The mixed problem for the Laplacian in Lipschitz domains

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Abstract

We consider the mixed boundary value problem, or Zaremba's problem for the Laplacian in a bounded Lipschitz domain Ω in $\mathbf{R}^n$, $n \geq 2$. We decompose the boundary $\partial\Omega = D \cup N$ with D and N disjoint. The boundary between D and N is assumed to be a Lipschitz surface in $\partial\Omega$. We specify Dirichlet data on D in the Sobolev space $W^{1,p}(D)$ and Neumann data in $L^p(N)$. Under these conditions, we find $q_0 > 1$ so that the mixed problem has a unique solution and the non-tangential maximal function of the gradient lies in $L^p(\partial\Omega)$ for $1 < p < q_0$. We also obtain results when the data comes from Hardy spaces for $p = 1$.

Keywords: Mixed boundary value problem, Laplacian

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1 Introduction

Over the past thirty years, there has been a great deal of interest in studying boundary value problems for the Laplacian in Lipschitz domains. A fundamental paper of Dahlberg [8] treated the Dirichlet problem. Jerison and Kenig [15] treated the Neumann problem and provided a regularity result for the Dirichlet problem. Another boundary value problem of interest is the mixed problem or Zaremba's problem where we specify Dirichlet data on part of the boundary and Neumann data on the remainder of the boundary. To state this boundary value problem, we let Ω be a bounded open set in $\mathbf{R}^n$ and suppose that we have written $\partial\Omega = D \cup N$ where D is an open subset of the boundary and

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$N = \partial\Omega \setminus D$. We consider the boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = f_D, & \text{on } D \\ \dfrac{\partial u}{\partial \nu} = f_N, & \text{on } N. \end{cases} \tag{1.1}$$

The study of the mixed problem in Lipschitz domains appears as problem 3.2.15 in Kenig's CBMS lecture notes [16]. Recall that simple examples show that we cannot expect to find solutions with the gradient in L^2 of the boundary. For example, the function $\mathrm{Re}\sqrt{z}$ on the upper half-plane has zero Neumann data on the positive real axis and zero Dirichlet data on the negative real axis but the gradient is not locally square integrable on the boundary of the upper half-space. This appears to present a technical problem as the standard technique for studying boundary value problems has been the Rellich identity which produces estimates in L^2.

In 1994, one of the authors observed that the Rellich identity could be used to study the mixed problem in a restricted class of Lipschitz domains. Based on this work and the methods used by Dahlberg and Kenig to study the Neumann problem [9], J. Sykes [30, 31] established results for the mixed problem in Lipschitz graph domains. I. Mitrea and M. Mitrea [24] have studied the mixed problem for the Laplacian with data taken from a large family of function spaces. Results have also been obtained for the Lamé system [3] and the Stokes system [4]. More recently, Lanzani, Capogna and Brown [18] used a variant of the Rellich identity to establish an estimate for the mixed problem in two dimensional graph domains when the data comes from weighted L^2 spaces and the Lipschitz constant is less than one. The present work also relies on weighted estimates, but uses a simpler, more flexible approach that applies to all Lipschitz domains.

Several other authors have treated the mixed problem in various settings. Verchota and Venouziou [32] treat a large class of three dimensional polyhedral domains under the condition that the Neumann and Dirichlet faces meet at an angle of less than π. Maz'ya and Rossman [20, 21, 22] have studied the Stokes system in polyhedral domains. Finally, we note that Savaré [28] has shown that solutions may be found in the Besov space $B^{2,\infty}_{3/2}$ in smooth domains. This result seems to be very close to optimal. The example $\mathrm{Re}\sqrt{z}$ described above shows that we cannot hope to obtain an estimate in the Besov space $B^{2,2}_{3/2}$.

We outline the rest of the paper and describe the main tools of the proof. Our first main result is an existence result for the mixed problem when the Neumann data is an atom for a Hardy space. We begin with the weak solution and use Jerison and Kenig's results for the Dirichlet problem and Neumann problem [15] to obtain estimates for the gradient of the solution on the interior of D or N. This leads to a weighted estimate where the weight is a power of the distance to the common boundary between D and N. The estimate involves a term in the interior of the domain. We handle this term by showing that the gradient of a weak solution lies in $L^p(\Omega)$ for some $p > 2$. The $L^p(\Omega)$ estimates for the gradient of a weak solution are proven in section 3 using the reverse Hölder technique of Gehring and Giaquinta. Using this weighted estimate for solutions of the mixed problem, we obtain existence for solutions with Hardy space data by extending the methods of Dahlberg and Kenig [9]. Uniqueness of solutions is proven in section 5.

With the Hardy space results in hand, we establish the existence of solutions to the mixed problem when the Neumann data is in $L^p(N)$ and the Dirichlet data is in the Sobolev space $W^{1,p}(D)$. This is done in sections 6 and 7 by adapting the reverse Hölder technique used by Shen to study boundary value problems for elliptic systems [29]. The novel feature in our work is that we are able to substitute an estimate in Hardy spaces while Shen's work begins with existence in L^2.

2 Definitions and preliminaries

We will consider several formulations of the mixed problem (1.1). Our goal is to obtain solutions where the gradient lies $L^p(\partial\Omega)$ for $1 < p < q_0$ for some $q_0 > 1$. However, our argument begins with a weak solution where the gradient lies in $L^2(\Omega)$. We will show that under appropriate assumptions on the data, this solution will have a gradient in $L^p(\partial\Omega)$.

We describe a weak formulation of the boundary value problem (1.1). The results of section 3 will hold for solutions of divergence form operators. Thus, we define weak solutions in this more general setting. For $k = 1, 2, \dots,$ we use $W^{k,p}(\Omega)$ to denote the Sobolev space of functions having k derivatives in $L^p(\Omega)$. For D a subset of the boundary, we let $W^{1,2}_D(\Omega)$ be the closure in $W^{1,2}(\Omega)$ of smooth functions which are zero on D. In the weak formulation, we take the Dirichlet data f_D from the Sobolev space $W^{1,2}(\Omega)$. We let $W^{1/2,2}_D(\partial\Omega)$ be the restrictions to $\partial\Omega$ of the space $W^{1,2}_D(\Omega)$. We define $W^{-1/2,2}_D(\partial\Omega)$ to be the dual of $W^{1/2,2}_D(\partial\Omega)$. The Neumann data f_N will be taken from the space $W^{-1/2,2}_D(\partial\Omega)$. If $A(x)$ is a symmetric matrix with bounded, measurable entries and satisfies the ellipticity condition $M|\xi|^2 \geq A(x)\xi \cdot \xi \geq M^{-1}|\xi|^2$ for some $M > 0$ and all $\xi \in \mathbf{R}^n$, we consider the problem

$$\begin{cases} \operatorname{div} A\nabla u = 0, & \text{in } \Omega \\ u = f_D, & \text{on } D \\ A\nabla u \cdot \nu = f_N, & \text{on } N. \end{cases}$$

We say that u is a *weak solution* of this problem if $u - f_D \in W^{1,2}_D(\Omega)$ and we have

$$\int_\Omega A\nabla u \cdot \nabla v \, d\sigma = -\langle f_N, v\rangle_{\partial\Omega}, \qquad v \in W^{1,2}_D(\Omega).$$

To establish existence of weak solutions of the mixed problem, we will assume the coercivity condition

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}, \qquad u \in W^{1,2}_D(\Omega). \tag{2.1}$$

Under this assumption, the existence and uniqueness of weak solutions to (1.1) is a consequence of the Lax-Milgram theorem. In our applications, Ω will be a connected, bounded domain whose boundary is locally the graph of a Lipschitz function and D will be an open subset of the boundary. These assumptions are sufficient to ensure that (2.1) holds.

If f_N is a function on N, then we may identify f_N with an element of the space $W_D^{-1/2,2}(\partial\Omega)$ by

$$\langle f_N, \phi\rangle_{\partial\Omega} = \int_N f_N \phi \, d\sigma, \qquad \phi \in W_D^{1/2,2}(\partial\Omega).$$

From Sobolev embedding we have $W_D^{1/2,2}(\partial\Omega) \subset L^p(\partial\Omega)$, where $p = 2(n-1)/(n-2)$ if $n \geq 3$ or $p < \infty$ when $n = 2$. Thus the integral on the right-hand side will be well-defined if we have f_N in $L^{2(n-1)/n}(\partial\Omega)$ when $n \geq 3$ or $L^p(\partial\Omega)$ for any $p > 1$ when $n = 2$.

We say that a bounded, connected open set Ω is a *Lipschitz domain* with Lipschitz constant M if the boundary is locally the graph of a Lipschitz function. To make this precise, we define a *coordinate cylinder* $Z_r(x)$ to be a set of the form $Z_r(x) = \{y : |y' - x'| < r,\ |y_n - x_n| < (1+M)r\}$. We use coordinates $(x', x_n) = (x_1, x'', x_n) \in \mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}$ and assume that this coordinate system is a translation and rotation of the standard coordinates. For each x in the boundary we assume that we may find a coordinate cylinder and a Lipschitz function ϕ with Lipschitz constant M so that

$$\begin{aligned} \Omega \cap Z_r(x) &= \{(y', y_n) : y_n > \phi(y')\} \cap Z_r(x) \\ \partial\Omega \cap Z_r(x) &= \{(y', y_n) : y_n = \phi(y')\} \cap Z_r(x). \end{aligned}$$

In the mixed problem, the boundary between D and N is another important feature of the domain. We assume that D is a relatively open subset of $\partial\Omega$ and let Λ be the boundary (relative to $\partial\Omega$) of D. For each x in Λ, we require that a coordinate cylinder centered at x have some additional properties. We ask that there be a coordinate system (x_1, x'', x_n), a coordinate cylinder $Z_r(x)$, a function ϕ as above and also a function $\psi : \mathbf{R}^{n-2} \to \mathbf{R}$ with $\|\nabla\psi\|_\infty \leq M$ so that

$$\begin{aligned} Z_r(x) \cap D &= \{(y_1, y'', y_n) : y_1 > \psi(y''),\ y_n = \phi(y')\} \cap Z_r(x) \\ Z_r(x) \cap N &= \{(y_1, y'', y_n) : y_1 \leq \psi(y''),\ y_n = \phi(y')\} \cap Z_r(x). \end{aligned}$$

We fix a covering of the boundary by coordinate cylinders $\{Z_{r_i}(x_i)\}_{i=1}^N$ so that each $Z_{100r_i}(x_i)$ is also a coordinate cylinder and let $r_0 = \min\{r_i : i = 1, \ldots, N\}$ be the smallest radius in the collection. We will use $\delta(y) = \operatorname{dist}(y, \Lambda)$ to denote the distance from a point y to Λ. We will let $B_r(x) = \{y : |y - x| < r\}$ denote the standard ball in $\mathbf{R}^n$ and then $\Delta_r(x) = B_r(x) \cap \partial\Omega$ will denote a *surface ball.* Throughout this paper we will need to be careful of several points. The surface balls may not be connected and we will use the notation $\Delta_r(x)$ where

x may not be on the boundary. We use $\Psi_r(x)$ to stand for $B_r(x) \cap \Omega$. Since Λ is a Lipschitz graph, it has the property

$$\text{If } x \in \Lambda \text{ and } 0 < r < r_0, \text{ then } \sigma(\Delta_r(x) \cap D) > M^{-1} r^{n-1}. \tag{2.2}$$

Here and throughout this paper, we use σ for surface measure.

