Homology of Newtonian Coalgebras

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Abstract

Given a Newtonian coalgebra we associate to it a chain complex. The homology groups of this Newtonian chain complex are computed for two important Newtonian coalgebras arising in the study of flag vectors of polytopes: $R\langle \mathbf{a}, \mathbf{b} \rangle$ and $R\langle \mathbf{c}, \mathbf{d} \rangle$. The homology of $R\langle \mathbf{a}, \mathbf{b} \rangle$ corresponds to the homology of the boundary of the n-crosspolytope. In contrast, the homology of $R\langle \mathbf{c}, \mathbf{d} \rangle$ depends on the characteristic of the underlying ring R. In the case the ring has characteristic 2, the homology is computed via cubical complexes arising from distributive lattices. This paper ends with a characterization of the integer homology of $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.

1 Introduction

This paper introduces homological algebra to Newtonian coalgebras. Newtonian coalgebras were first introduced by Joni-Rota [13] and then studied systematically by Hirschhorn-Raphael [12] for the polynomial ring $k[x]$. For more recent work, we refer to Aguiar's papers [2, 1].

The renaissance of Newtonian coalgebras has been its connection with studying the difficult question of characterizing face incidence information of polytopes. Beginning in [11], Ehrenborg and Readdy first introduced the ideas of coalgebras to polytopal theory. Their main result is that the **cd**-index, a noncommutative polynomial which encodes polytopal face incidence data without linear redundancies, is in fact a coalgebra homomorphism. As one consequence, geometric operations on a polytope can be expressed as derivation operations on the **cd**-index. With Billera, Ehrenborg and Readdy continued this line of work by characterizing the flag vector data of polytopes arising from central hyperplane arrangements [8]. The power of Newtonian coalgebras was again exploited when Billera and Ehrenborg [7] settled the Stanley conjecture for Gorenstein* lattices [16] in the case of polytopes.

For A a Newtonian coalgebra, the central object we will work with is the Newtonian chain complex $\mathcal{N}_n(A)$; see (2.2). The two interesting cases we consider are when A is $R\langle \mathbf{a}, \mathbf{b} \rangle$, the Newtonian coalgebra generated by the two non-commutative variables $\mathbf{a}$ and $\mathbf{b}$, and $R\langle \mathbf{c}, \mathbf{d} \rangle$, the Newtonian subalgebra generated by the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Both are geometrically motivated as they arise in the seminal coalgebraic study of flag vectors [11]. Surprisingly, the problem of determining the homology of their respective Newtonian chain complexes reduces to computing the homology of well-known objects.

The homology of the Newtonian algebra $R\langle \mathbf{a}, \mathbf{b} \rangle$ case corresponds to the reduced homology of the boundary of the crosspolytope. For the $R\langle \mathbf{c}, \mathbf{d} \rangle$ case, the behavior of the homology groups is dependent on the underlying ring R. When the element 2 is a unit in the ring R, the Newtonian chain complex $\mathcal{N}_n(R\langle \mathbf{c}, \mathbf{d}\rangle)$, abbreviated $\mathcal{C}_n(R)$, is the direct sum of chain complexes corresponding to the reduced homology of simplices of various dimensions. The case when the ring R has characteristic 2 is quite different. The corresponding topological objects are then cubical complexes built from distributive lattices. By a theorem of Kalai and Stanley (see Theorem 4.3), these cubical complexes are contractible and thus only have zeroth homology.

The next important ring to consider is the integers. By first considering $\mathbb{Z}_4$ we are able to obtain the homology groups for $\mathcal{N}_n(\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle) = \mathcal{C}_n(\mathbb{Z})$ using homological algebra techniques. However, even more can be obtained. Namely, in Theorem 5.2 we describe how the image of the boundary map ∂_{i+1} is contained in the kernel of ∂_i, thus giving a larger understanding of the chain complex.

We hope that this paper will spur interest in Newtonian coalgebras and their homology groups.

2 Preliminaries on Newtonian coalgebras

Let R be a commutative ring with unit. Let A be an R-module with a coassociative coproduct $\Delta : A \longrightarrow A \otimes A$, that is, Δ satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. Define a map $d_n : A^{\otimes n} \longrightarrow A^{\otimes(n+1)}$ by

$$d_n = \sum_{i+j=n-1} (-1)^i \cdot \mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes j}.$$

Then we have the following lemma.

