} \theta_i$$

by the previous case the estimate (2.5) for S follows since $\text{rank}(S) = 1 + \text{rank}(S') \geq \theta_{4m+2} + \text{rank}(S')$.

Finally, we consider the case when $e_1 - e_2 + \cdots + e_{2m+1} \in \text{span}(S')$. In this case, we write $S' = \cup_{i=0}^k S_i$ as in Lemma 3.4. Using Lemma 3.2 we obtain

$$\text{rank}(S) = \text{rank}(S') \geq \sum_{f_i \in S'} \theta_i + k(1/2 - 3\delta) + L(1/2 - \delta).$$

If $L = 0$, then $S = M$ by Proposition 3.5 and thus we have $\text{rank}(S) = \sum_{f_i \in S} \theta_i = 2m + 1$ from the definition of P_δ .

In the remaining cases, we want $k(1/2 - 3\delta) + L(1/2 - \delta) \geq \theta_{4m+2}$ which is implied by

$$k(1/2 - 3\delta) + L(1/2 - \delta) \geq 1/2 + \delta. \quad (3.6)$$

If $L = 1$, then $k \geq 1$ as otherwise S' contains only one vector and we cannot have $e_1 - e_2 + \cdots + e_{2m+1} \in \text{span}(S')$. If $L = k = 1$, then we have (3.6) if $\delta \leq 1/10$. If $k \geq 2$, then we have (3.6) if $\delta \leq 1/6$. ■

In order to apply Corollary 2.2 to the form associated to the matroid M , we will need to compute the determinants arising in Proposition 2.1 for the matroid M .

Lemma 3.7 *Let $B \subset M$ be a basis and let \hat{B} be the matrix whose rows are the vectors in B . We have $|\det \hat{B}| = 1$.*

Proof. We begin by ordering the vectors in B in the following way. We let f_{j_1} be the first vector on the list $e_1, e_1 - e_2, e_1 - e_2 + \cdots + e_{2m+1}$ that appears in B . Now given f_{j_1}, \dots, f_{j_k} , we choose $f_{j_{k+1}}$ to be the first vector on the list $e_k - e_{k+1}, e_{k+1}, e_{k+1} - e_{k+2}, e_1 - e_2 + \cdots + e_{2m+1}$ that is an element in the set $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$. We claim that this procedure continues until all of the vectors in B have been chosen.

To establish the claim, we argue by contradiction. Suppose that for some k , there is no choice for $f_{j_{k+1}}$. We claim that $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$ is contained in the span of $\{e_{k+2}, \dots, e_{2m+1}\}$. If we have this containment, then the rank of $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$ is at most $2m - k$ and the rank of $\{f_{j_1}, \dots, f_{j_k}\}$ is at most k and we obtain a contradiction with our assumption that B is a basis. Because we are assuming there is no choice for $f_{j_{k+1}}$, the vectors $\{e_k - e_{k+1}, e_k, e_{k+1} - e_{k+2}, e_1 - e_2 + \cdots + e_{2m+1}\}$ are not in $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$. In addition, none of the vectors $e_{i-1} - e_i$, $i = 2, \dots, k$ can be in $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$ as the vector $e_{i-1} - e_i$ has first priority when we choose f_{j_i} . For the same reason, we do not have e_1 in $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$. Finally, suppose for some i , $2 \leq i \leq k$, e_i is in $B \setminus \{f_{j_1}, \dots, f_{j_k}\}$. This implies $f_{j_i} = e_{i-1} - e_i$ as this is the only vector with higher priority than e_i . Working backwards, we see that $f_{j_{i-1}}$ is either e_{i-1} or $e_{i-2} - e_{i-1}$ and continuing we find that for some j with $1 \leq j < i$, we have the vectors $e_j, e_j - e_{j+1}, e_{j+1} - e_{j+2}, \dots, e_{i-1} - e_i, e_i$ in B . This is a dependent set of vectors and contradicts our assumption that B is basis. Thus our claim holds.

We let \hat{B} be the matrix whose rows are the vectors $f_{j_1}, \dots, f_{j_{2m+1}}$. We claim that $|\det \hat{B}| = 1$ and consider several cases to give the proof.