Our main tool for estimating solutions will be the non-tangential maximal function. For $\alpha > 0$ and $x \in \partial\Omega$, we define a *non-tangential approach region* by

$$\Gamma(x) = \{y \in \partial\Omega : |x - y| \leq (1 + \alpha)\, \mathrm{dist}(y, \partial\Omega)\}.$$

Given a function u defined on Ω, we define the *non-tangential maximal function* by

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|, \qquad x \in \partial\Omega.$$

It is well-known that for different values of α, the non-tangential maximal functions have comparable L^p-norms. Thus, the dependence on α is not important for our purposes and we suppress the value of α in our notation.

Many of our estimates will be of a local, scale invariant form and hold for r less than a multiple of r_0 and with a constant that depends only on M and the dimension, n. Global estimates will also depend on the collection of coordinate cylinders which cover $\partial\Omega$ and the constant in (2.1).

Before stating the main theorem, we recall the definition of atoms and atomic Hardy spaces. We say that a is an *atom for the boundary* $\partial\Omega$ if a is supported in a surface ball $\Delta_r(x)$ for some x in $\partial\Omega$, $\|a\|_{L^\infty(\partial\Omega)} \leq 1/\sigma(\Delta_r(x))$ and $\int_{\partial\Omega} a\, d\sigma = 0$.

When we consider the mixed problem, we will want to consider atoms for the subset N. We say that *a is an atom for N* if a is the restriction to N of a function A which is an atom for $\partial\Omega$. For a subset of $N \subset \partial\Omega$, the Hardy space $H^1(N)$ is the collection of functions f which can be represented as $\sum \lambda_j a_j$ where each a_j is an atom for N and the coefficients satisfy $\sum |\lambda_j| < \infty$. This includes, of course, the case where $N = \partial\Omega$ and then we obtain the standard definition. It is easy to see that the Hardy space $H^1(N)$ is the restriction to N of elements of the Hardy space $H^1(\partial\Omega)$. We give a similar definition for the Hardy-Sobolev space $H^{1,1}$ of functions which have one derivative in H^1. We say A is an *atom for $H^{1,1}(\partial\Omega)$* if A is supported in a surface ball $\Delta_r(x)$ and $\|\nabla_t A\|_{L^\infty(\partial\Omega)} \leq 1/\sigma(\Delta_r(x))$. We say that A is an *atom for $H^{1,1}(D)$* if A is the restriction to D of an atom for $\partial\Omega$. Again, the space $H^{1,1}(D)$ is the collection generated by taking sums of atoms with coefficients in ℓ^1. See the article of Coifman and Weiss [7] for more information about Hardy spaces.

We are now ready to state our main theorem.

Theorem 2.3 *Let Ω, N and D be as described above.*

a) There exists $q_0 > 1$ so that for p in the range $1 < p < q_0$, given $f_N \in L^p(N)$ and $f_D \in W^{1,p}(D)$ we may find a solution u to (1.1) with $(\nabla u)^* \in L^p(\partial\Omega)$. The solution u satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{L^p(N)} + \|f_D\|_{W^{1,p}(D)}).$$

The solution is unique in the class of functions with $(\nabla u)^* \in L^p(\partial\Omega)$.

b) If f_N *lies in the Hardy space* $H^1(N)$ *and* f_D *lies in the Hardy-Sobolev space* $H^{1,1}(D)$*, then we may find a solution of the mixed problem* u *which satisfies*

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \leq C(\|f_N\|_{H^1(N)} + \|f_D\|_{H^{1,1}(D)}).$$

The solution is unique in the class of functions with $(\nabla u)^* \in L^1(\partial\Omega)$.

Proof. We begin by recalling that for the Dirichlet problem with data from a Sobolev space, we obtain non-tangential maximal function estimates for the gradient. This is treated for $p = 2$ by Jerison and Kenig [15] and for $1 < p < 2$ by Verchota [33, 34]. The Hardy space problem was studied by Dahlberg and Kenig [9]. Using these results, it suffices to prove Theorem 2.3 in the case when the Dirichlet data is zero.

The existence for Neumann data from the atomic Hardy space and zero Dirichlet data is given in Theorem 4.14 and for L^p data appears in section 7. It suffices to establish uniqueness when $p = 1$ and this is treated in Theorem 5.1. ∎

3 Higher integrability of the gradient of a weak solution.

It is well-known that one can obtain higher integrability of the gradient of weak solutions. An early result of this type is due to Meyers [23]. We use the reverse Hölder technique introduced by Gehring [13] and, in particular, the formulation from Giaquinta [14, p. 122]. At a few points of the proof it will be simpler if we are working in a coordinate cylinder Z where we have that $\partial\Omega \cap Z$ lies in a hyperplane. Thus, we will establish results for divergence form elliptic operators with bounded measurable coefficients as this class is preserved by a change of variable that will flatten part of the boundary.

We define an operator P which takes functions on $\partial\Omega$ to functions in Ω by

$$Pf(x) = \sup_{s>0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| \, d\sigma$$

and a local version of P by

$$P_r f(x) = \sup_{r>s>0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| \, d\sigma.$$

On the boundary, we have that Pf is the Hardy-Littlewood maximal function

$$Mf(x) = Pf(x) = \sup_{s>0} \frac{1}{s^{n-1}} \int_{\Delta_s(x)} |f| \, d\sigma, \qquad x \in \partial\Omega.$$

The following result is probably well-known, but we could not find a reference.

Lemma 3.1 *For $p > 1$, $1 \le q \le pn/(n-1)$, $x \in \partial\Omega$ and $r < r_0$, we have*

$$\left(\fint_{\Psi_r(x)} |P_r f|^q \, dy \right)^{1/q} \le C \left(\frac{1}{r^{n-1}} \int_{\Delta_{2r}(x)} f^p \, d\sigma \right)^{1/p}. \tag{3.2}$$

The constant in this estimate depends only on the Lipschitz constant M and the dimension.

Proof. We begin by considering the case where $\Omega = \{(y', y_n) : y_n > 0\}$ is a half-space. We use coordinates $y = (y', y_n)$ and we claim that

$$\begin{aligned} Pf(y', y_n) &\le Mf(y', 0) && (3.3) \\ Pf(y) &\le C\|f\|_{L^p(\partial\Omega)} y_n^{(1-n)/p}, \qquad y_n > 0. && (3.4) \end{aligned}$$

The estimate (3.3) follows easily since $\Delta_s((y', y_n)) \subset \Delta_s((y', 0))$. To establish the second estimate, we observe that if $s < y_n$, then $\Delta_s(y) = \emptyset$ and hence

$$Pf(y) = \sup_{s > y_n} \frac{1}{s^{n-1}} \int_{\Delta_s(y)} |f| \, d\sigma \le C y_n^{(1-n)/p} \|f\|_{L^p(\partial\Omega)}.$$

The second inequality follows from Hölder's inequality.

We claim that we have the following weak-type estimate for Pf,

$$|\{x \in \Omega : Pf(x) > \lambda\}| \le C\|f\|^p_{L^p(\partial\Omega)} \lambda^{-pn/(n-1)}, \qquad \lambda > 0. \tag{3.5}$$

To prove (3.5), we may assume $\|f\|_{L^p(\partial\Omega)} = 1$. With this normalization, the observation (3.4) implies that $\{y' : Pf(y', y_n) > \lambda\} = \emptyset$ if $y_n > c\lambda^{-p/(n-1)}$. Thus, we may use Fubini's theorem to write

$$\begin{aligned} |\{Pf > \lambda\}| &= \int_0^{c\lambda^{-p/(n-1)}} \sigma(\{y' : Pf(y', y_n) > \lambda\}) \, dy_n \\ &\le C \int_0^{c\lambda^{-p/(n-1)}} \sigma(\{y' : Mf(y', 0) > c\lambda\}) \, dy_n \\ &= C\lambda^{-pn/(n-1)} \end{aligned}$$

where we used (3.3), the weak-type (p, p) inequality for the maximal operator on $\mathbf{R}^{n-1}$ and our normalization of the L^p-norm of f.

From the weak-type estimate (3.5) and the Marcinkiewicz interpolation theorem we obtain

$$\|Pf\|_{L^{pn/(n-1)}(\Omega)} \le \|f\|_{L^p(\mathbf{R}^{n-1})}. \tag{3.6}$$

To obtain the estimate (3.2), we observe that if $y \in B_r(x)$ then $B_r(y) \subset B_{2r}(x)$ and hence

$$P_r f(y) \le P_r(\chi_{\Delta_{2r}(x)} f)(y), \qquad y \in B_r(x).$$

Thus the result (3.2) with Ω a half-space follows from (3.6) and Hölder's inequality.

Finally, to obtain the local result on a general Lipschitz domain, one may change variables so that the boundary is flat near $\Delta_r(x)$. ∎

We recall several versions of the Poincaré and Sobolev inequalities.

Lemma 3.7 *Let Ω be a convex domain of diameter d. Suppose that $S \subset \bar{\Omega}$ is a set with the properties that for some r with $0 < d < r$ we have $\sigma(S \cap B_r(x)) = r^{n-1}$ and $\sigma(S \cap B_t(x)) \leq M t^{n-1}$ for $t > 0$. Then for $1 < p < n$, we have*

$$\left(\int_\Omega |u|^p \, dy \right)^{1/p} \leq \frac{C d^n}{|\Omega|^{1/p'}} r^{1-n/p} \left(\int_\Omega |\nabla u|^p \, dy \right)^{1/p}.$$

The constant depends only on M.

