Negative $q$-Stirling numbers

Wachs, fest

Margaret Readdy.

Joint with Yue Cai.
Michelle's abilities*

Topological might.

Combinatorial insight.

"Poset topology" she did write, which we use and cite.

* Not to be confused with "Michellability".
Let's count [$\sim 50,000 \text{ BC}^*$]

$$\sum_{\pi \in S_n} 1 = n!.$$ 

$$\sum_{S \subseteq \{1, \ldots, n\}, \mid S \mid = k} 1 = \binom{n}{k}.$$ 

*Source: Wikipedia*
Let's consider \( q \)-count \* [1700's Euler].

\( q \)-analogue of \( n \in \mathbb{Z}^+ \)

\[
[n]_q = [n] = 1 + q + \cdots + q^{n-1},
\]

\( q \) an indeterminate.

\[
\lim_{q \to 1} [n]_q = 1 + \cdots + 1 = n.
\]

\[
[n]! = [n] [n-1] \cdots [2] [1].
\]

\[\text{Theta functions} \quad \sum_{n=-\infty}^{\infty} \frac{(q^n)}{(a^n)} (\frac{q^n}{a^n}) \]

\[ f(a, b) = \sum_{n=-\infty}^{\infty} a^n b^n, \quad |ab| < 1 \]
Combinatorial interpretation

[MacMahon 1916]

\[ \sum_{\pi \in S_n} \text{inv}(\pi) q = [n]! , \]

where

\[ \text{inv}(\pi) = \{ (i,j) : i < j \text{ and } \pi_i > \pi_j \} . \]

for \ \pi = \pi_1 \ldots \pi_n \in S_n .
Gaussian polynomial. (the q-binomial)

\[ [n] = \begin{cases} \frac{[n]!}{[\kappa]! [n-\kappa]!} & 0 \leq \kappa \leq n \\ 0 & \kappa < 0 \text{ or } \kappa > n \end{cases} \]

Comb\'l interpretation:

\[ \sum_{\pi \in \mathfrak{S}(1^\kappa, 0^{n-\kappa})} \text{inv} \ \uparrow_{q} = [n]_{\kappa}. \]

[MacMahon 1916]
\[ \begin{array}{c|c}
\eta & \text{inv} \ \eta \\
0011 & 0 \\
0101 & 1 \\
0110 & 2 \\
1001 & 2 \\
1010 & 3 \\
1100 & 4 \\
\end{array} \]

\[
\sum_{\eta \in \{0^2,1^2\}} q^{\text{inv} \ \eta} = q^4 + q^3 + 2q^2 + q + 1.
\]

Check
\[
\left[ 4 \atop 2 \right] = \frac{\left[ 4 \atop 3 \right]}{\left[ 2 \atop 2 \right]} = \frac{(1+q)(1+q^2)(1+q+q^2)}{(1+q)}.
\]
The negative $q$-binomial \cite{Fur-Rem-Stan-Thie-2012}

\[
\begin{align*}
\left[ {n \atop \ell} \right]_q' & \triangleq (-1)^{\ell} [n]_{-q}^{(n-\ell)} \\
\end{align*}
\]

\text{ex.}
\[
\left[ {a \atop 4} \right]_q' = q^4 - q^3 + 2q^2 - q + 1.
\]
Theorem: [Fu - Reiner - Stanton - Thiem].

\[
\begin{align*}
\left[ \begin{array}{c}
\ell \\
\end{array} \right]_q &= \sum_{w \in \Omega(n, k)'} w^+(w) \\
&= \sum_{w \in \Omega(n, k)'} q^{a(w)} (q-1)^{p(w)}
\end{align*}
\]

where \( \Omega(n, k) \) is a certain subset of \( \{0, 1\}^k \cup 0^{n-k} \).