Lemma 2.1 *The identity $d_{n+1} \circ d_n = 0$ holds, that is, d is a boundary map.*

Proof: By linearity it is enough to prove that $d_{n+1} \circ d_n$ applied to an element $x_1 \otimes \cdots \otimes x_n$ is equal to zero. This follows from the fact that Δ is coassociative. □

Using the boundary map d we obtain a chain complex. The natural question to consider is to compute the homology of this chain complex. Our interest will be do to this in the case when A is a Newtonian coalgebra.

Definition 2.2 *An R-module A is a* Newtonian coalgebra *if A has an associative product $\mu : A \otimes A \longrightarrow A$ with unit 1 and coassociative coproduct $\Delta : A \longrightarrow A \otimes A$ such that*

$$\Delta \circ \mu = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) + (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id}). \tag{2.1}$$

In what follows we only consider Newtonian algebras A that have grading $A = \bigoplus_{n \geq 0} A_n$ such that A_0 is isomorphic to the ring R, $A_i \cdot A_j \subseteq A_{i+j}$ and $\Delta(A_n) \subseteq \bigoplus_{i+j=n-1} A_i \otimes A_j$.

The Newtonian coalgebras that we are interested in are $R\langle \mathbf{a}, \mathbf{b}\rangle$ and $R\langle \mathbf{c}, \mathbf{d}\rangle$. The first Newtonian coalgebra $R\langle \mathbf{a}, \mathbf{b}\rangle$ is generated by two non-commutative variables $\mathbf{a}$ and $\mathbf{b}$ such that $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) =$

$1 \otimes 1$. The second Newtonian coalgebra $R\langle \mathbf{c}, \mathbf{d}\rangle$ is a Newtonian subcoalgebra of $R\langle \mathbf{a}, \mathbf{b}\rangle$ where we set $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. Then $R\langle \mathbf{c}, \mathbf{d}\rangle$ is generated by the two non-commutative variables $\mathbf{c}$ and $\mathbf{d}$ such that $\Delta(\mathbf{c}) = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = \mathbf{c} \otimes 1 + 1 \otimes \mathbf{c}$.

From [11, Lemma 2.2 and Corollary 3.2] we have the two following lemmas.

Lemma 2.3 *Let* $R\langle \mathbf{a}, \mathbf{b}\rangle = \bigoplus_{n \geq 0} A_n$ *be the grading of* $R\langle \mathbf{a}, \mathbf{b}\rangle$*. Then the kernel of the coproduct* $\Delta : A_n \longrightarrow \bigoplus_{i+j=n-1} A_i \otimes A_j$ *is generated by the element* $(\mathbf{a} - \mathbf{b})^n$*.*

Lemma 2.4 *Let* $R\langle \mathbf{c}, \mathbf{d}\rangle = \bigoplus_{n \geq 0} A_n$ *be the grading of* $R\langle \mathbf{c}, \mathbf{d}\rangle$*. Then the kernel of the coproduct* $\Delta : A_n \longrightarrow \bigoplus_{i+j=n-1} A_i \otimes A_j$ *is* 0 *if* n *is odd and is generated by the element* $(\mathbf{c}^2 - 2\mathbf{d})^{n/2}$ *if* n *is even.*

Let $W_{n,i}(A)$ denote

$$W_{n,i}(A) = \bigoplus_{j_1 + \cdots + j_{n-i+1} = i} A_{j_1} \otimes \cdots \otimes A_{j_{n-i+1}}.$$

Observe that d_{n+1-i} maps $W_{n,i}$ to $W_{n,i-1}$. Thus setting $\partial_i = d_{n+1-i}$, we have the *Newtonian chain complex*

$$\mathcal{N}_n(A) : \quad 0 \longrightarrow W_{n,n}(A) \xrightarrow{\partial_n} W_{n,n-1}(A) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} W_{n,1}(A) \xrightarrow{\partial_1} W_{n,0}(A) \longrightarrow 0. \qquad (2.2)$$

Let $H_i(\mathcal{N}_n(A))$ denote the ith homology group of this chain complex, that is,

$$H_i(\mathcal{N}_n(A)) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

The tensor ring, $T(A)$, of the Newtonian coalgebra A is the direct sum

$$T(A) = \bigoplus_{i \geq 1} A^{\otimes i}.$$

If A is generated by k elements $x_1, \ldots, x_k$, it will be convenient to view the tensor ring generated by the $k+1$ elements $x_1, \ldots, x_k, \otimes$. Thus it is natural to consider monomials in these $k+1$ variables. Moreover, let $\otimes$ be a generator of degree 1, that is, the degree of an element $y_1 \otimes y_2 \otimes \cdots \otimes y_k$ is given by $\deg(y_1) + \deg(y_2) + \cdots + \deg(y_k) + k - 1$. Let $T_n(A)$ consists of all homogeneous elements of $T(A)$ of degree n. Then $T_n(A)$ is the direct sum of all the components appearing in the Newtonian chain complex $\mathcal{N}_n(A)$, that is, $T_n(A) = \bigoplus_k W_{n,k}(A)$.

