

HOMEWORK

1. (due Friday, January 16th)

Let $f(x)$ be the ordinary generating function of the sequence $\{a_n\}_{n \geq 0}$. Express using $f(x)$ the ordinary generating function of the sequences

(a) $\{na_n\}_{n \geq 0}$

(b) $\{p(n)a_n\}_{n \geq 0}$ where $p(x)$ is any polynomial. (Hint: consider the special case when $p(x) = x^k$ and then prove for a general polynomial.)

(c) $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$

2. (due Wednesday, January 21st)

For each of the following expressions, find a simple formula which only involves one Fibonacci number. Then prove each identity by induction.

(a) $F_0 + F_1 + \dots + F_n$

(b) $F_0 + F_2 + \dots + F_{2n}$

(c) $F_1 + F_3 + \dots + F_{2n+1}$

3. (due Friday, January 23rd)

(a) Use Taylor's formula to show the power series expansion

$$(1+x)^\alpha = \sum_{k \geq 0} \frac{\alpha \cdot (\alpha - 1) \cdots (\alpha - k + 1)}{k!} \cdot x^k,$$

where α is any complex number. This is known as THE BINOMIAL THEOREM.

(b) Use partial fractions to find the power series expansion of $\frac{1+5x}{1-2x-3x^2}$.

4. (due Monday, January 26th)

Prove the identity

$$F_0 + F_1 + \dots + f_n = F_{n+2} - 1$$

using generating functions.

5. (due Wednesday, January 28th)

Consider the sequence \mathcal{A}_{n+1} of compositions of $n+1$ into odd parts, that is, compositions of the form $n+1 = c_1 + \dots + c_k$ where $c_i \in \{1, 3, 5, 7, \dots\}$.

(a) Give a generating function proof to show that $|\mathcal{A}_{n+1}|$ equals the Fibonacci number F_n .

(b) Give a bijective proof that $|\mathcal{A}_{n+1}| = F_n$.

6. (due Friday, February 6th)

Recall the n th derangement number D_n is the number of permutations in the symmetric group on n elements having no fixed points. Give a combinatorial proof (that is, work with permutations directly) of

$$D_{n+1} = n \cdot (D_n + D_{n-1}) \quad (n \geq 1)$$

with $D_0 = 1, D_1 = 0$.

7. (due Wednesday, February 11th)

Give a lattice path proof of the identity

$$\binom{n+m}{n} = \binom{n+m-1}{n} + \binom{n+m-1}{n-1}$$

with boundary conditions $\binom{m}{0} = 1, \binom{n}{n} = 1$.

8. (due Monday, February 16th)

For $m, n \in \mathbb{N}$, the q -binomial coefficient (*Gaussian polynomial*) is defined as

$$\left[\begin{matrix} m+n \\ n \end{matrix} \right] = \sum_p q^{\text{Area}(p)},$$

where the sum is over all lattice paths p from $(0, 0)$ to (m, n) having North and East steps and $\text{Area}(p)$ denotes the area under the path p .

(a) Prove for $n = 0$, $\left[\begin{matrix} 0 \\ k \end{matrix} \right] = \delta_{0,k}$. (Here $\delta_{0,k}$ denotes the Kronecker delta function, where $\delta_{x,x} = 1$ and for $x \neq y$ $\delta_{x,y} = 0$.)

(b) Prove for $n \geq 1$

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + q^k \left[\begin{matrix} n-1 \\ k \end{matrix} \right] \quad (1)$$

$$= q^{n-k} \cdot \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k \end{matrix} \right] \quad (2)$$

9. (due Wednesday, February 18th)

Give a direct counting argument to show that $\left(\binom{n}{k} \right) = \binom{n+k-1}{k}$ using dot and slash diagrams.

10. (due Monday, February 23rd)

Count the number of permutations in the symmetric group S_n which have

(a) $n - 2$ cycles

(b) $n - 3$ cycles

(c) $n - 4$ cycles

11. (due Monday, February 23rd)

(a) Find a recurrence for τ_n , the number of permutations $\pi \in S_n$ satisfying $\pi^3 = \text{id}$. Give a bijective proof of your recurrence.

(b) Find the exponential generating function for τ_n .

12. (due Wednesday, February 25th)

Let $f(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ and $g(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$ be exponential generating functions.

(a) Find an expression for the coefficients c_n of $f(x) \cdot g(x) = \sum_{n \geq 0} c_n \frac{x^n}{n!}$.

(b) Use part (a) and the exponential generating function for involutions to find an expression for inv_n , the number of involutions in the symmetric group S_n .

13. (due Friday, February 27th)

(a) Give a bijective proof of the identity

$$\binom{\binom{n}{2}}{2} = 3 \binom{n}{4} + n \binom{n-1}{2}.$$

(b) Find an expansion for

$$\binom{\binom{n}{2}}{3}$$

similar to the identity in part (a).