\( p(w) = \text{number of 10 pairs in } w \)  
\( a(w) = \text{inv}(w) - p(w) \).
Corollary: \([F-R-S-T]\) 

The \(q\)-binomial can be expressed as 

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \sum_{\omega \in \mathcal{Q}(n,k)} q^{|\omega|} (1+q)^{p(\omega)}.
\]
def. Given \( w = w_1 \cdots w_n \in \mathbb{B} \cdot \{1^w, 0^{n-w}\} \), pair

t. \( n = 1 \). Leave letter unpaired.

tt. \( n \geq 2 \) and \( w \) odd:

Pair \( w_1 \overline{w} \)

Repeat on \( w_2 \cdots w_n \)

ttt. \( n \geq 2 \) and \( w \) even:

Pair \( w_1 \)

Repeat on \( w_2 \cdots w_n \).

ex. \[
\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
\]
Define

\[ \omega_{n,k} = \begin{cases} \text{we get } 0^n1^k, \text{only if } & \text{w has no paired 01 y}. \end{cases} \]

ex. \[ \underline{\omega} \]

\begin{align*}
0011 & \quad \text{No.} \\
0101 & \quad \text{No.} \\
0110 & \quad \text{No.} \\
1001 & \quad \text{No.} \\
1010 & \quad \text{No.} \\
1100 &
\end{align*}
ex. (cont’d)

\[\Omega(2,2)\]

<table>
<thead>
<tr>
<th></th>
<th>(q)</th>
<th>(w^+(w))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0011</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0110</td>
<td>(q^2)</td>
<td>(q(1+q))</td>
</tr>
<tr>
<td>1001</td>
<td>(q^2)</td>
<td>(q^2)</td>
</tr>
<tr>
<td>1100</td>
<td>(q^4)</td>
<td>(q^3(1+q))</td>
</tr>
</tbody>
</table>

\[\sum = 1 + (q+q^3)(1+q) + q^2\]

\[= q^4 + q^3 + 2q^2 + q + 1.\]

Recall

\[w^+(w) = q \cdot (1+q) p(w)\]

\[p(w) = \# \text{ 10 pairs in } w, \quad a(w) = \]

\[a(w) = \text{inv}(w) - p(w).\]
L: What about other combinatorial objects with $q$-analogues?
Given a $q$-analogue

$$f[\vec{x}]_q = \sum_{w \in S} w^+(w),$$

when can we find a subset $T \subseteq S$ and statistics $A(\cdot) + B(\cdot)$ s.t.

$$f[\vec{x}]_q = \sum_{w \in T} q^{-A(w)} (1+q)^B(w).$$
The Stirling numbers of the second kind

\[ S(n, k) = \# \text{ partitions of } \{1, \ldots, n\} \into k \text{ blocks}. \]

\[ \text{ex. } S(4, 2) : \quad 1/234 \quad 13/24 \]
\[ \quad 134/2 \quad 13/24 \quad (\text{written in standard form}) \]
\[ \quad 124/3 \quad 14/23 \]
\[ \quad 123/4 \]

The \( q \)-Stirling numbers

\[ S_q^{[n, k]} = S_q^{[n-1, k-1]} + [k] \, S_q^{[n-1, k]} \]

with \( S_q^{[n, n]} = 1 = S_q^{[n, 1]} \).
**RG-words**  [Milne, Rota].

Encoder partition \( \Pi \) using a restricted growth word \( w \).

\[ w = w_1 \ldots w_n \text{ where } w_{z_i} = j \text{ if the elt } z_i \text{ is in the } j^{th} \text{ block of } \Pi. \]

Ex. \( \Pi = 125/36/47 \leftrightarrow 1123123 \).

Let \( \mathcal{R}_n(n, k) = \text{ set of all RG-words which encode an partition of } \{1, \ldots, n\} \text{ into } k \text{ parts.} \)
For \( w \in \mathcal{Q}_n(n,w) \) let

\[
    w^+(w) = \prod_{\varepsilon=1}^{n} w^+_\varepsilon(w),
\]

where \( m_\varepsilon = \max \{ w_1, \ldots, w_\varepsilon \} \),

\[
    w^+_1(w) = 1 \quad \text{and} \quad \text{for} \quad 2 \leq \varepsilon \leq n
\]

\[
    w^+\varepsilon(w) = \begin{cases} 
        q \cdot \varepsilon^{W\varepsilon-1} & \text{if} \quad w_\varepsilon \leq m_{\varepsilon-1} \\
        1 & \text{if} \quad w_\varepsilon > m_{\varepsilon-1}
    \end{cases}
\]