Proof. We follow the proof of Corollary 8.2.2 in the book of Adams and Hedberg [1], except that we have substituted the Riesz capacity for the Bessel capacity in order to obtain a scale-invariant estimate. Following their arguments, we obtain that if u vanishes on S, then

$$|u(x)| \leq \frac{d^n}{|\Omega|} (I_1(|\nabla u|)(x) + \|\nabla u\|_{L^p(\Omega)} \|I_1(\mu)\|_{L^{p'}(\Omega)}). \tag{3.8}$$

Here $I_1(f)(x) = \int_\Omega f(y)|x-y|^{1-n} \, dy$ is the first order fraction integral and μ is any measure on S normalized so that $\mu(S) = 1$. To estimate $\|I_1(\mu)\|_{L^{p'}(\Omega)}$ we use Theorem 4.5.4 of Adams and Hedberg [1] which gives that

$$\int_{\mathbf{R}^n} (I_1(\mu))^{p'} \, dy \leq C \int_{\mathbf{R}^n} \dot{W}^\mu_{1,p} \, d\mu$$

where $\dot{W}^\mu_{1,p}(x)$ is the Wolff potential of μ defined by

$$\int_0^\infty (\mu(B(x,t)) t^{p-n})^{1/(p-1)} \, dt/t.$$

Our assumptions imply that with $\mu = r^{1-n}\sigma$ denoting normalized surface measure on S, we have $I_1(\mu)(x) \leq C r^{(p-n)/(p-1)}$ where C depends only on M. Using this, the Lemma follows from (3.8). ∎

The next inequality is also taken from Adams and Hedberg [1, Corollary 8.1.4]. Let $1/q + 1/n < 1$ and assume that Ω is a convex domain of diameter d. We let $\bar{u} = \fint_\Omega u \, dy$ and then we have

$$\int_\Omega |u - \bar{u}|^q \, dy \leq C_q \frac{d^n}{|\Omega|} \left(\int_\Omega |\nabla u|^{nq/(n+q)} \, dy \right)^{(n+q)/n}. \tag{3.9}$$

Finally, we suppose that Ω is a domain and $\Psi_r(x)$ lies in a coordinate cylinder Z so that $\partial\Omega \cap Z$ lies in a hyperplane and let $\bar{u} = \fint_{\Psi_r(x)} u \, dy$. Provided $\Psi_r(x) \subset Z$, we have

$$\left(\int_{\Delta_r(x)} |u - \bar{u}|^q \, d\sigma \right)^{1/q} \leq C \left(\int_{\Psi_r(x)} |\nabla u|^p \, dy \right)^{1/p}. \tag{3.10}$$

In this inequality, p and q are related by $1/q = 1/p - (1 - 1/p)/(n-1)$ and $p > 1$.

Lemma 3.11 *Let Ω, N, D be a Lipschitz domain for the mixed problem as defined above, suppose that (2.2) holds, and let $0 < r < r_0$. Let u be a weak solution of the mixed problem for a divergence form elliptic operator with zero Dirichlet data and Neumann data f_N. We have the estimate*

$$\left(\fint_{\Psi_r(x)} |\nabla u|^2\, dy\right)^{1/2} \le C\left[\fint_{\Psi_{Cr}(x)} |\nabla u|\, dy + \left(\frac{1}{r^{n-1}}\int_{\Delta_{Cr}(x)} f_N^p\, d\sigma\right)^{1/p}\right].$$

Here, $p = 2$ if $n = 2$ and $p = 2(n-1)/(n-2)$ for $n \ge 3$. The constant C depends only on M and the dimension n.

Proof. Changing variables to flatten the boundary of a Lipschitz domain preserves the class of elliptic operators with bounded measurable coefficients, thus it suffices to consider the case where the ball $B_r(x)$ lies in a hyperplane. We may rescale to set $r = 1$. We claim that for $1/2 \le s \le t \le 1$, we have

$$\left(\int_{\Psi_s(x)} |\nabla u|^2\, dy\right)^{1/2} \le \frac{C}{(t-s)^a}\left(\int_{\Psi_t(x)} |\nabla u|^q\, dy\right)^{1/q} + \left(\int_{\Delta_1(x)} f_N^p\, d\sigma\right)^{1/p} \tag{3.12}$$

where we may choose the exponents $p = 2(n-1)/(n-2)$ and $q = 2n/(2n+2)$ if $n \ge 3$ or $p = 2$ and $q = 4/3$ if $n = 2$.

We give the details when $n = 3$. In the argument that follows, let $\epsilon = (t-s)/2$ and choose η to be a cut-off function which is one on $B_s(x)$, supported in $B_{s+\epsilon}(x)$ and satisfies $|\nabla\eta| \le C/\epsilon$. We let $v = \eta^2(u - E)$ where E is a constant. If we choose E so that $v \in W^{1,2}_D(\Omega)$, the weak formulation of the mixed problem and Hölder's inequality gives for $1 < p < \infty$

$$\begin{aligned}\int_\Omega |\nabla u|^2\eta^2\, dy \;\; &\le \;\; C\left[\int_\Omega |u-E|^2|\nabla\eta|^2\, dy + \left(\int_{\Delta_{s+\epsilon}(x)} |u-E|^{p'}\, d\sigma\right)^{2/p'}\right.\\ &\qquad\left. + \left(\int_{\Delta_{s+\epsilon}(x)} f_N^p\, d\sigma\right)^{2/p}\right].\end{aligned} \tag{3.13}$$

We consider two cases: a) $B_{s+\epsilon}(x) \cap D = \emptyset$ and b) $B_{s+\epsilon}(x) \cap D \ne \emptyset$. In case a) we may chooose $E = \bar u = \fint_{\Psi_{s+\epsilon}(x)} u\, dy$. We use the Poincaré-Sobolev inequality (3.9) and the inequality (3.10) to estimate the first two terms on the right of (3.13) and conclude that

$$\begin{aligned}\int_{\Psi_s(x)} |\nabla u|^2\, dy \;\; &\le \;\; C\left[\frac{1}{(t-s)^2}\left(\int_{\Psi_{s+\epsilon}(x)} |\nabla u|^{\frac{2n}{n+2}}\, dy\right)^{\frac{n+2}{n}}\right.\\ &\qquad\left. + \left(\int_{\Psi_{s+\epsilon}(x)} |\nabla u|^{\frac{np}{np-n+1}}\, dy\right)^{\frac{2(np-n+1)}{pn}} + \left(\int_{\Delta_1(x)} f_N^p d\sigma\right)^{1/p}\right].\end{aligned}$$

If $n \geq 3$, we may choose $p = 2(n-1)/(n-2)$ and then we have that $np/(np - n + 1) = 2n/(n+2)$ to obtain the claim.

We now turn to case b). Since $B_{s+\epsilon}(x)$ meets the set D, we cannot subtract a constant from u and remain in the space of test functions, $W^{1,2}_D(\Omega)$. Thus, we let $E = 0$ in (3.13). We let $\bar{u}$ be the average value of u on $\Psi_{s+2\epsilon}(x)$ and obtain

$$\int_{\Psi_{s+\epsilon}(x)} u^2 |\nabla \eta|^2 \, dy \leq \frac{C}{\epsilon^2} \left[\int_{\Psi_{s+2\epsilon}(x)} |u - \bar{u}|^2 \, dy + \bar{u}^2 \right].$$

Since $B_{s+\epsilon}(x) \cap D \neq \emptyset$, our assumption on the boundary (2.2) implies that we may find a point $\tilde{x}$ so that $B_\epsilon(\tilde{x}) \subset B_t(x)$ and so that $\sigma(B_\epsilon(\tilde{x}) \cap D) \geq M^{-1}\epsilon^{n-1}$. Using (3.9) and the Poincaré inequality of Lemma 3.7 we conclude that

$$\begin{aligned} \int_{\Psi_{s+\epsilon}(x)} u^2 |\nabla \eta|^2 \, dy \quad &\leq \quad C \left[\frac{1}{\epsilon^2} \left(\int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^{2n/(n+2)} \, dy \right)^{(n+2)/n} \right. \\ &\qquad \left. + \epsilon^{2-2n/q} \left(\int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^q \, dy \right)^{2/q} \right] \end{aligned} \tag{3.14}$$

for $1 < q < n$. A similar argument using (3.10) and Lemma 3.7 gives us

$$\begin{aligned} \left(\int_{\Delta_{s+2\epsilon}(x)} |u|^{p'} \, d\sigma \right)^{1/p'} \quad &\leq \quad \left(\int_{\Delta_{s+2\epsilon}(x)} |u - \bar{u}|^{p'} \, d\sigma \right)^{1/p'} + |\bar{u}| \\ &\leq \quad C \left[\left(\int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^{np/(np-n+1)} \, dy \right)^{(np-n+1)/(np)} \right. \\ &\qquad \left. + \epsilon^{1-n/q} \left(\int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^q \, dy \right)^{1/q} \right] \end{aligned} \tag{3.15}$$

where the use of Lemma 3.7 requires that we have $1 < q < n$. We use (3.14) and (3.15) in (3.13) and choose $q = 2n/(n+2)$ and $p = 2(n-2)/(n-1)$ if $n \geq 3$. Once we recall that $t - s = 2\epsilon$, we obtain (3.12).

Finally, we observe that the claim (3.12) implies the estimate

$$\left(\int_{\Psi_{1/2}(x)} |\nabla u|^2 \, dy \right)^{1/2} \leq C \left[\int_{\Psi_1(x)} |\nabla u| \, dy + \left(\int_{\Delta_1(x)} f_N^p \right)^{1/p} \right] \tag{3.16}$$

with p as in (3.12). This follows from an argument of Dahlberg and Kenig that can be found in Fabes and Stroock [12, pp. 1004-5]. The only change in their argument is to keep track of the extra term $(\int_{\Delta_1(x)} f_N^p \, dy)^{1/p}$ which we denote by A. To give the details, suppose that we have $\int_{\Psi_1(x)} |\nabla u| \, dy = 1$ and define $I(t) = \left(\int_{\Psi_t(x)} |\nabla u|^q \, dy \right)^{1/q}$ where $q < 2$ is as in (3.12). From (3.12), our normalization in L^1 and Hölder's inequality, we conclude that with $\theta = 2 - 2/q$ we have

$$I(s) \leq \frac{C}{(t-s)^{a\theta}} (I(t) + A)^{\theta}.$$

If $I(t) \leq A$ for some t in $[1/2, 1]$, then we have $I(1/2) \leq A$. If $I(t) \geq A$ for all t, then we have $I(s) \leq CI(t)^{\theta}/(t-s)^{\theta a}$. Taking the logarithm of this expression, setting $s = t^b$ and integrating as in Fabes and Stroock [12] implies $I(1/2) \leq C$.

