

HOMEWORK

1. (due Friday, January 13th)

(a) If  $f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$  then it has a multiplicative inverse if and only if  $a_0 \neq 0$ .

(b) Let  $f(x)$  be the ordinary generating function of the sequence  $\{a_n\}_{n \geq 0}$ . Express using  $f(x)$  the ordinary generating function of the sequences

(i)  $\{na_n\}_{n \geq 0}$

(ii)  $\{n^2 a_n\}_{n \geq 0}$

(iii)  $\{p(n)a_n\}_{n \geq 0}$ , where  $p(x) = c_0 + c_1x + \cdots + c_kx^k$  is a polynomial.

2. (due Wednesday, January 18th)

Prove the identity

$$\sum_{i=0}^n iF_i = nF_{n+2} - F_{n+3} + 3$$

using:

(a) induction

(b) generating functions.

As usual,  $F_n$  denotes the  $n$ th Fibonacci number.

3. Prove the Fibonacci identity

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$$

using:

(a) generating functions (due Monday, January 23rd)

(b) a bijective proof (due Wednesday, January 25th).

4. (due Friday, January 27th)

(a) Use Taylor's formula to show the power series expansion

$$(1+x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} \cdot x^k,$$

where  $\alpha$  is any complex number.

(b) Use part (a) to prove:

$$(i) \sum_{k \geq 0} \binom{n}{k} (-1)^k = 0.$$

$$(ii) \sum_{k \geq 0} \binom{n}{k} = 2^n.$$

$$(iii) \frac{1}{\sqrt{1-4x}} = \sum_{k \geq 0} \binom{2k}{k} x^k.$$

5. (due Monday, January 30th)

Prove

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \text{ where } 0 < k < n$$

with boundary conditions  $\binom{n}{0} = \binom{n}{n} = 1$  using generating functions.

6. (due Wednesday, February 1st)

Let  $f(m, n)$  be the number of paths from  $(0, 0)$  to  $(m, n) \in \mathbb{N} \times \mathbb{N}$ , where each step is of the form  $N = (1, 0)$ ,  $E = (0, 1)$ ,  $D = (1, 1)$ .

a. Show

$$\sum_{m \geq 0} \sum_{n \geq 0} f(m, n) x^m y^n = (1 - x - y - xy)^{-1}.$$

b. Find a simple expression for

$$\sum_{n \geq 0} f(n, n) x^n.$$

(This is Exercise 9 in Chapter 1 of EC I, 2nd Edition.)

7. (due Friday, February 3rd)

Prove directly that

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \quad (2)$$

8. (due Wednesday, February 8th)

For  $m, n \in \mathbb{N}$ , a combinatorial interpretation of the  $q$ -binomial coefficient (*Gaussian polynomial*) is

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_p q^{\text{Area}(p)},$$

where the sum is over all lattice paths  $p$  from  $(0, 0)$  to  $(m, n)$  having North and East steps and  $\text{Area}(p)$  denotes the area under the path  $p$ . Verify this interpretation satisfies one of the identities in Exercise 7. (Choose one to prove.)

9. (due Friday, February 10th)

For the symmetric group  $\mathfrak{S}_n$  prove bijectively that the number of permutations with an odd number of cycles equals the number of permutations with an even number of cycles.

10. (due Monday, February 13th)

Prove  $\text{inv}(\pi) = \text{inv}(\pi^{-1})$ , where  $\text{inv}$  denotes the inversion number.

11. (due Wednesday, February 15th)

Prove bijectively that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})$$

by finding a bijection between the set  $\{0\} \times \{0, 1\} \times \{0, 1, 2\} \times \cdots \times \{0, 1, \dots, n\}$  and  $\mathfrak{S}_n$ . As usual,  $\text{inv}(\cdot)$  denotes the inversion number.