Theorem: \([\text{Cai-Rea/dley}]\)

The \( q \)-Stirling number of the second kind is given by

\[
    S_q[n,w] = \sum_{w \in \mathcal{Q}_n(n,w)} w^+(w).
\]
\[ w \bar{w} \]

\[
\begin{array}{cccc}
\text{w} & \text{w} & \text{w}_1 (w) \\
1/234 & 1222 & q' \cdot q' = q^2 \\
134/2 & 1211 & 1 \\
124/3 & 1121 & 1 \\
123/4 & 1112 & 1 \\
12/34 & 1122 & q' \\
13/24 & 1212 & q' \\
14/23 & 1221 & q' \\
\end{array}
\]

\[
\sum = q^2 + 3q + 3.
\]

\[ S_q [4,2] \]
Remark: See Garrett-Remmel, Milne, and especially Waage-White for a multitude of statistics that generate $S_q[n, u, v]$.

The $W+(\cdot)$ statistic is related to Waage-White's $L+(\cdot)$ statistic.
Let $w^*(w) = \prod_{\varepsilon=1}^{n} w_{\varepsilon}^*(w)$, $m_{\varepsilon} = \max R w_1, \ldots, w_{\varepsilon, y}$, and

$$w_{\varepsilon}^*(w) = \begin{cases} \frac{w_{\varepsilon, y} - 1}{q_{\varepsilon} (1 + q_{\varepsilon})} & \text{if } w_{\varepsilon} < m_{\varepsilon, y - 1} \\ \frac{w_{\varepsilon, y} - 1}{q_{\varepsilon}} & \text{if } w_{\varepsilon} = m_{\varepsilon, y - 1} \\ 1 & \text{if } w_{\varepsilon} > m_{\varepsilon, y - 1} \text{ or } \varepsilon = 1. \end{cases}$$

Write $A(w) = \sum_{\varepsilon=1}^{n} A_{\varepsilon}(w)$ and $B(w) = \sum_{\varepsilon=1}^{n} B_{\varepsilon}(w)$.

where

$$A_{\varepsilon}(w) = \begin{cases} w_{\varepsilon, y} - 1 & \text{if } w_{\varepsilon} \leq m_{\varepsilon, y - 1} \\ 0 & \text{if } w_{\varepsilon} > m_{\varepsilon, y - 1} \text{ or } \varepsilon = 1 \end{cases} \quad B_{\varepsilon}(w) = \begin{cases} 1 & \text{if } w_{\varepsilon} < m_{\varepsilon, y - 1} \\ 0 & \text{otherwise.} \end{cases}$$
Allowable RG-words

def. An RG-word \( w \in A(n,w) \) is allowable if it is of the form

\[
\begin{array}{ccccccc}
1 & \cdots & 1 & 2 & \cdots & 4 & 6 & \cdots \\
& & & & & & & \\
\frac{1}{2} & \frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{3}{5} & \frac{5}{5} \\
\end{array}
\]

Ex. \( w = 1121331435 \in A(10,5) \).

\( wt(w) = 1 \cdot 1 \cdot 1 \cdot (1 + q) \cdot 1 \cdot q^2 \cdot \cdots \cdot (1 + q) \cdot 1 \cdot q^2 (1 + q) \cdot 1 \)

Allowable words are denoted by \( A(n,w) \).
<table>
<thead>
<tr>
<th>( \text{w} )</th>
<th>( \text{w}^+(w) )</th>
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<tbody>
<tr>
<td>1222</td>
<td>-</td>
</tr>
<tr>
<td>1211</td>
<td>((1+q)^2)</td>
</tr>
<tr>
<td>1121</td>
<td>((1+q))</td>
</tr>
<tr>
<td>1112</td>
<td>1</td>
</tr>
<tr>
<td>1122</td>
<td>-</td>
</tr>
<tr>
<td>1212</td>
<td>-</td>
</tr>
<tr>
<td>1221</td>
<td>-</td>
</tr>
</tbody>
</table>

\[
\mathbb{W} = (1+q)^2 + (1+q) + 1
= q^2 + 3q + 3
\leq S_q [4,2]
\]
Example: $S_q[5,3]$. 