When the dimension $n = 2$, the exponent $2n/(n+2)$ is 1 and it is not clear that we have (3.9) as used to obtain (3.14). However, from (3.9) and Hölder's inequality we can show

$$\left(\int_{\Psi_{s+2\epsilon}(x)} |u - \bar{u}|^2 \, dy \right)^{1/2} \leq C \left(\int_{\Psi_{s+2\epsilon}(x)} |\nabla u|^{4/3} \, dy \right)^{3/4}.$$

This may be substituted for (3.9) in the above argument to obtain (3.12) when $n = 2$. ∎

Lemma 3.17 *Let u be a weak solution of the mixed problem (1.1) with zero Dirichlet data and Neumann data f in $L^p(N)$ which is supported in $N \cap \Delta_r(x)$ with $r < r_0$. There exists $p_0 > 2$ so that for t in $[2, p_0)$, we have the estimate*

$$\left(-\!\!\!\!\!\!\int_{\Psi_r(x)} |\nabla u|^t \, dy \right)^{1/t} \leq C \left[-\!\!\!\!\!\!\int_{\Psi_{Cr}(x)} |\nabla u| \, dy + \left(\frac{1}{r^{n-1}} \int_{\Delta_{Cr}(x)} f^{t(n-1)/n} \, d\sigma \right)^{n/(t(n-1))} \right].$$

The constant in this estimate depends on M and n.

Proof. According to Lemma 3.11, ∇u satisfies a reverse Hölder inequality and thus we may apply a result of Giaquinta [14, p. 122] to conclude that there exists $p_0 > 2$ so that we have

$$\left(-\!\!\!\!\!\!\int_{\Psi_r(x)} |\nabla u|^t \, dy \right)^{1/t} \leq C \left[-\!\!\!\!\!\!\int_{\Psi_{Cr}(x)} |\nabla u| \, dx + \left(-\!\!\!\!\!\!\int_{\Psi_{Cr}(x)} (Pf^p)^{t/p} \, dy \right)^{1/p} \right]$$

for t in $[2, p_0)$ and p as in lemma 3.11. From this, we may use Lemma 3.1 to conclude the estimate of the Lemma. ∎

4 Estimates for solutions with atomic data

We establish an estimate for the solution of the mixed problem when the Neumann data is an atom and the Dirichlet data is zero. The key step is to establish decay of the solution as we move away from the support of an atom. We will measure the decay by taking L^q-norms in dyadic rings around the support of the atom. Thus, given a surface ball $\Delta_r(x)$, we define $\Sigma_k = \Delta_{2^k r}(x) \setminus \Delta_{2^{k-1} r}(x)$ and define $S_k = \Psi_{2^k r}(x) \setminus \Psi_{2^{k-1} r}(x)$ and $\tilde{S}_k = \cup_{j=k-1}^{k+1} S_j$.

Theorem 4.1 *Let u be a weak solution of the mixed problem (1.1) with Neumann data $f_N = a$ an atom for N which is supported in $\Delta_r(x)$ and zero Dirichlet data. For $1 < q < p_0/2$, we have the following estimates*

$$\left(\int_{\Delta_{8r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q} \leq C\sigma(\Delta_r(x))^{-1/q'} \tag{4.2}$$

and for $k \geq 3$

$$\left(\int_{\Sigma_k} |\nabla u|^q \, d\sigma\right)^{1/q} \leq C 2^{-\alpha k}\sigma(\Sigma_k)^{-1/q'}. \tag{4.3}$$

Here, α is as Lemma 4.7, p_0 is the exponent from Lemma 3.17, and the constant C depends on Ω.

If $r < r_0$ and x is in $\partial\Omega$, then we may construct a star-shaped Lipschitz domain $\Omega_r(x) = Z_r(x) \cap \Omega$ where $Z_r(x)$ is the coordinate cylinder defined above. Given a function u defined in Ω, $x \in \partial\Omega$, and $r > 0$, we define a truncated non-tangential maximal function v_r^* by

$$v_r^*(x) = \sup_{y \in \Gamma(x) \cap B_r(x)} |v(y)|.$$

Lemma 4.4 *Suppose that $x \in \partial\Omega$ and $r < r_0$. Let u be a harmonic function in $\Omega_{4r}(x)$. If $\nabla u \in L^2(\Omega_{4r}(x))$ and $\partial u/\partial\nu$ is in $L^2(\partial\Omega \cap \partial\Omega_{4r}(x))$, then we have $\nabla u \in L^2(\Delta_r(x))$ and*

$$\int_{\Delta_r(x)} ((\nabla u)_r^*)^2 \, d\sigma \leq C \left(\int_{\partial\Omega\cap\partial\Omega_{4r}(x)} \left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma + \frac{1}{r}\int_{\Omega_{4r}(x)} |\nabla u|^2 \, dy\right).$$

Proof. Since the estimate only involves ∇u, we may subtract a constant from u so that $\int_{\Omega_r(x)} u \, dy = 0$. We pick a smooth cut-off function η which is one on $Z_{3r}(x)$ and zero outside $Z_{4r}(x)$. Since we assume that ∇u is in $L^2(\Omega)$, it follows that $\Delta(\eta u) = \eta\Delta u + 2\nabla u \cdot \nabla\eta$ is in $L^2(\Omega)$. Thus, with Ξ the usual fundamental solution of the Laplacian, $w = \Xi * (\Delta(\eta u))$ will be in the Sobolev space $W^{2,2}(\mathbf{R}^n)$. Next, we let v be the solution of the Neumann problem

$$\begin{cases} \Delta v = 0, & \text{in } \Omega_{4r}(x) \\ \dfrac{\partial v}{\partial\nu} = \dfrac{\partial\eta u}{\partial\nu} - \dfrac{\partial w}{\partial\nu}, & \text{on } \partial\Omega_{4r}(x). \end{cases}$$

According to Jerison and Kenig [15], the solution v will have non-tangential maximal function in $L^2(\partial\Omega)$. By uniqueness of weak solutions to the Neumann problem, we may add a constant to v so that we have $\eta u = v + w$. As w and all its derivatives are bounded in $\Omega_{2r}(x)$ and the non-tangential maximal function of ∇v is in $L^2(\partial\Omega_{4r}(x))$, we obtain the Lemma. ∎

In the next lemma and below, we let $\nabla_t u = \nabla u - \nu \nabla u \cdot \nu$ denote the tangential component of the gradient. The proof of this lemma is identical to the proof of Lemma 4.4.

Lemma 4.5 *Suppose that $x \in \partial\Omega$ and $r < r_0$. Let u be a harmonic function in $\Psi_{2r}(x)$. If $\nabla u \in L^2(\Omega_{4r}(x))$ and $\nabla_t u$ is in $L^2(\partial\Omega \cap \partial\Omega_{4r}(x))$, then we have $\nabla u \in L^2(\Delta_r(x))$ and*

$$\int_{\Delta_r(x)} ((\nabla u)_r^*)^2 \, d\sigma \leq C \left(\int_{\partial\Omega\cap\partial\Omega_{4r}(x)} |\nabla_t u|^2 \, d\sigma + \frac{1}{r} \int_{\Omega_{4r}(x)} |\nabla u|^2 \, dy \right).$$

The following weighted estimate will be an intermediate step towards our estimates for solutions with atomic data. In the next lemma Ω is a bounded Lipschitz domain and the boundary is written $\partial\Omega = D \cup N$. Recall that $\delta(x)$ denotes the distance from x to the set Λ.

Lemma 4.6 *Let $\epsilon \in \mathbf{R}$, $r < r_0$, and suppose that u is a weak solution of the mixed problem with $f_D = 0$ and f_N in $L^2(N)$. There is a constant C so that the solution u satisfies*

$$\int_{\Delta_r(x)} ((\nabla u)_{c\delta}^*)^2 \, \delta^{1-\epsilon} d\sigma \leq C \left(\int_{\Delta_{2r}(x)} |f_N|^2 \delta^{1-\epsilon} \, d\sigma + \int_{\Psi_{Cr}(x)} |\nabla u|^2 \, \delta^{-\epsilon} \, dy \right).$$

for constants c and C which depend only on n and M.

The next lemma uses a Whitney decomposition and thus it is simpler if we use surface cubes, rather than the surface balls used elsewhere. A *surface cube* is the image of a cube in $\mathbf{R}^{n-1}$ under the mapping $x' \to (x', \phi(x'))$. Obviously, each cube will lie in a coordinate cylinder.

Proof. Note that if $\delta(y) \geq r$ for all y in $\Delta_r(x)$, then this Lemma follows directly from Lemma 4.4 or Lemma 4.5. The Lemma is more interesting when $\Delta_r(x)$ is close to Λ. To prove the Lemma, we make a Whitney decomposition of $\partial\Omega \setminus \Lambda$. Thus we write $\partial\Omega = \Lambda \cup (\cup_j Q_j)$. The collection of surface cubes $\{Q_j\}$ has the following properties: 1) for each j, we have either $Q_j \subset D$ or $Q_j \subset (N \setminus \Lambda)$, 2) we have $c\delta(x) \leq \mathrm{diam}(Q) \leq \delta(x)/4$ for each x in Q, 3) if we define $T(Q) = \{x \in \bar{\Omega} : \mathrm{dist}(x, Q) < \mathrm{diam}(Q)\}$, then the sets $\{T(Q_j)\}$ have bounded overlaps and thus

$$\sum \chi_{T(Q_j)} \leq C(n, M).$$

We let $r_j = \mathrm{diam}(Q_j)$ and use the estimates of Lemma 4.5 and 4.4 to obtain

$$\int_{Q_j} (\nabla u)_{r_j}^* |\nabla u|^2 \, d\sigma \leq C \left[\int_{2Q_j \cap N} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + \frac{1}{r_j} \int_{T(Q_j)} |\nabla u|^2 \, dy \right].$$

Now we multiply by $r_j^{-\epsilon}$, use that $r_j \approx \delta(x)$ for $x \in T(Q_j)$, sum on j and use that the family $\{T(Q_j)\}$ has bounded overlaps to obtain the Lemma. ∎

An important part of the proof of our estimate for the mixed problem is to show that a solution with Neumann data an atom will decay as we move away from the support of the atom. This decay is encoded in estimates for the Green function for the mixed problem. These estimates rely in large part on the work of de Giorgi [10], Moser [25] and Nash [26] on Hölder continuity of weak solutions of elliptic equations with bounded and measurable coefficients, and the work of Littman, Stampacchia and Weinberger [19] who constructed the fundamental solution of such operators. Also see Kenig and Ni [17] for the construction of a fundamental solution in two dimensions. Given the free space fundamental solution, the Green function may be constructed by reflection in a manner similar to the construction given for graph domains in [18]. A similar argument was used by Dahlberg and Kenig [9] in their study of the Neumann problem.