3 Homology of ab-polynomials

Theorem 3.1 *The homology of the Newtonian chain complex* $\mathcal{N}_n(R\langle \mathbf{a}, \mathbf{b}\rangle)$ *vanishes everywhere except for the top homology, which is spanned by* $(\mathbf{a} - \mathbf{b})^n$*.*

First proof: Observe that we are interested in monomials of length n in the two variables $\mathbf{a}$ and $\mathbf{b}$ and in the tensor sign $\otimes$. These monomials are in bijection with the faces of the boundary of the n-dimensional crosspolytope. For instance, the 2^n monomials in $\mathbf{a}$ and $\mathbf{b}$ correspond with the 2^n facets of the crosspolytope and the term $1 \otimes \cdots \otimes 1$ corresponds to the empty face. Also the boundary map of the chain complex corresponds to the geometric boundary map under this bijection. Hence, the question of computing the homology of $R\langle \mathbf{a}, \mathbf{b} \rangle_n$ is equivalent to computing the relative homology of the boundary of the crosspolytope. This homology vanishes everywhere but in the top homology, where it is one-dimensional. This top homology group is the kernel of Δ, which is determined in Lemma 2.3. $\square$

Second proof: This proof works under the assumption that the element 2 is a unit in the ring R. Begin by making a change of basis into $\mathbf{c}/2 = (\mathbf{a} + \mathbf{b})/2$ and $\mathbf{e} = \mathbf{a} - \mathbf{b}$, that is, $R\langle \mathbf{a}, \mathbf{b} \rangle = R\langle \mathbf{c}/2, \mathbf{e} \rangle$. Observe that $\Delta(\mathbf{c}/2) = 1 \otimes 1$ and $\Delta(\mathbf{e}) = 0$. For S a subset of $\{1, \ldots, n\}$ let $V_{n,k,S}$ be the subspace of $W_{n,k}(R\langle \mathbf{c}/2, \mathbf{e} \rangle)$ spanned by the $\binom{|S|}{n-k}$ monomials where $\mathbf{e}$ appears in every position not in the set S. Hence we have the direct sum decomposition $W_{n,k}(R\langle \mathbf{c}/2, \mathbf{e} \rangle) = \bigoplus_S V_{n,k,S}$. Moreover, the boundary map $\partial_k : W_{n,k}(R\langle \mathbf{c}/2, \mathbf{e} \rangle) \longrightarrow W_{n,k-1}(R\langle \mathbf{c}/2, \mathbf{e} \rangle)$ restricts to a map $\partial_k : V_{n,k,S} \longrightarrow V_{n,k-1,S}$. Hence associated with the subset S we have the chain complex

$$\mathcal{M}_{n,S} : \quad 0 \longrightarrow V_{n,n,S} \xrightarrow{\partial_n} V_{n,n-1,S} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} V_{n,1,S} \xrightarrow{\partial_1} V_{n,0,S} \longrightarrow 0.$$

For the Newtonian chain complex $\mathcal{N}_n(R\langle \mathbf{a}, \mathbf{b} \rangle)$ we have the decomposition $\mathcal{N}_n(R\langle \mathbf{a}, \mathbf{b} \rangle) = \bigoplus_S \mathcal{M}_{n,S}$.

Observe that $\mathcal{M}_{n,S}$ is isomorphic to the chain complex that computes the reduced homology of an $(|S| - 1)$-dimensional simplex. This reduced homology vanishes everywhere except when S is the empty set. In this case the homology is one-dimensional and is generated by $\mathbf{e}^n = (\mathbf{a} - \mathbf{b})^n$. $\square$

Let $R\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle$ denote the Newtonian coalgebra of non-commutative polynomials in the variables $\mathbf{a}_1, \ldots, \mathbf{a}_k$ such that $\Delta(\mathbf{a}_i) = 1 \otimes 1$. Similar to Theorem 3.1 we have the following result, the proof of which we omit.