<table>
<thead>
<tr>
<th>$w$</th>
<th>$w^+'(w)$</th>
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<tbody>
<tr>
<td>12311</td>
<td>$(1+q)^2$</td>
</tr>
<tr>
<td>12131</td>
<td>$(1+q)^2$</td>
</tr>
<tr>
<td>12113</td>
<td>$(1+q)^2$</td>
</tr>
<tr>
<td>12133</td>
<td>$(1+q)q^2$</td>
</tr>
<tr>
<td>12313</td>
<td>$(1+q)q^2$</td>
</tr>
<tr>
<td>12331</td>
<td>$q^2(1+q)$</td>
</tr>
<tr>
<td>12333</td>
<td>$q^2q^2$</td>
</tr>
<tr>
<td>11231</td>
<td>$(1+q)$</td>
</tr>
<tr>
<td>11233</td>
<td>$(1+q)$</td>
</tr>
<tr>
<td>11123</td>
<td>$q^2$</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

$$\sum = q^4 + 3q^3(1+q) + q^2 + 3 \cdot (1+q)^2 + 2(1+q) + 1.$$ 

$$S_q[5,3] = q^4 + 3q^3 + 7q^2 + 8q + 6.$$
Theorem: [Cari-Reardon]

\[ S_q[n, \omega] = \sum_{w \in A(n, \omega)} w^+ (w) \]

\[ = \sum_{w \in A(n, \omega)} q^w \cdot (1+q) \]
Stembridge's $q = -1$

phenomenon

$B$ finite set

$$X(q) = \sum_{b \in B} q^{\text{wt}(b)}.$$ 

Set $q = -1$ to count fixed pts in an involution.

Corollary: [Cari - Readdy]

$S_q [n, \nu]$ when $q = -1$

counts the # of weakly increasing allowable words in $(n, \nu)$. 

Fam: $1 \cdots 1 2 3 \cdots 3 + 5 \cdots 5 6 \cdots$

(No $(1+q)$ terms).
The Stirling poset of the second kind $T(n, \lambda)$

For $\pi, \sigma \in \mathfrak{R}_n(n, \lambda)$ let $\pi \leq \sigma$ if

$$\sigma = \pi_1 \pi_2 \ldots (\pi_{i_{\star}} + 1) \ldots \pi_n.$$ 

for some index $i_{\star}$.

Clearly, $\pi \leq \sigma \Rightarrow \text{wt}(\sigma) = q \cdot \text{wt}(\pi)$.

Thus, $T(n, \lambda)$ is a graded poset.
Theorem: [Cari-Readdy].
The Stirling poset of the second kind has the decomposition:

\[ T(n, \nu) \cong \bigcup_{w \in A(n, \nu)} \bigcup_{|\text{Inv}(w)|} B_{|\text{Inv}(w)|} \]

where \( B_j \) is the Borel algebra on \( j \) elts, \( \text{Inv}(w) = \{ w_i : w_j > w_i \text{ for some } j < i \} \) is the set of all entries in \( w \) that contribute to an inversion, and \( A(n, \nu) \) are allowable \( \nu \)-words in \( T(n, \nu) \).
Homological $q = -1$

phenomenon [*Hersh-Sharbaugh-Stanton*]

Claim: Stembridge's $q = -1$ phenomenon is the same Euler characteristic computation.

Idea: Define a chain complex $(\mathcal{C}, \partial)$.

- Ranks of chain groups are coefficients in the polynomial $X(q)$.
- Euler characteristic is $X(-1)$.
- Also, Euler characteristic is an alternating sum of ranks of homology groups.