Lemma 4.7 *We consider the mixed problem in a Lipschitz domain with D and N satisfying our standard hypotheses. There exists a Green function $G(x,y)$ for (1.1) which satisfies: 1) If $G_x(y) = G(x,y)$, then G_x is in $W^{1,2}_D(\Omega \setminus B_r(x))$ for all $r > 0$, 2) $\Delta G_x = \delta_x$, the Dirac δ-measure at x, 3) If f_N lies in $W^{-1/2,2}_D(\partial\Omega)$, then the solution of the mixed problem with $f_D = 0$ can be represented by*

$$u(x) = -\langle f_N, G_x \rangle_{\partial\Omega},$$

4) The Green function is Hölder continuous away from the pole and satisfies the estimates

$$|G(x,y) - G(x,y')| \le \frac{C|y-y'|^\alpha}{|x-y|^{n-2+\alpha}}, \qquad |x-y| > 2|y-y'|,$$

$$|G(x,y)| \le \frac{C}{|x-y|^{n-2}}, \qquad n \ge 3,$$

and

$$|G(x,y)| \le C(1 + |\log|x-y||), \qquad n = 2.$$

Lemma 4.8 *Let u be a weak solution of the mixed problem with Neumann data f in $L^p(N)$ where $p = (2n-2)/n$ for $n \ge 3$. Then we have the estimate*

$$\int_\Omega |\nabla u|^2 \, dy \le C\|f\|^2_{L^p(N)}.$$

If $n = 2$, we have

$$\int_\Omega |\nabla u|^2 \, dy \le C\|f\|^2_{H^1(N)}.$$

In each case, the constant C depends on Ω.

Proof. When $n \ge 3$, we use that $W^{1/2,2}_D(\partial\Omega) \subset L^{2(n-1)/(n-2)}(\partial\Omega)$. Transposing to the dual tells us that $L^{2(n-1)/n}(\partial\Omega) \subset W^{-1/2,2}_D(\partial\Omega)$ and since the weak solution of the mixed problem satisfies

$$\int_\Omega |\nabla u|^2 \, dy \le C\|f\|^2_{W^{-1/2,2}_D(\partial\Omega)}$$

the Lemma follows.

When $n = 2$, the proof is the same except that we do not have that $W_D^{1/2,2}(\partial\Omega) \subset L^\infty(\partial\Omega)$. However, we do have the embedding $W_D^{1/2,2}(\partial\Omega) \subset BMO(\partial\Omega)$ and this gives the conclusion of the Lemma for $n = 2$. ∎

Proof of Theorem 4.1. We begin by considering

$$\int_{\Delta_{8r}(x)} |\nabla u|^q \, d\sigma$$

to obtain an estimate for the gradient of the solution near the support of the atom. To begin we let $q > 1$ and use Hölder's inequality to estimate

$$\begin{aligned}\left(\int_{\Delta_{8r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q} &\leq \left(\int_{\Delta_{8r}(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma\right)^{1/2} \left(\int_{\Delta_{8r}(x)} \delta^{\frac{q(\epsilon-1)}{2-q}} \, d\sigma\right)^{\frac{2-q}{2q}} \\ &\leq Cr^{(n-1)(\frac{1}{q}-\frac{1}{2})+\frac{\epsilon-1}{2}} \left(\int_{\Delta_{8r}(x)} |\nabla u|^2 \delta^{1-\epsilon} \, d\sigma\right)^{1/2}.\end{aligned}$$

This requires that q and ϵ satisfy $q(\epsilon-1)/(2-q) > -1$ or $q < 1/(1-\epsilon/2)$. Next we use Lemma 4.6 to bound the weighted $L^2(\delta^{1-\epsilon} d\sigma)$-norm of ∇u. This gives us

$$\begin{aligned}\left(\int_{\Delta_{8r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q} &\leq C \left[\left(\int_{\Delta_r(x)} a^2 \delta^{1-\epsilon} \, d\sigma\right]^{1/2}\right. \\ &\quad \left. + \left(\int_{\Psi_{Cr}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy\right)^{1/2}\right) r^{(n-1)(\frac{1}{q}-\frac{1}{2})+\frac{\epsilon-1}{2}}.\end{aligned}$$

To estimate the solid integral in this last expression, we use Hölder's inequality again and find

$$\begin{aligned}\left(\int_{\Psi_{Cr}(x)} |\nabla u|^2 \delta^{-\epsilon} \, dy\right)^{1/2} &\leq \left(\int_{\Psi_{Cr}(x)} |\nabla u|^p \, dy\right)^{1/p} \left(\int_{\Psi_{Cr}(x)} \delta^{-\epsilon p/(p-2)} \, dy\right)^{1-2/p} \\ &\leq Cr^{n(\frac{1}{p}-\frac{1}{2})-\epsilon/2} \left(\int_{\Psi_{Cr}(x)} |\nabla u|^p \, dy\right)^{1/p}.\end{aligned}$$

The last inequality holds when $\epsilon p/(p-2) < 2$ or $p > 2/(1-\epsilon/2)$. Now we may use the previous two displayed inequalities, Hölder's inequality and Lemma 3.17 to obtain

$$\left(\frac{1}{r^{n-1}} \int_{\Delta_{8r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q} \leq C \left[\left(\frac{1}{r^n} \int_{\Psi_{Cr}(x)} |\nabla u|^2 \, dy\right)^{1/2} + r^{1-n}\right].$$

We have used the normalization of the atom to estimate the term involving the Neumann data in Lemma 3.17. Finally, we may use Lemma 4.8 and the normalization of the atom a to obtain the estimate $(r^{-n}\int |\nabla u|^2\,dy)^{1/2} \le Cr^{1-n}$ which gives the estimate (4.2). Examining the conditions on $p > 2/(1-\epsilon/2)$ and $q < 1/(1-\epsilon/2)$ we see that we obtain this estimate precisely when $1 < q < p_0/2$ and p_0 is the exponent from the reverse Hölder argument in Lemma 3.17.

Our next goal is to estimate $\int_{\Sigma_k} |\nabla u|^q\,d\sigma$ for k sufficiently large. If $1 < q < 2$ and $q < 1/(1-\epsilon/2)$, we may use Hölder's inequality to obtain

$$\begin{aligned}
\left(\int_{\Sigma_k} |\nabla u|^q\,d\sigma\right)^{1/q} &\le \left(\int_{\Sigma_k} |\nabla u|^2 \delta^{1-\epsilon}\,d\sigma\right)^{1/2} \left(\int_{\Sigma_k} \delta^{\frac{(\epsilon-1)q}{2-q}}\,d\sigma\right)^{(2-q)/2} \\
&\le C\left(\int_{\Sigma_k} |\nabla u|^2\delta^{1-\epsilon}\,d\sigma\right)^{1/2} (2^k r)^{(n-1)(\frac{1}{q}-\frac{1}{2})+(\epsilon-1)/2}
\end{aligned} \tag{4.9}$$

This requires that $(\epsilon-1)q/(2-q) > -1$. Next, Lemma 4.6 gives that

$$\left(\int_{\Sigma_k} |\nabla u|^2\delta^{1-\epsilon}\,d\sigma\right)^{1/2} \le C\left(\int_{\tilde{S}_k} |\nabla u|^2\delta^{-\epsilon}\,dy\right)^{1/2}. \tag{4.10}$$

To estimate the integral over Ω, we use Hölder's inequality and then the reverse Hölder estimate in Lemma 3.17 to obtain

$$\begin{aligned}
\left(\int_{S_k} |\nabla u|^2\delta^{-\epsilon}\,dy\right)^{1/2} &\le C\left(\int_{S_k} |\nabla u|^p\,dy\right)^{1/p} (2^k r)^{n(\frac{1}{p}-\frac{1}{2})-\frac{\epsilon}{2}} \\
&\le C\left(\int_{S_k} |\nabla u|^2\,dy\right)^{1/2} (2^k r)^{-\frac{\epsilon}{2}}.
\end{aligned} \tag{4.11}$$

Here we have used that $(\int_{S_k} \delta^{-\epsilon p/(p-2)}\,dy)^{1/p} \le C(2^k r)^{n(1/p-1/2)-\epsilon/2}$ provided $p > 2/(1-\epsilon/2)$. We appeal to the Caccioppoli inequality, recall that the mixed data for u is zero outside of $\Delta_r(x)$, and obtain that

$$\left(\int_{S_k} |\nabla u|^2\,dy\right)^{1/2} \le C\frac{1}{2^k r}\left(\int_{\tilde{S}_k} |u|^2\,dy\right)^{1/2}. \tag{4.12}$$

From Lemma 4.7, we represent u by

$$u(y) = -\int_{N\cap\Delta_r(x)} G(y,z)a(z)\,dz.$$

If $\Delta_r(x)$ intersects D, we have that G is zero at some point in $\Delta_r(x)$. Thus, the normalization of a, $\|a\|_{L^1(\partial\Omega)} = 1$, and the Hölder continuity of G in Lemma 4.7 part 4) imply that

$$|u(y)| \le Cr^\alpha/|x-y|^{n-2+\alpha}, \qquad \text{if } |x-y| > 2r. \tag{4.13}$$

If $\Delta_r(x) \subset N$, then we may use that a has mean value zero to write

$$u(y) = \int_{N\cap\Delta_r(x)} (G(y,z) - G(y,x))a(z)\,dz$$

and obtain the estimate (4.13) from part 4) of Lemma 4.7. Using the estimate (4.13) in (4.12) and the estimates (4.9–4.11) gives the result of the theorem. We note that if p_0 is the exponent for the reverse Hölder inequality then the conditions on ϵ allow us to again obtain results for $1 < q < p_0/2$. ∎

We now show that the non-tangential maximal function of our weak solutions lies in L^1 when the Neumann data is an atom.

Theorem 4.14 *Let f_N be in $H^1(N)$, then there exists u a solution of the mixed problem with Neumann data f_N and zero Dirichlet data and this solution satisfies*

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \le C\|f\|_{H^1(N)}.$$

Proof. We begin by considering the case when f_N is an atom and we let u be the weak solution of the mixed problem with Neumann data an atom a and zero Dirichlet data. The result for data in $H^1(N)$ follows easily from the result for an atom.

