Math 114
Spring 2009
Professor Readdy

Answers to Review III (sketch)

1a. The nth partial sum of the series is $s_n = a_1 + a_2 + \cdots + a_n$.
1b. The series is convergent if the sequence of partial sums $\{s_n\}$ is convergent to a finite limit, that is, $\lim_{n\to+\infty} s_n = s$, where s is a finite value. A divergent series is a series which is not convergent.

2a. This is a geometric series with ratio $r = 2/5$ with $|r| = 2/5 < 1$. Hence it converges. (In fact, it converges to 5/3.)
2b. This is a geometric series with ratio $r = -1/5$ with $|r| = 1/5 < 1$. Hence it converges. (In fact, it converges to 5/6.)
2c. Limit of the terms approach $3/5 \neq 0$, so the series diverges.
2d. This is a p-series with $p = 1/2$. Since $p \leq 1$, the series diverges. (One could also show this with the integral test using $\int_1^{+\infty} x^{-1/2} dx$.)
2e. This is the divergent harmonic series.
2f. The absolute value of this series can be compared with the series $\sum_{n\geq 1} \frac{1}{n^2}$ and $\sum_{n\geq 1} \frac{-1}{n^2}$. Both are convergent p-series, hence the original series also converges.
2g. $\sin n\pi = 0$ for $n \geq 1$, so all the terms in the series equal zero. Hence the series converges to zero.
2h. This is the convergent alternating harmonic series.
2i. Converges (use limit comparison test with the convergent series $\sum_{n\geq 1} \frac{5n^2}{n^{18}} = \sum_{n\geq 1} \frac{5}{n^{16}}$).
2j. The ratio test gives

$$\lim_{n\to+\infty} \left| \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} \right)^2 \cdot \frac{e^{-(n+1)}}{e^{-n}} \right| = \frac{1}{e} < 1$$

so the series converges absolutely.
2k. The ratio test gives

$$\lim_{n\to+\infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n\to+\infty} \left| \frac{1}{(n+1)} \right| = 0 < 1,$$

so the series converges absolutely.
2l. The ratio test gives

$$\lim_{n\to+\infty} \left| \left(\frac{n}{n+1} \right)^n \right| = e^{-1} < 1,$$

so the series converges absolutely.

3a. $\sum_{n=0}^{+\infty} x^{3n}$
3b. $\sum_{n=2}^{+\infty} (-1)^n n \cdot (n-1) x^{n-2}$

4a. Use the fact that $\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$ to show the nth partial sum is $s_n = 1 - \frac{1}{n+1}$ and hence the series converges to 1.
4b. Convergent geometric series with $r = -2/7$ and sums to $7/9 - 1 + 2/7 = 4/63$
4c. This is the series $6 + \sum_{n\geq 0} 4 \cdot (\frac{2}{3})^n$. This sums to $6 + 4 \cdot 3 = 18$.

5a. Diverges by the integral test.
5b. Converges since is a p-series with $p = 7$. (Also can show convergence with integral test.)
5c. Diverges since is a p-series with $p = -7$.
5d. Converges using comparison test with the convergent geometric series $\sum_{n\geq 1} \frac{1}{3}$.
5e. Converges absolutely using limit comparison test with the convergent p-series $\sum_{n\geq 1} \frac{n^2}{n^4} = \sum_{n\geq 1} \frac{1}{n^2}$
5f. Does not converge absolutely as the limit of the absolute value of the terms is $1 \neq 0$. Converges as an alternating series.
5g. Converges absolutely using the limit

comparison test with the convergent p-series $\sum_{n\geq 1}\frac{1}{n^3}$.

6. By the integral test we have $\int_2^{+\infty}\frac{1}{x(\ln x)^2}dx = \frac{1}{\ln 2}$. The remainder R_n can be estimated by $R_n \leq \int_n^{+\infty}\frac{1}{x(\ln x)^2}dx = \frac{1}{\ln n}$. We want $\frac{1}{\ln n} \leq .01$ or $\ln n \geq 100$ or $n \geq e^{100}$ (many terms).

7. This is an alternating series with decreasing terms. However, the limit of the terms is $1 \neq 0$, so the alternating series diverges.

8a. By the root test the limit $\lim_{n\to+\infty}|a_n| = 1/2 < 1$, so the series converges absolutely.
8b. We have

$$\frac{1}{2^{n+(-1)^n}} \leq \frac{1}{2^{n+1}}.$$

Since the geometric series $\sum_{n\geq 0}\frac{1}{2^{n+1}}$ is a convergent geometric series (converges absolutely), the original series also converges absolutely using the comparison test.

9.

$$\begin{aligned}
&\int_0^1 \sum_{n\geq 0}(-1)^n\frac{x^{2n}}{(2n+2)!}dx \\
= \;& \sum_{n\geq 0}(-1)^n\frac{x^{2n+1}}{(2n+1)\cdot(2n+2)!}\Bigg]_0^1 \\
= \;& \sum_{n\geq 0}(-1)^n\frac{1}{(2n+1)\cdot(2n+2)!}
\end{aligned}$$

This is an alternating series and we want the error $< .000005$. The smallest such n is 3 and the integral is approximated by $1/2 - 1/(3\cdot 4!) + 1/(5\cdot 6!) = .486389$, so up to 5 decimal places is .48639.