Best scenario: $(\mathcal{C}, \partial)$ has homology concentrated in ranks of same parity if has basis indexed by fixed points of involution = $X(-1)$. 
def. P graded poset

\[ W_i = \text{rank } i \text{ elts of } P \]

The poset P supports a chain complex \((\mathcal{C}, d)\)
of \(F\)-vector spaces \(C_i\) if:

- \(C_i\) has basis indexed by the elts \(W_i\)
- \(C_i \neq 0 \iff W_i \neq \emptyset\)
- a boundary map

For \(x \in W_{i-1}\), \(y \in W_i\) the coeff of
\(\partial y, x\) of \(x\) in \(\partial_i(y)\) is zero unless \(x = y\).
The algebraic complex $(\mathcal{C}, \partial)$ supported by the poset $\Pi(n, \mathcal{A})$.

For we $\mathcal{A}(n, \mathcal{A})$ let
\[ E(w) = \{ w_{i_1}, \ldots, w_{i_j} : i_1 < \cdots < i_j \}, \]
with $w_r \geq w_{i_k}$ for some $r < i_k$,
be the set of all repeated entries in $w$ arranged by index.

( $w = 122344 \Rightarrow E(w) = \{ w_3, w_4 \} = \{ 2, 4 \}$.)

The boundary map $\partial$ on $\mathcal{A}(n, \mathcal{A})$:
\[
\partial(w) = \begin{cases} 
\sum_{w_{i_k} \in E(w)} (-1)^{i_k-1} w_1 \cdots w_{i_k-1} (w_{i_k} - 1) w_{i_k+1} \cdots w_n & \text{if } w \notin \mathcal{A}(n, \mathcal{A}) \\
0 & \text{if } w \in \mathcal{A}(n, \mathcal{A}).
\end{cases}
\]
\[ w = 122344 \]
\[ E(w) = \{ w_5, w_6 \mid y = 24 \} \]
\[ \vartheta(w) = 12344 - 122343. \]

Lemma: \[ \vartheta^a = 0. \]
Algebraic Morse Theory.


Page

Orient edges in Hasse diagram downwards.

A partial matching is a subset \( M \subseteq P \times P \) s.t.

(i) \( (a, b) \in M \implies a \leq b \)

(ii) Each elt \( a \in P \) belongs to at most one elt in \( M \).

For \( (a, b) \in M \) write \( b = w(a) \), \( a = d(b) \) "up" "down".

A partial matching is acyclic if there are no cycles in the directed Hasse diagram.
Matching $M$ on $\Pi(n, k)$:

Let $\psi_i^2$ be first entry in $\psi_i = \psi_1 \cdots \psi_n \in \Pi(n, k)$ s.t. $\psi_i$ is weakly decreasing:

$$\psi_1 \leq \psi_2 \leq \cdots \leq \psi_{i-1} \geq \psi_i \cdots$$

and $\psi_{i-1} \geq \psi_i$ is strict unless both $\psi_{i-1} + \psi_i$ even.

For $\psi_i$ even:

$$d(\psi_i) = \psi_1 \psi_2 \cdots \psi_{i-1} (\psi_i - 1) \psi_{i+1} \cdots \psi_n.$$

For $\psi_i$ odd:

$$u(\psi_i) = \psi_1 \psi_2 \cdots \psi_{i-1} (\psi_i + 1) \psi_{i+1} \cdots \psi_n.$$
Lemma: The unmatched words in $T^*(n,\omega)$ are of the form $1 \cdots 1, 2 3 \cdots 3 4 5 \cdots 5 6 \cdots$

Theorem: [Kozlov]
A partial matching on $P$ is acyclic $\iff$
There exists a linear extension $L$ of $P$ such that the elites $v$ and $w(a)$ follow consecutively in $L$.

Theorem: [Cai–Reavddory]
The matching described for $T^*(n,\omega)$ is an acyclic matching.
Lemma: \cite{hersh-shareshian-stanton}.

A graded poset supporting an algebraic complex \((\mathcal{C}, \partial)\).

Assume \(P\) has a Morse matching \(M\) s.t. for all \(q = M(p)\) with \(q < p\) one has \(\partial_p, q \in F^*\).