We establish a representation for the gradient of u in terms of the boundary values of u. Let $x \in \Omega$ and j be an index ranging from 1 to n. We claim

$$\begin{aligned} \frac{\partial u}{\partial x_j}(x) &= -\int_{\partial\Omega} \sum_{i=1}^n \frac{\partial \Xi}{\partial y_i}(x-\cdot)(\nu_i \frac{\partial u}{\partial y_j} - \frac{\partial u}{\partial y_i}\nu_j) \\ &\qquad + \frac{\partial \Xi}{\partial y_j}(x-\cdot)\frac{\partial u}{\partial \nu}\, d\sigma. \end{aligned} \tag{4.15}$$

If u is smooth up to the boundary, the proof is a straightforward application of the divergence theorem. However, it takes a bit more work to establish this result when we only have that u is a weak solution.

Thus, we suppose that η is a smooth function which is zero in a neighborhood of Λ and supported in a coordinate cylinder. Using the coordinate system for our coordinate cylinder, we set $u_\tau(y) = u(y+\tau e_n)$. Applying the divergence theorem gives

$$\begin{aligned} -\int_{\partial\Omega} \eta(\frac{\partial \Xi}{\partial \nu}(x-\cdot)\frac{\partial u_\tau}{\partial y_j} - \nabla\Xi(x-\cdot)\cdot\nabla u_\tau \nu_j + \frac{\partial \Xi}{\partial y_j}(x-\cdot)\frac{\partial u_\tau}{\partial \nu})\, d\sigma \\ = \eta(x)\frac{\partial u_\tau}{\partial x_j}(x) - \int_\Omega \nabla\eta\cdot\nabla\Xi(x-\cdot)\frac{\partial u_\tau}{\partial y_j} \\ -\nabla_y\Xi(x-\cdot)\cdot\nabla u_\tau \frac{\partial \eta}{\partial y_j} \\ +\frac{\partial \Xi}{\partial y_j}(x-\cdot)\nabla u_\tau\cdot\nabla\eta\, dy. \end{aligned} \tag{4.16}$$

Thanks to the truncated maximal function estimate in Lemma 4.6, we may let τ tend to zero from above and conclude that the same identity holds with u_τ replaced by u. Next, we suppose that η is of the form $\eta\phi_\epsilon$ where $\phi_\epsilon = 0$ on $\{x : \delta(x) < \epsilon\}$ and $\phi_\epsilon(x) = 1$ on $\{x : \delta(x) > 2\epsilon\}$ and we have the estimates

$|\nabla\phi_\epsilon(x)| \le C/\epsilon$. Since we assume the boundary between D and N is a Lipschitz surface, we have the following estimate for ϵ sufficiently small

$$|\{x : \delta(x) \le 2\epsilon\}| \le C\epsilon^2. \tag{4.17}$$

Using our estimate for $\nabla\phi_\epsilon$ and the estimate (4.17), we have

$$|\int_\Omega \eta\nabla\phi_\epsilon \cdot \nabla\Xi(x-\cdot)\frac{\partial u}{\partial y_j}\,dy| \le C\left(\int_{\{y:\delta(y)<2\epsilon\}} |\nabla u|^2\,dy\right)^{1/2}$$

and the last term tends to zero with ϵ since the gradient of a weak solution lies in $L^2(\Omega)$. Using this and similar estimates for the other terms in (4.16), gives

$$\begin{aligned}\lim_{\epsilon\to 0^+} -\int_\Omega \nabla(\phi_\epsilon\eta)\cdot\nabla\Xi(x-\cdot)\frac{\partial u}{\partial y_j} - \nabla_y\Xi(x-\cdot)\cdot\nabla u\frac{\partial\phi_\epsilon\eta}{\partial y_j}&\\ +\frac{\partial\Xi}{\partial y_j}(x-\cdot)\nabla u\cdot\nabla(\phi_\epsilon\eta)\,dy \quad &= \quad -\int_\Omega \nabla(\phi_\epsilon\eta)\cdot\nabla\Xi(x-\cdot)\frac{\partial u}{\partial y_j}\\ &\qquad -\nabla\Xi(x-\cdot)\cdot\nabla u\frac{\partial\phi_\epsilon\eta}{\partial y_j}\\ &\qquad +\frac{\partial\Xi}{\partial y_j}(x-\cdot)\nabla u\cdot\nabla\eta j\,dy.\end{aligned}$$

Thus we obtain the identity (4.16) with u_τ replaced by u and without the support restriction on η. Finally, we choose a partition of unity which consists of functions that are either supported in a coordinate cylinder, or whose support does not intersect the boundary of Ω. Summing as η runs over this partition gives us the representation formula (4.15) for u. As we have $\nabla u \in L^q(\partial\Omega)$, it follows from the theorem of Coifman, McIntosh and Meyer [6] that $(\nabla u)^*$ lies in $L^q(\partial\Omega)$. However, a bit more work is needed to obtain the correct L^1 estimate for $(\nabla u)^*$.

We claim

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\,d\sigma &= 0\\ \int_{\partial\Omega} \nu_j\frac{\partial u}{\partial y_i} - \nu_i\frac{\partial u}{\partial y_j}\,d\sigma &= 0.\end{aligned}$$

Since $(\nabla u)^*$ lies in $L^q(\partial\Omega)$, the proof of these two identities is a standard application of the divergence theorem. Using these results and the estimates for ∇u in Theorem 4.1, allows us to show that $\partial u/\partial\nu$ and $\nu_j\partial u/\partial y_i - \nu_i\partial u/\partial y_j$ are molecules on the boundary and hence it follows from the representation formula (4.15) that $(\nabla u)^*$ lies in $L^1(\partial\Omega)$ and satisfies the estimate

$$\|(\nabla u)^*\|_{L^1(\partial\Omega)} \le C.$$

∎

5 Uniqueness of solutions

In this section we establish uniqueness of solutions to the mixed problem (1.1). We use the existence result established in section 4 and argue by duality that if u is a solution of the mixed problem with zero data, then u is also a solution of the Dirichlet problem with zero data and hence is zero.

Theorem 5.1 *Suppose that u solves the mixed problem (1.1) with data $f_N = 0$ and $f_D = 0$. If $(\nabla u)^* \in L^1(\partial\Omega)$, then $u = 0$.*

We recall that G. Verchota constructed a sequence of approximating domains in his dissertation ([34, Theorem 1.12] [33, Appendix A]). We will need this approximation scheme and a few extensions. Given a Lipschitz domain Ω, Verchota constructs a family of smooth domains $\{\Omega_k\}$ with $\bar{\Omega}_k \subset \Omega$. In addition, he finds bi-Lipschitz homeomorphisms $\Lambda_k : \partial\Omega_k \to \partial\Omega$ which are constructed as follows.

We choose a smooth vector field V so that for some $\delta = \delta(M)$, $V \cdot \nu \leq -\delta$ a.e. on $\partial\Omega$ and define a flow for t small by $\frac{d}{dt} f(x,t) = V(f(x,t))$, $f(x,0) = x$. One may find $\xi > 0$ so that

$$\mathcal{O} = \{f(x,t) : x \in \partial\Omega, -\xi < t < \xi\} \tag{5.2}$$

is an open set and the map $(x,t) \to f(x,t)$ from $\partial\Omega \times (-\xi,\xi) \to \mathcal{O}$ is bi-Lipschitz. Since the vector field V is smooth, we have

$$Df(x,t) = I_n + O(t) \tag{5.3}$$

where I_n is the $n \times n$ identity matrix and DF denotes the derivative of a map F. In addition, we have a Lipschitz function $t_k(x)$ defined on $\partial\Omega$ so that $\Lambda_k(x) = f(x, t_k(x))$ is a bi-Lipschitz homeomorphism $\Lambda_k : \partial\Omega \to \partial\Omega_k$. We have a collection of coordinate cylinders $\{Z_i\}$ so that each Z_i serves as a coordinate cylinder for $\partial\Omega$ and for each of the approximating domains $\partial\Omega_k$. If we fix a coordinate cylinder Z, we have functions ϕ and ϕ_k so that $\partial\Omega \cap Z = \{(x', \phi(x')) : x' \in \mathbf{R}^{n-1}\} \cap Z$ and $\partial\Omega_k \cap Z = \{(x', \phi_k(x')) : x' \in \mathbf{R}^{n-1}\} \cap Z$. The functions ϕ_k are C^∞ and $\|\nabla'\phi_k\|_{L^\infty(\mathbf{R}^{n-1})}$ is bounded in k, $\lim_{k\to\infty} \nabla'\phi_k(x') = \nabla'\phi(x')$ a.e. and ϕ_k converges to ϕ uniformly. Here we are using ∇' to denote the gradient on $\mathbf{R}^{n-1}$.

We let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection $\pi(x', x_n) = x'$ and define $S_k(x') = \pi(\Lambda_k(x', \phi(x')))$. According to Verchota, the map S_k is bi-Lipschitz and has a Jacobian which is bounded away from 0 and ∞. We let T_k denote S_k^{-1} and assume that both are defined in a neighborhood of $\pi(Z)$. We claim that

$$\lim_{k\to\infty} DT_k(S_k(x')) = I_{n-1}, \qquad \text{a.e. in } \pi(Z), \tag{5.4}$$

and the sequence $\|DT_k\|_{L^\infty(\pi(Z))}$ is bounded in k.