Theorem 3.2 *The homology of the Newtonian chain complex $\mathcal{N}_n(R\langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle)$ vanishes everywhere except for the top homology, which is spanned by the $(k-1)^n$ elements of the form $(\mathbf{a}_{i_1+1} - \mathbf{a}_{i_1}) \cdots (\mathbf{a}_{i_n+1} - \mathbf{a}_{i_n})$, where $1 \leq i_j \leq k-1$.*

4 Homology of cd-polynomials

We now begin the study of the homology of the Newtonian coalgebra $R\langle \mathbf{c}, \mathbf{d} \rangle$. Since the homology varies for different rings R, let us introduce the notation that $\mathcal{C}_n(R)$ stands for the Newtonian chain complex $\mathcal{N}_n(R\langle \mathbf{c}, \mathbf{d} \rangle)$. Similarly, let $\mathcal{C}_n$ denote the Newtonian chain complex $\mathcal{N}_n(\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle)$. In this section we obtain the homology groups of $\mathcal{C}_n(R)$ when 2 is invertible in R or the ring R has characteristic 2. In the next section we compute the homology groups for the case R is the integers.

Theorem 4.1 *Let R be a ring such that 2 is a unit in R. When n is odd the homology of the Newtonian* *chain complex $\mathcal{C}_n(R)$ vanishes everywhere. When n is even the homology of $\mathcal{C}_n(R)$ vanishes everywhere except for the top homology, which is spanned by* $(\mathbf{c}^2 - 2\mathbf{d})^{n/2}$.

Proof: We use the observation $R\langle \mathbf{c}, \mathbf{d}\rangle = R\langle \mathbf{c}/2, \mathbf{e}^2\rangle$. Now use the second proof of Theorem 3.1. Pick the chain complexes corresponding to the sets S where the complement of S is the disjoint union of pairs of consecutive elements. □

Corollary 4.2 *For p an odd prime number the homology of the Newtonian chain complex $\mathcal{C}_n(\mathbb{Z}_p)$ is given by*

$$H_i(\mathcal{C}_n(\mathbb{Z}_p)) \cong \begin{cases} \mathbb{Z}_p & \text{if } i = n \text{ and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

To every poset P we can associate a cubical complex $\mathcal{C}(P)$ by letting the vertices of $\mathcal{C}(P)$ be the elements of the poset and the faces of $\mathcal{C}(P)$ be the intervals in the poset that are isomorphic to Boolean algebras. A finite meet-semilattice is called *meet-distributive* if $[x, y]$ is an interval in P with x equal to the meet of the coatoms in $[x, y]$ implies the interval $[x, y]$ is isomorphic to a Boolean algebra. The most natural examples of finite meet-distributive meet-semilattices are finite distributive lattices.

Kalai and Stanley [15, Exercise 3.19b] proved the following topological result.

Theorem 4.3 *Let P be a finite meet-distributive meet-semilattice. Then the associated cubical complex $\mathcal{C}(P)$ is collapsible and in fact contractible.*

We next consider the characteristic 2 case.

Theorem 4.4 *Let R be a ring such that $2 = 0$. The homology of the Newtonian chain complex $\mathcal{C}_n(R)$ is given by $H_k(\mathcal{C}_n(R)) \cong R$, where $0 \leq k \leq n$. Moreover, the homology group $H_k(\mathcal{C}_n(R))$ is generated by any $\mathbf{c}\otimes$-monomial with k $\mathbf{c}$'s and $(n-k)$ $\otimes$'s.*

Proof: For $0 \leq s \leq n$ let $V_{n,k,s}$ be the subgroup of $W_{n,k}(R\langle \mathbf{c}, \mathbf{d}\rangle)$ generated by monomials consisting of $2s - k$ $\mathbf{c}$'s, $k - s$ $\mathbf{d}$'s and $n - k$ $\otimes$'s. That is, we have the direct sum decomposition $W_{n,k}(R\langle \mathbf{c}, \mathbf{d}\rangle) = \bigoplus_s V_{n,k,s}$.

Since $2 = 0$ we have that $\Delta(\mathbf{c}) = 0$, that is, the coproduct only acts on $\mathbf{d}$. Hence if a monomial m consists of x $\mathbf{c}$'s, y $\mathbf{d}$'s and z $\otimes$'s, then the boundary map of m consists of a sum of monomials with $x + 1$ $\mathbf{c}$'s, $y - 1$ $\mathbf{d}$'s and $z + 1$ $\otimes$'s. Thus the chain complex $\mathcal{C}_n(R)$ decomposes as a direct sum of the following chain complexes:

$$\mathcal{M}_{n,s}: \quad 0 \longrightarrow V_{n,n,s} \xrightarrow{\partial_n} V_{n,n-1,s} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} V_{n,1,s} \xrightarrow{\partial_1} V_{n,0,s} \longrightarrow 0.$$

Thus we have $\mathcal{C}_n(R) = \bigoplus_s \mathcal{M}_{n,s}$.