14. (due Monday, March 2nd)

Recall that for $k \in \mathbb{Z}$, $n \in \mathbb{N}$, the q -binomial coefficient (*Gaussian polynomial*) is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{[n]!}{[k]![n-k]!}, & 0 \leq k \leq n \\ 0, & k < 0 \text{ or } k > n. \end{cases}$$

Using this definition, prove: For $n = 0$, $\begin{bmatrix} 0 \\ k \end{bmatrix} = \delta_{0,k}$. For $n \geq 1$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{3}$$

$$= q^{n-k} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \tag{4}$$

15. (due Wednesday, March 4th)

Prove the noncommutative binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k},$$

where $yx = q \cdot xy$.

16. (due Friday, March 6th)

Give a bijective proof of the noncommutative binomial theorem.

17. (due Monday, March 9th)

Let $V = \mathbb{F}_q^n$ and $W = \mathbb{F}_q^k$, where \mathbb{F}_q is the finite field with q elements. Determine the number of functions $f : V \rightarrow W$ which are

(a) one-to-one

(b) onto

(c) all functions.

18. (due Wednesday, March 11th)
 Let $V = \mathbb{F}_q^n$ and $W = \mathbb{F}_q^k$, where \mathbb{F}_q is the finite field with q elements. Determine the number of linear transformations $f : V \rightarrow W$ which are
- one-to-one
 - onto
 - all linear transformations.
19. (due Monday, March 23rd)
 Find the exponential generating function for:
- The number of permutations π of an n -set such that $\pi^4 = id$.
 - The number of partitions of an n -set such that one element from each block is marked.
 - The number of permutations of an n -set where each k -cycle has one of 2 colors.
 - The number of ways to partition an n -set and then color each block with one of two colors.
20. (due Friday, March 27th)
 Use Lagrange Inversion Formula to count the number of labeled rooted trees on n vertices such that each parent has 0 or 2 children.
21. (due Monday, March 30th)
 Use Prüfer sequences to count the number of labeled rooted trees on n vertices such that each parent has 0 or 2 children.
22. (due Wednesday, April 15th)
 Exercise 3a from Chapter 2 of Stanley's text.
23. (due Wednesday, April 15th)
 Use the Principle of Inclusion and Exclusion to count the number of permutations in the symmetric group on n elements having no cycle of length 2.
24. (due Wednesday, April 15th)
 Give a sign-reversing involution to prove the identity

$$1 + \sum_{k=0}^n \binom{n}{k} (-1)^{k+1} k! (n - k + 1) = 0.$$

25. (due Monday, April 20th)
 Let P and Q be two graded posets with rank-generating functions $F(P, q)$ and $F(Q, q)$. Prove the Cartesian product $P \times Q$ is graded and

$$F(P \times Q, q) = F(P, q)F(Q, q).$$

26. (due Monday, April 20th)
 For the three posets pictured in Figure 3-40, Exercise 11 of Chapter 3 from Stanley's text, explain why each is not a lattice.

27. (due Monday, April 20th)
For the lattices pictured in Figure 3-5 of Stanley's text, determine which are not finite upper semimodular lattices.

28. (due Monday, April 20th)
Prove Proposition 3.3.3 from Stanley's text, that is,

Proposition 1 *Let L be a finite semimodular lattice. The following two conditions are equivalent:*

(i) *L is relatively complemented.*

(ii) *L is atomic.*

29. (due Wednesday, April 22nd)
For $L_n(q)$, the lattice of subspaces of an n -dimensional vector space over a finite field with q elements:

(a) Verify that $L_n(q)$ is rank-symmetric.

(b) Determine the number of elements covered and co-covered by a given element $x \in L_n(q)$ of rank k .

30. (due Friday, April 24th)
Prove that for a poset P the size of the largest antichain equals the minimal number of disjoint chains whose union is all the elements of P . As usual, please assume P is finite.

31. (due Friday, May 1st)
Recall the n -dimensional cube can be described as the convex hull of the points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $x_i \in \{0, 1\}$. The face lattice of the n -dimensional cube is the poset formed by taking the faces of the n -dimensional cube ordered by inclusion.

(a) Prove that \mathcal{C}_n is isomorphic to the poset $(\Lambda)^n \cup \{\hat{0}\}$, where Λ is the three element poset consisting of two elements covered by one element.

(b) Use the product theorem for the Möbius function to prove that the Möbius function of the face lattice of the n -dimensional cube is given by $\mu(\mathcal{C}_n) = (-1)^{n+1}$.

(Note: If you cannot prove (a), you can assume (a) is true and prove part (b).)