To establish (5.4), it suffices to show that DS_k converges to I_{n-1} and that the Jacobian determinant of DS_k is bounded away from zero and infinity. The bound on the Jacobian is part of Verchota's construction (see [33,

p. 119]). As a first step, we compute the derivatives of $t_k(x', \phi(x'))$. Since $f((x', \phi(x')), t_k(x', \phi(x')))$ lies in $\partial\Omega_k$, the derivative is tangent to $\partial\Omega_k$ and we have

$$(\frac{\partial}{\partial x_i} f(x, \phi(x'), t_k(x', \phi(x')))) \cdot \nu_k(y) = 0, \qquad \text{a.e. in } \pi(Z),$$

where $y = (S_k(x'), \phi_k(S_k(x')))$ and ν_k is the normal to $\partial\Omega_k$. Solving this equation for $\frac{\partial}{\partial x_i} t_k$ gives

$$\begin{aligned} \frac{\partial}{\partial x_i} t_k(x', \phi(x')) &= -(V(y) \cdot \nu_k(y))^{-1} \Bigg(\frac{\partial f}{\partial x_i}((x', \phi(x')), t_k(x', \phi(x'))) \\ &\qquad + \frac{\partial \phi}{\partial x_i}(x') \frac{\partial f}{\partial x_n}((x', \phi(x')), t_k(x', \phi(x'))) \Bigg) \cdot \nu_k(y). \end{aligned}$$

Since $\lim_{k\to\infty} t_k = 0$ uniformly in $\pi(Z)$, (5.3), and $\nu_k(y)$ converges pointwise a.e. and boundedly to $\nu(x)$, we obtain that

$$\lim_{k\to\infty} \frac{\partial}{\partial x_i} t_k(x', \phi(x')) = 0, \qquad \text{a.e. in } \pi(Z). \tag{5.5}$$

Computing

$$\begin{aligned} &\frac{\partial}{\partial x_i} f((x', \phi(x')), t_k(x', \phi(x'))) \\ &\quad = \frac{\partial f}{\partial x_i}(x', \phi(x'), t_k(x', \phi(x'))) + \frac{\partial \phi}{\partial x_i}(x') \frac{\partial f}{\partial x_n}(x', \phi(x'), t_k(x', \phi(x'))) \\ &\qquad + V(f((x', \phi(x')), t_k(x', \phi_k(x')))) \frac{\partial}{\partial x_i} t_k(x', \phi(x')). \end{aligned}$$

Given (5.3), (5.5), and recalling that $S_k(x') = \pi(f(x', \phi(x')), t_k(x', \phi(x')))$, (5.4) follows.

Lemma 5.6 *If w is the weak solution of the mixed problem with Neumann data an atom and zero Dirichlet data, then we have*

$$\int_{\partial\Omega_k} u \frac{\partial w}{\partial \nu}\, d\sigma \le C_w \|u\|_{W^{1,1}(\partial\Omega_k)}.$$

Proof. This may be proven using generalized Riesz transforms as in [34, Section 5]. Verchota's argument uses square function estimates to show that the generalized Riesz transforms are bounded on L^p. In the proof of this Lemma, we need that the Riesz transforms of w are bounded. From the estimate for the Green function in Lemma 4.7 and the representation of $w = -\langle G, a\rangle_{\partial\Omega}$, we conclude that w is Hölder continuous. The Hölder continuity, and hence boundedness, of the Riesz transforms of w follow from the following characterization of Hölder continuous harmonic functions. A harmonic function u in a Lipschitz domain Ω is Hölder continuous of exponent α, $0 < \alpha < 1$, if and only if $\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)^{1-\alpha} |\nabla u(x)|$ is finite. ∎

We will need the following technical lemma on approximation. The proof relies on the approximation scheme of Verchota outlined above. In our application, we are interested in studying functions in Sobolev spaces on the family of approximating domains. Working with derivatives makes the argument fairly intricate.

Lemma 5.7 *If u satisfies $(\nabla u)^* \in L^1(\partial\Omega)$ and ∇u has non-tangential limits a.e. on $\partial\Omega$, then we may find a sequence of Lipschitz functions U_j so that*

$$\lim_{k\to\infty} \|u - U_j\|_{W^{1,1}(\partial\Omega_k)} \leq C/j.$$

If $u|_{\partial\Omega}$ is supported in N, then $U_j|_{\partial\Omega}$ is also supported in N.

The constant C may depend on Ω.

Proof. To prove the Lemma, it suffices to consider a function u which is zero outside one of the coordinate cylinders Z as given in Verchota's approximation scheme. We have $u(x', \phi(x')) \in W^{1,1}(\mathbf{R}^{n-1})$. Hence, there exists a sequence of Lipschitz functions u_j so that $\int_{\mathbf{R}^{n-1}} |\nabla' u(x', \phi(x')) - \nabla' u_j(x', \phi(x'))|\, dx' \leq 1/j$ where ∇' denotes the gradient in $\mathbf{R}^{n-1}$. We extend u_j to a neighborhood of $\partial\Omega$ by

$$U_j(f(x,t)) = \eta(f(x,t))u_j(x), \qquad x \in \partial\Omega$$

where η is a smooth cutoff function which is one $\partial\Omega$ and supported in the set $\mathcal{O}$ defined in (5.2).

We consider

$$\begin{aligned}
\int_{\pi(Z)} &|\nabla' u(x', \phi_k(x')) - \nabla' U_j(x', \phi_k(x'))|\, dx' \\
&\leq \int_{\pi(Z)} |\nabla' u(x', \phi_k(x')) - \nabla' u(x', \phi(x'))|\, dx' \\
&\quad + \int_{\pi(Z)} |\nabla' u(x', \phi(x')) - \nabla' u_j(x', \phi(x'))|\, dx' \\
&\quad + \int_{\pi(Z)} |\nabla' u_j(x', \phi(x')) - \nabla' U_j(x', \phi_k(x'))|\, dx' \\
&= A_k + B + C_k.
\end{aligned}$$

We have that $\lim_{k\to\infty} A_k = 0$ since we assume that $(\nabla u)^* \in L^1(\partial\Omega)$, ∇u has non-tangential limits a.e., and $\nabla' \phi_k$ converges pointwise and boundedly to $\nabla' \phi$. By our choice of u_j, we have $B \leq C/j$. Finally, our construction of U_j and our definition of T_k imply that $U_j(x', \phi_k(x')) = u_j(T_k(x'), \phi(T_k(x')))$ and hence we have

$$\begin{aligned}
C_k &\leq \int_{\pi(Z)} |(I_{n-1} - DT_k(x'))\nabla' u_j(x', \phi(x'))|\, dx' \\
&\quad + \int_{\pi(Z)} |DT_k(x')(\nabla' u_j(x', \phi(x'))) - \nabla' u_j(T_k(x'), \phi(T_k(x')))|\, dx' \\
&= C_{k,1} + C_{k,2}.
\end{aligned}$$

We have that $\lim_{k\to\infty} C_{k,1} = 0$ since $\nabla' u_j$ is bounded and (5.4) holds. Since $T_k(x')$ converges uniformly to x', DT_k is bounded, and the Jacobian of S_k is bounded, we have that $\lim_{k\to\infty} C_{k,2} = 0$. ∎

Proof of Theorem 5.1. We let u be a solution of (1.1) with zero data f_N and f_D and we wish to show that u is zero. We fix a an atom for N and let w be a solution of the mixed problem with Neumann data a and zero Dirichlet data. Our goal is to show that

$$\int_N au\, d\sigma = 0. \tag{5.8}$$

This implies that u is zero on $\partial\Omega$ and then Dahlberg and Kenig's result for uniqueness in the regularity problem [9] imply that $u = 0$ in Ω.

We turn to the proof of (5.8). Applying Green's second identity in one of the approximating domains Ω_k gives us

$$\int_{\partial\Omega_k} w\frac{\partial u}{\partial\nu}\, d\sigma = \int_{\partial\Omega_k} u\frac{\partial w}{\partial\nu}\, d\sigma, \qquad k = 1, 2 \dots . \tag{5.9}$$

We have $(\nabla u)^*$ is in $L^1(\partial\Omega)$ and w Hölder continuous and hence bounded. Recalling that w is zero on D and $\partial u/\partial\nu$ is zero on N, we may use the dominated convergence theorem to obtain

$$\lim_{k\to\infty} \int_{\partial\Omega_k} w\frac{\partial u}{\partial\nu}\, d\sigma = 0. \tag{5.10}$$

Thus, our claim will follow if we can show that

$$\lim_{k\to\infty} \int_{\partial\Omega_k} u\frac{\partial w}{\partial\nu}\, d\sigma = \int_{\partial\Omega} ua\, d\sigma. \tag{5.11}$$

Note that the existence of the limit in (5.11) follows from (5.9) and (5.10). We let U_j be the sequence of functions from Lemma 5.7 and consider

$$\begin{aligned}\left|\int_{\partial\Omega} ua\, d\sigma - \lim_{k\to\infty}\int_{\partial\Omega_k} u\frac{\partial w}{\partial\nu}\, d\sigma\right| \quad \leq \quad & \left|\int_{\partial\Omega} ua\, d\sigma - \lim_{k\to\infty}\int_{\partial\Omega_k} U_j\frac{\partial w}{\partial\nu}\, d\sigma\right| \\ & + \limsup_{k\to\infty}\left|\int_{\partial\Omega_k} (u - U_j)\frac{\partial w}{\partial\nu}\, d\sigma\right| .\end{aligned}$$

Because we have that $(\nabla w)^*$ is in $L^1(\partial\Omega)$ and U_j is bounded, we may take the limit of the first term on the right of (5.12) and obtain

$$\left|\int_{\partial\Omega} ua\, d\sigma - \lim_{k\to\infty}\int_{\partial\Omega_k} u\frac{\partial w}{\partial\nu}\, d\sigma\right| \leq \left|\int_{\partial\Omega} (u - U_j)a\, d\sigma\right| + C/j \leq C/j.$$

According to Lemmata 5.6 and 5.7, the second term on the right of (5.12) is bounded by C_w/j. As j is arbitrary, we obtain (5.11) and hence the Theorem. ∎

6 A Reverse Hölder inequality at the boundary

In this section we establish an estimate in $L^p(\partial\Omega)$ for the gradient of a solution to the mixed problem. This is the key estimate that is used in section 7 to establish L^p-estimates for the mixed problem.

Lemma 6.1 *Let u be a solution of the mixed problem (1.1) with zero Dirichlet data and Neumann data f_N. Let p_0 be as in Lemma 3.17 and assume $f_N \in L^{p_0(n-1)/n}(N)$. Then for $1 < q < p_0/2$, $x \in \Omega$ and $r < r_0$, we have*

$$\left(\fint_{\Delta_r(x)} (\nabla u)_r^{*q}\, d\sigma\right)^{1/q} \le C\left[\fint_{\Psi_{Cr}(x)} |\nabla u|\, dy + \left(\fint_{N\cap\Delta_{Cr}(x)} f_N^{p_0(n-1)/n}\, d\sigma\right)^{(n-1)/(np_0)}\right].$$

The constant in this local estimate depends on n and M.