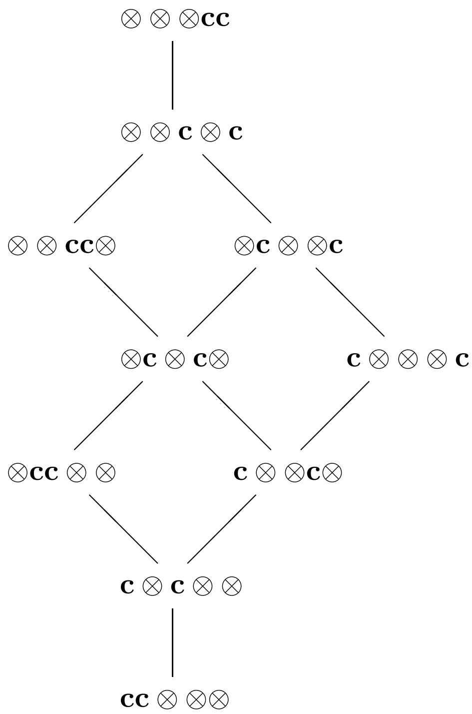

Figure 1: The distributive lattice $L_{5,2}$ associated with 2 $\mathbf{c}$'s and 3 $\otimes$'s. Observe that the three squares in the lattice correspond to the monomials $\mathbf{dd}\otimes$, $\mathbf{d} \otimes \mathbf{d}$ and $\otimes\mathbf{dd}$.

Let $L_{n,s}$ be the set of monomials consisting of s $\mathbf{c}$'s and $n - s$ $\otimes$'s. Define a partial order on $L_{n,s}$ by $u\mathbf{c} \otimes v$ is covered by $u \otimes \mathbf{c}v$. See Figure 1 for the case when $n = 5$ and $s = 2$. The poset $L_{n,s}$ is a distributive lattice. In fact, it is the lattice of lower order ideals of a product of an s-element chain with an $(n - s)$-element chain.

Let w be a monomial in $\mathbf{c}$'s, $\mathbf{d}$'s and $\otimes$'s having i $\mathbf{d}$'s. Write w as the product $u_1\mathbf{d}u_2\mathbf{d} \cdots \mathbf{d}u_{i+1}$. Let $F(w)$ be the i-dimensional cubical face in $\mathcal{C}(L_{n,s})$ associated to the interval between $u_1\mathbf{c} \otimes u_2\mathbf{c} \otimes \cdots \mathbf{c} \otimes u_{i+1}$ and $u_1 \otimes \mathbf{c}u_2 \otimes \mathbf{c} \cdots \otimes \mathbf{c}u_{i+1}$. Observe that F is a bijection between monomials and faces. Moreover, the boundary of the face $F(w)$ is F applied to each monomial in $\partial(w)$. Thus the homology of the chain complex $\mathcal{M}_{n,s}$ is equal to the homology of the cubical complex $\mathcal{C}(L_{n,s})$. However this homology is shifted, that is, the zeroth homology of the cubical complex appears as the $(n - s)$th homology group of $\mathcal{M}_{n,s}$.

By Theorem 4.3 we know that the cubical complex $\mathcal{C}(L_{n,s})$ is collapsible to a point. Thus this cubical complex has the same homology as a point and this homology is generated by one of its vertices. Hence the $(n - s)$th homology of $\mathcal{M}_{n,s}$ is generated by any monomial with s $\mathbf{c}$'s and $n - s$ $\otimes$'s. By direct summing the chain complexes over $0 \leq s \leq n$, the result follows. $\square$

We now have the immediate corollary.

Corollary 4.5 *The homology of the Newtonian chain complex $\mathcal{C}_n(\mathbb{Z}_2)$ is given by $H_k(\mathcal{C}_n(\mathbb{Z}_2)) \cong \mathbb{Z}_2$ for $0 \leq k \leq n$.*

We now turn our attention to the ring $\mathbb{Z}_4$.