If all unmatched elts occur in ranks of the same parity then:

\[
\dim H_\ast(\mathcal{C}, \partial) = \left| \{ p \in P : \text{un} M \} \right|, \text{ that is, the \# of unmatched elts of rank } \alpha.
\]
Lemma:\ The weighted generating function of the unmatched words in $T(n, \psi)$ is given by the $q^2$-binomial coefficient

$$
\sum_{\text{well } (n, \psi)} w^+ (w) = \left[ \begin{array}{l} n-1 - \left\lfloor \frac{\psi}{2} \right\rfloor \\ \left\lfloor \frac{\psi-1}{2} \right\rfloor \end{array} \right]_{q^2}.
$$

Theorem:\ [Cai–Readdy]
The algebraic complex $(\mathcal{C}, \partial)$ supported by $T(n, \psi)$ has basis for homology given by the increasing allowable RG-words in $A(n, \psi)$.

Furthermore

$$
\sum_{\varepsilon \geq 0} \dim (H_{\varepsilon}) q^{\varepsilon} = \left[ \begin{array}{l} n-1 - \left\lfloor \frac{\psi}{2} \right\rfloor \\ \left\lfloor \frac{\psi-1}{2} \right\rfloor \end{array} \right]_{q^2}.
$$
q-Stirling number of the first kind

\[ c[n, \psi] = c[n-1, \psi-1] + \binom{n-1}{\psi} c[n-1, \psi] \]

with \( c[n, 0] = S_n, 0 \).

Recall Stirling number \( c(n, \psi) \) counts \# \( \psi \in S_n \) with \( \psi \) disjoint cycles.

Theorem: [de Médicis-Leroux]

\[ c[n, \psi] = \sum_{T \in \mathcal{P}(n-1, n-\psi)} q_\psi \]

\( \mathcal{P}(m,n) \) = set of ways to place \( n \) rocks on a \( \psi \) length \( m \) stair-case board with no two rocks in same column.

For \( T \in \mathcal{P}(m,n) \), \( s(T) = \# \) of squares to the south of the rocks in \( T \).
\[ a \alpha \cdot c_{[4,2]} = q^3 + 3q^2 + 4q + 3 \]
To find a subgraph $Q(n-1, n-k)$ of $P(n-1, n-k)$:

\[ q^2 \]

\[ q^2(1+q) \]

\[ (1+q) \]

\[ (1+q)^2 \]

\[ (1+q) \]

\[ q^2 (1+q) + (1+q)^2 + q^2 + 2 \cdot (1+q) \]

\[ q^3 + 3q^2 + 4q + 3. \]
Theorem: \[ \text{[Cai-Readdy]} \]

\[ c[n, k] = \sum_{T \in Q(n-1, n-k)} q^{|s(T)|} (1+q)^{-r(T)} \]

where \( Q(n-1, n-k) \subseteq P(n-1, n-k) \) are rook placements on the alternating shaded staircase board (shaded alternatingly starting from lowest diagonal),

\( s(T) = \# \text{ squares to the south of the rooks in } T \)

\( r(T) = \# \text{ rooks not in first row} \).
The Stirling poset of the first kind $P(m,n)$

For $T, T' \in P(m,n)$ let $T \leq T'$ if $T'$ can be obtained from $T$ by moving one rook to the left (west) or up (north).
Define a matching $m$:

For $T \in P(m,n)$, let $r$ be the first rock (reading left to right) that is not in a shaded square in first row.

Match $T$ to $T'$ where $T'$ is obtained from $T$ by moving $r$ one square down if $r$ is not in a shaded square, or one square up if $r$ is in a shaded square but not in first row.
Lemma: The unmatched row placements in $\pi(m,n)$ have all of the rocks occur in shaded squares in the first row.