Case 1. Suppose $e_1 - e_2 + \cdots + e_{2m+1}$ is not in B .

In this case, we show how to use column operations to reduce \hat{B} to a lower triangular matrix. Suppose $\hat{B}_{i,i+1} = 0$ for $i = 1, \dots, k-1$ and that $\hat{B}_{k,k+1} \neq 0$. In this case, we have $f_{j_k} = e_k - e_{k+1}$ and $\hat{B}_{k+1,k} = 0$ since we have $f_{j_{k+1}} \neq e_k - e_{k+1}$ and $f_{j_{k+1}} \neq e_1 - e_2 + \dots + e_{2m+1}$. We replace the $(k+1)$ st column, $\hat{B}_{\cdot,k+1}$ by the sum $\hat{B}_{\cdot,k} + \hat{B}_{\cdot,k+1}$ and obtain a matrix with $\hat{B}_{k,j} = 0$ for $j \leq k-1$ and $\hat{B}_{k,k} = \pm 1$. Continuing in this manner, we obtain a lower triangular matrix with entries of $+1$ or -1 on the diagonal. It follows that $\det \hat{B} = \pm 1$.

Case 2. Suppose $e_1 - e_2 + \dots + e_{2m+1}$ is in B .

In this case, we fix k so that $f_{j_k} = e_1 - e_2 + \dots + e_{2m+1}$ and write the matrix

$$\hat{B} = \begin{pmatrix} A & C \\ 0 & D \end{pmatrix}$$

where the block A is of size $k \times k$, C is of size $k \times (2m+1-k)$ and D is of size $(2m+1-k) \times (2m+1-k)$. Note that our ordering of the basis guarantees that the lower left block is 0. We may apply the same argument used above and find column operations which reduce the matrix A to a lower triangular matrix with diagonal entries of ± 1 . Observe that as we are either leaving column i unchanged or replacing column i by the sum of column i and $i-1$, the entries in the k th row $\hat{B}_{k,i}$, $i = 1, \dots, k$ will be either 0, 1 or -1 . Since we assume that B is a basis, we cannot have $\hat{B}_{k,k} = 0$. We apply the same procedure to reduce the block D to a lower triangular matrix with diagonal entries of $+1$ or -1 . Since the blocks A and D have determinant ± 1 , it follows that the determinant of the matrix $\hat{B} = 0$. ■

As a consequence of the previous lemma, we immediately obtain the following estimate for the form (1.3) specialized to the matroid we are studying in this section

$$\Lambda(a_1, \dots, a_{4m+2}) \leq \prod_{i=1}^{4m+2} \|a_i\|_{\theta_i}, \quad \theta \in \Omega_M.$$

Finally, we are ready to give the proof of our main theorem.

Theorem 3.8 *Suppose that θ lies in the interior of P_δ and that the indices q_1, \dots, q_{4m+2} satisfy $\sum_{j=1}^4 1/q_{4k+j} \geq 1$ for $k = 0, \dots, m-1$. Then we may find a constant $c = c_\theta$ so that*

$$\begin{aligned} \Lambda(a_1, \dots, a_{4m+2}) &\leq c^n \|a_{4m+1}\|_{\theta_{4m+1}} \|a_{4m+2}\|_{\theta_{4m+2}} \\ &\quad \cdot \prod_{j=0}^{m-1} (\|a_{4j+1}\|_{\theta_{4j+1}, q_{4j+1}} \|a_{4j+2}\|_{\theta_{4j+2}, q_{4j+2}} \|a_{4j+3}\|_{\theta_{4j+3}, q_{4j+3}} \|a_{4j+4}\|_{\theta_{4j+4}, q_{4j+4}}) \end{aligned}$$

The constant c depends on $\max\{1/(1/10 - |\theta_i - 1/2|) : i = 1, \dots, 4m+2\}$.