Proof. By rescaling, we may assume $r = 1$. Fix $x \in \Omega$ let $\Delta_r = \Delta_r(x)$, $B_r = B_r(x)$ and $\Psi_r = B_r \cap \Omega$ to simplify the notation. We let ϵ satisfy $\epsilon < 2 - 4/p_0$ where p_0 is as in Lemma 3.17. Using Hölder's inequality, we have

$$\left(\int_{\Delta_2} |\nabla u|^q\, d\sigma\right)^{1/q} \le \left(\int_{\Delta_2} |\nabla u|^2 \delta^{1-\epsilon}\, d\sigma\right)^{1/2} \left(\int_{\Delta_2} \delta^{(\epsilon-1)q/(2-q)}\right)^{1/q-1/2}$$

where the last integral will be finite if $q < 1/(1-\epsilon/2)$. From Lemma 4.6, we obtain

$$\left(\int_{\Delta_2} |\nabla u|^2 \delta^{1-\epsilon}\, d\sigma\right)^{1/2} \le C\left(\int_{\Delta_4} f^2\, \delta^{1-\epsilon}\, d\sigma\right)^{1/2} + \left(\int_{\Psi_4} \delta^{-\epsilon} |\nabla u|^2\, dy\right)^{1/2}.$$

By Hölder's inequality and Lemmata 3.17 and 3.11, we may find an exponent $p > 2$ so that

$$\begin{aligned}\left(\int_{\Psi_4} |\nabla u|^2 \delta^{-\epsilon}\, dy\right)^{1/2} &\le \left(\int_{\Psi_4} |\nabla u|^p\, dy\right)^{1/p} \left(\int_{\Psi_4} \delta^{-\epsilon p/(p-2)}\, dy\right)^{1/2-n/p} \\ &\le C\left[\int_{\Psi_8} |\nabla u|\, dy + \left(\int_{\Delta_8} f_N^{p_0(n-1)/n}\, d\sigma\right)^{(n-1)/(np)}\right],\end{aligned}$$

where our choice of ϵ implies that the integral of $\delta^{-\epsilon p/(p-2)}$ is finite. Combining the previous two displayed equations, we obtain

$$\left(\int_{\Delta_2} |\nabla u|^q\, d\sigma\right)^{1/q} \le C\left[\int_{\Psi_8} |\nabla u|\, dy + \left(\int_{\Delta_8} f_N^{p_0(n-1)/n}\, d\sigma\right)^{n/((n-1)p_0)}\right]. \tag{6.2}$$

We now obtain an estimate for $(\nabla u)_c^*$ in $\Delta_{1/2}$. Towards this end, let η be a cut-off function which is one on the ball B_1 and is supported in B_2. As the

Neumann data for our solution u is an atom, we have that $(\nabla u)^* \in L^1(\partial\Omega)$ from Theorem 4.1. Hence, we have the representation formula

$$\begin{aligned}
&\int_{\partial\Omega} \eta \left(\frac{\partial \Xi}{\partial y_i}(z-\cdot)\nu_i \frac{\partial u}{\partial y_j} - \frac{\partial \Xi}{\partial y_i}(z-\cdot)\frac{\partial u}{\partial y_i}\nu_j + \frac{\partial \Xi}{y_j}(z-\cdot)\frac{\partial u}{\partial y_j}\nu_i \right) d\sigma \\
&= \; -(\eta u)(z) + \int_\Omega \frac{\partial \eta}{\partial y_i}\frac{\partial u}{\partial y_j}\frac{\partial \Xi}{\partial y_i}(z-\cdot) - \frac{\partial \Xi}{\partial y_i}(z-\cdot)\frac{\partial u}{\partial y_i}\frac{\partial \eta}{\partial y_j} + \frac{\partial \Xi}{y_j}(z-\cdot)\frac{\partial u}{\partial y_j}\frac{\partial \eta}{\partial y_i}\, dy.
\end{aligned}$$

This representation formula and the theorem of Coifman, McIntosh and Meyer on the Cauchy integral [6] imply that

$$\|(\nabla u)^*_c\|_{L^q(\Delta_{1/2})} \le C(\|\nabla u\|_{L^q(\Delta_2)} + \int_{\Psi_2} |\nabla u|\, dy).$$

The Lemma follows from this estimate and (6.2). ∎

7 Estimates for solutions with data from L^p, $p > 1$

In this section, we use the following variant of an argument developed by Shen [29] to establish L^p-estimates for elliptic problems in Lipschitz domains. This argument appeared earlier in work of Peral and Caffarelli [5].

As the argument depends on a Calderón-Zygmund decomposition into dyadic cubes, it will be stated using surface cubes rather than the surface balls $\Delta_r(x)$ used elsewhere in this paper.

Let Q_0 be a cube in the boundary and let F be defined on $4Q_0$. Let the exponents p and q satisfy $1 < p < q$. Assume that for each $Q \subset Q_0$, we may find two functions F_Q and R_Q defined in $2Q$ such that for

$$|F| \le |F_Q| + |R_Q|, \tag{7.1}$$

$$\fint_{2Q} |F_Q|\, d\sigma \le C\left(\fint_{4Q} |f|^p\, d\sigma\right)^{1/p}, \tag{7.2}$$

$$\left(\fint_{2Q} |R_Q|^q\, d\sigma\right)^{1/q} \le C\left[\fint_{4Q} |F|\, d\sigma + \left(\fint_{4Q} |f|^p\, d\sigma\right)^{1/p}\right]. \tag{7.3}$$

Under these assumptions, for r in the interval (p, q), we have

$$\left(\fint_{Q_0} |F|^r\, d\sigma\right)^{1/r} \le C\left[\fint_{4Q_0} |F|\, d\sigma + \left(\fint_{4Q_0} |f|^r\, d\sigma\right)^{1/r}\right].$$

The constant in this estimate will depend on the Lipschitz constant of the domain and the constants in the estimates in the conditions (7.2–7.3). The argument to obtain this conclusion is more or less the same as in Shen [29, Theorem 3.2]. We are not able to use Shen's result directly as we have results in Hardy spaces rather than L^p-spaces.

In our application, we will let $4Q_0$ be a cube with sidelength comparable to r_0. We let u be a solution of the mixed problem with Neumann data f in $L^p(N)$ and Dirichlet data zero. Since $L^p(N)$ is contained in the Hardy space $H^1(N)$, we may use Theorem 4.1 to obtain a solution of the mixed problem with Neumann data f and zero Dirichlet data. Let $F = (\nabla u)^*$ and given a cube $Q \subset Q_0$ and with diameter r, define F_Q and R_Q as follows. We let $\bar f_{4Q} = 0$ if $4Q \cap D \neq 0$ and $\bar f_{4Q} = \fint_{4Q} f\, d\sigma$ if $4Q \subset N$. Set $g = \chi_{4Q}(f - \bar f_{4Q})$ and $h = f - g$. As both g and h are elements of the Hardy space $H^1(N)$, we may solve the mixed problem with Neumann data g or h. We let v be the solution with Neumann data g and w be the solution with Neumann data h. According to the uniqueness result Theorem 5.1 we have $u = v + w$. We let $R_Q = (\nabla w)^*$ and $F_Q = (\nabla v)^*$ so that (7.1) holds. We turn our attention to establishing (7.2) and (7.3).

To establish (7.2), observe that the H^1-norm of g satisfies the bound

$$\|g\|_{H^1(N)} \le C\|f\|_{L^p(4Q)}\sigma(Q)^{1/p'}.$$

With this, the estimate (7.2) follows from Theorem 4.1. Now we turn to the estimate (7.3) for $F_Q = (\nabla w)^*$. We note that the Neumann data h is constant on $4Q \cap N$. We define a maximal operator by taking the supremum over that part of the cone that is far from the boundary,

$$(\nabla w)^*_+(x) = \sup_{y\in\Gamma(x)\cap B_{Cr}(x)} |\nabla w(y)|$$

where C is to be chosen.

We first observe that we have

$$(\nabla w)^*_+(x) \le C \fint_{4Q} (\nabla w)^*\, d\sigma, \qquad x \in 2Q. \tag{7.4}$$

The estimate for $(\nabla w)^*_{Cr}$ uses the local estimate for the mixed problem in Lemma 6.1 to conclude that

$$\begin{aligned}\left(\fint_{2Q} (\nabla w)^{*\,q}_{Cr}\, d\sigma\right)^{1/q} &\le C\left[\left(\fint_{4Q} f^p\, d\sigma\right)^{1/p} + \fint_{T(3Q)} |\nabla w|\, d\sigma\right]\\ &\le C\left[\left(\fint_{4Q} f^p\, d\sigma\right)^{1/p} + \fint_{4Q} (\nabla w)^*\, d\sigma\right]. \end{aligned}\tag{7.5}$$

We choose the constant C in the definition of $(\nabla w)^*_+$ large in order that (7.5) holds. Recall that $T(Q)$ was defined at the beginning of the proof of Lemma 4.6. From the estimates (7.4) and (7.5), we conclude that

$$\left(\fint_{2Q} (R_Q)^q\, d\sigma\right)^{1/q} \le C\left[\left(\fint_{4Q} f^p\, d\sigma\right)^{1/p} + \left(\fint_{4Q} (\nabla w)^*\, d\sigma\right)^{1/p}\right]. \tag{7.6}$$

We have $(\nabla w)^* \le (\nabla v)^* + (\nabla u)^*$ and hence we may estimate the term involving the $(\nabla w)^*$ by

$$\fint_{4Q} (\nabla w)^* \, d\sigma \le \fint_{4Q} (\nabla u)^* \, d\sigma + \fint_{4Q} (\nabla v)^* \, d\sigma \le \fint_{4Q} (\nabla u)^* \, d\sigma + C \left(\fint_{4Q} f^p \, d\sigma \right)^{1/p}.$$

Combining this with (7.6) gives (7.3).

Applying the technique of Shen outlined above gives the L^p-estimate and thus we obtain the following theorem.

Theorem 7.7 *Suppose that $1 < p < p_0/2$ where p_0 is as in Lemma 3.17. Let $f_N \in L^p(N)$ and let u be the solution of the mixed problem with Neumann data f_N and zero Dirichlet data.*

The solution satisfies

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C \|f_N\|_{L^p(\partial\Omega)}.$$

The constant depends on the number of coordinate cylinders and p as well as M.

8 Further questions

This work adds to our understanding of the mixed problem in Lipschitz domains. However, there are several avenues which are not yet explored.

1. Can we study the inhomogeneous mixed problem and obtain results similar to those of Fabes, Mendez and Mitrea [11] and M. Mitrea and I. Mitrea [24]?
2. Is there an extension to $p < 1$ as the work of Brown [2]?
3. Can we study the mixed problem for more general decompositions of the boundary, $\partial\Omega = D \cup N$? To what extent is the condition that the boundary between D and N be a Lipschitz graph needed?
4. Can we extend these techniques to elliptic systems and higher order elliptic equations?

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