Theorem: [Cai- Readdy].

\[ \sum_{T \in \pi(m,n)} wt(T) = q^\frac{n(n-1)}{2} \begin{bmatrix} L_{m+1/2} \end{bmatrix} \begin{bmatrix} n \end{bmatrix} q^2. \]

$T \text{ unmatched}$
For $T \in P(m, n)$, let

$$N(T) = \{ \tau_j : \text{the rook } \tau_j \text{ in } T \text{ is not in a shaded square } j \}.$$ 

$$I(T) = \{ \tau_j : \tau_j \in N(T) \text{ and } \tau_j < \tau_{j_2} < \ldots < \tau_{j_{\text{in}(T)}} \}.$$ 

The boundary map $\partial$ on $P(m, n)$:

$$\partial(T) = \sum (-1)^{j-1} T_{\tau_j}.$$ 

where $T_{\tau_j}$ is obtained by moving the rook $\tau_j$ in $T$ down by one square.
Theorem: [Cai–Readdy]

The algebraic complex \((\mathcal{G}, \partial)\) supported by \(P(m,n)\) has bases for homology 
given by the rock placements in 
\(\mathcal{P}(m,n)\) having all of the rocks occur 
in shaded squares in the first row.

Furthermore,

\[
\sum_{i \geq 0} \dim \left( H_{i, i} \right) q^i = q^{\binom{n}{2}} \left[ \frac{L^m}{q^2} \right]_n.
\]
Orthogonality

Recall the signed $q$-Stirling numbers of the first kind.

$$s_q[n, \nu] = (-1)^{n-\nu} c[n, \nu].$$

Known generating polynomials.

$$(\nu x)_n, q = \sum_{\nu k = 0}^{n} s_q[n, \nu k] \nu \nu x^k$$

$$\nu x^n = \sum_{\nu k = 0}^{n} s_q[n, \nu k] (\nu x)^k, q$$

where

$$(\nu x)_n, q = \prod_{m=0}^{n-1} (\nu x - [m]_q).$$
Define the \((q,t)\) Stirling numbers of the first and second kind by

\[
S_{q,t}(n,\beta) = (-1)^{n-\beta} \sum_{T \in \mathcal{P}(n-1, n-\beta)} s_T r_T
\]

and

\[
S_{q,t}(n,\beta) = \sum_{\text{w.d.a.}(n,\beta)} A(w) B(w)
\]

respectively, where \(t = q+1\).
Let 
\[
[\nu^k]_{q,t} = \begin{cases} 
(q^{\nu-2} + q^{\nu-4} + \cdots + 1) \cdot t & \text{for } \nu \text{ even} \\
q^{\nu-1} + (q^{\nu-3} + q^{\nu-5} + \cdots + 1) t & \text{for } \nu \text{ odd}
\end{cases}
\]

Theorem: [Cai- Readdy].

The generating polynomials for the \((q,t)\)-Stirling numbers are 

\[
(vx)_n^{q,t} = \sum_{\nu=0}^{n} S_{\nu,^{\nu}} \cdot q^{\nu} 
\]

\[
v^n x = \sum_{\nu=0}^{n} S_{\nu,^{\nu}} (vx)^{\nu} 
\]

where 

\[
(vx)_n^{q,t} = \prod_{m=0}^{n-1} (vx - [m]_{q,t}).
\]
Theorem: [de Médicis - Leroux].
The signed q-Stirling numbers $s_q [n, k]$ and the q-Stirling numbers $S_q [n, k]$
are orthogonal, that is,

$$\sum_{k=m}^{n} s_q [n, k] S_q [k, m] = S_{m,n}$$

and

$$\sum_{k=m}^{n} S_q [n, k] s_q [k, m] = S_{m,n}$$

Furthermore, this orthogonality holds bijectively.
Theorem: [Cai-Readdy].

The $(q,t)$-Stirling numbers are orthogonal, that is,

\[ \sum_{\nu, \tau} s_{q,t}^{[\nu, \tau]} \cdot S_{q,t}^{[\nu, \tau]} = S_{m,n} \]

and

\[ \sum_{\nu, \tau} s_{q,t}^{[\nu, \tau]} s_{q,t}^{[\nu, \tau]} = S_{m,n} \]

Furthermore, this orthogonality holds bijectively.
Thank you!