Proof. By Theorem 3.1, we have $P_\delta \subset \Omega_M$ if $\delta \leq 1/10$. Thus, we have that θ is an interior point of Ω_M . We will prove by induction that if $\eta \in P_\delta$ and

$\eta_i = \theta_i$ for $i = 1, \dots, 4k$, then we have

$$\Lambda(a_1, \dots, a_{4m+2}) \leq c^k \prod_{i=4k+1}^{4m+2} \|a_i\|_{\eta_i} \quad (3.9)$$

$$\cdot \prod_{j=0}^{k-1} (\|a_{4j+1}\|_{\theta_{4j+1}, q_{4j+1}} \|a_{4j+2}\|_{\theta_{4j+2}, q_{4j+2}} \|a_{4j+3}\|_{\theta_{4j+3}, q_{4j+3}} \|a_{4j+4}\|_{\theta_{4j+4}, q_{4j+4}})$$

We use $k = 0$ as the base case. The estimate we need holds for $\theta \in \Omega_M$ and follows from Corollary 2.2 and Lemma 3.7.

Now suppose that the estimate (3.9) holds for $k < m$ and we show how to obtain the same result for $k + 1$. Fix $a_1, \dots, a_{4k}, a_{4k+5}, \dots, a_{4m+2}$ and set

$$\Lambda_0(a_{4k+1}, \dots, a_{4k+4}) = \Lambda(a_1, \dots, a_{4m+2}).$$

We consider the three directions

$$\begin{aligned} u^1 &= (1, 1, -1, -1) \\ u^2 &= (1, -1, 1, -1) \\ u^3 &= (1, -1, -1, 1). \end{aligned}$$

We will need the six points

$$(\theta_{4k+1}, \theta_{4k+2}, \theta_{4k+3}, \theta_{4k+4}) \pm \tau u^j, \quad j = 1, 2, 3$$

where $\tau = \min\{1/10 - |\theta_{4k+i} - 1/2| : i = 1, \dots, 4\}$. Each of these six points lies in P_δ . As the vectors u_j give three linearly independent directions, the convex hull of these six points give us a neighborhood of $(\theta_{4k+1}, \theta_{4k+2}, \theta_{4k+3}, \theta_{4k+4})$ in P_δ . Applying our induction hypothesis and then Theorem 2.6 gives that

$$\Lambda_0(a_{4k+1}, \dots, a_{4k+4}) \leq c^{k+1} \prod_{i=4k+4}^{4m+2} \|a_i\|_{\eta_i}$$

$$\cdot \prod_{j=0}^k (\|a_{4j+1}\|_{\theta_{4j+1}, q_{4j+1}} \|a_{4j+2}\|_{\theta_{4j+2}, q_{4j+2}} \|a_{4j+3}\|_{\theta_{4j+3}, q_{4j+3}} \|a_{4j+4}\|_{\theta_{4j+4}, q_{4j+4}})$$

The Theorem now follows by induction. \blacksquare

Finally, we observe that this theorem implies the following estimate for the form Λ_n defined in (1.1).

Corollary 3.10 *If $1/p + 1/p' = 1$, and $|1/p - 1/2| < 1/10$ we have*

$$\Lambda_n(t, q_0, q_1, \dots, q_{2n}) \leq c^n \|t\|_{1/p} \|q_0\|_{1/p'} \prod_{j=0}^{2n} \|q_j\|_{1/2}.$$

Proof. We observe that in our previous Theorem, we may let $\theta_j = 1/2$ for $j = 1, \dots, 4m$. The functions a_{2j} , $j = 1, \dots, m$ are chosen to be $1/x$ or $1/\bar{x}$ which lie in $L^{2,\infty}(\mathbb{C})$. With these choices, the estimate follows immediately. \blacksquare

Let \mathcal{T} be the map that takes a potential Q to the scattering data S as defined, for example, in Beals and Coifman [2] or Sung [18, 19, 20]. Combining the estimate of Corollary 3 with the method of proof in the work of Brown [8], we obtain the following result.

Corollary 3.11 *Let $1/10 > 1/p - 1/2 \geq 0$, then there exists N , a neighborhood of 0 in $L^p(\mathbf{C}) \cap L^2(\mathbf{C})$ so that*

$$\|\mathcal{T}(q)\|_{1/p'} \leq \frac{C}{1 - c^2 \|q\|_{1/2}^2} \|q\|_{1/p}.$$

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