Theorem 4.6 *The homology of the Newtonian chain complex* $\mathcal{C}_n(\mathbb{Z}_4)$ *is given by*

$$H_i(\mathcal{C}_n(\mathbb{Z}_4)) \cong \begin{cases} \mathbb{Z}_4 & \text{if } i = n, \\ \mathbb{Z}_2 & \text{if } 0 \leq i < n. \end{cases}$$

Proof: Tensor product the exact sequence $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ with the chain complex $\mathcal{C}_n$ to obtain the following exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{C}_n(\mathbb{Z}_2) \xrightarrow{\phi} \mathcal{C}_n(\mathbb{Z}_4) \xrightarrow{\psi} \mathcal{C}_n(\mathbb{Z}_2) \longrightarrow 0.$$

By the zig-zag lemma, we have the following exact sequence of homology groups:

$$\cdots \longrightarrow H_{i+1}(\mathcal{C}_n(\mathbb{Z}_2)) \xrightarrow{\partial_*} H_i(\mathcal{C}_n(\mathbb{Z}_2)) \xrightarrow{\phi_*} H_i(\mathcal{C}_n(\mathbb{Z}_4)) \xrightarrow{\psi_*} H_i(\mathcal{C}_n(\mathbb{Z}_2)) \xrightarrow{\partial_*} H_{i-1}(\mathcal{C}_n(\mathbb{Z}_2)) \longrightarrow \cdots \tag{4.3}$$

where ∂_* is induced by the boundary map in $\mathcal{C}_n(\mathbb{Z}_4)$. Let us determine the map $\partial_* : H_{i+1}(\mathcal{C}_n(\mathbb{Z}_2)) \longrightarrow H_i(\mathcal{C}_n(\mathbb{Z}_2))$. By Corollary 4.5 we know that both groups are isomorphic to $\mathbb{Z}_2$. The homology in $H_{i+1}(\mathcal{C}_n(\mathbb{Z}_2))$ is generated by $m = \mathbf{c}^{i+1} \otimes^{n-i-1} \in V_{n,i+1}(\mathbb{Z}_2)$. We can lift m to $V_{n,i+1}(\mathbb{Z}_4)$ to obtain $\mathbf{c}^{i+1} \otimes^{n-i-1}$. Applying the boundary map ∂ gives $2 \cdot \sum_{i_1+i_2=i} \mathbf{c}^{i_1} \otimes \mathbf{c}^{i_2} \otimes^{n-i-1}$ in $V_{n,i}(\mathbb{Z}_4)$. Lastly, this element can be lifted to $V_{n,i}(\mathbb{Z}_2)$ to obtain $\partial_*(m) = \sum_{i_1+i_2=i} \mathbf{c}^{i_1} \otimes \mathbf{c}^{i_2} \otimes^{n-i-1}$. If i is even then $\partial_*(m)$ is a sum of an odd number of generators of the homology group $H_i(\mathcal{C}_n(\mathbb{Z}_2)) \cong \mathbb{Z}_2$. Thus $\partial_*(m)$ is a generator and we conclude that ∂_* is the identity map. When i is odd the argument follows the same outline and we obtain that ∂_* is the zero map.

Next consider the exact sequence (4.3) when i is even. By exactness at $H_i(\mathcal{C}_n(\mathbb{Z}_2))$, we have that $\mathrm{im}(\psi_*) \cong \mathbb{Z}_2$. Thus ψ_* is surjective. By the exactness at $H_i(\mathcal{C}_n(\mathbb{Z}_2))$, we know that $\ker(\phi_*) \cong \mathbb{Z}_2$. Hence ϕ_* is the zero map, that is, $\mathrm{im}(\phi_*) \cong 0$. By the exactness at $H_i(\mathcal{C}_n(\mathbb{Z}_4))$, we have that ψ_* is injective and hence ψ_* is an isomorphism. We conclude that $H_i(\mathcal{C}_n(\mathbb{Z}_4))$ is isomorphic to $\mathbb{Z}_2$. A similar argument holds when i is odd. □

5 Integer homology of cd-polynomials

In this section we complete our analysis of the Newtonian chain complex $\mathcal{C}_n = \mathcal{C}_n(\mathbb{Z})$ by computing its homology.

Theorem 5.1 *The homology of the Newtonian chain complex* $\mathcal{C}_n$ *is given by*

$$H_i(\mathcal{C}_n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \text{ and } i \text{ is even}, \\ \mathbb{Z}_2 & \text{if } 0 \leq i < n \text{ and } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases}$$

Proof: The top homology, $i = n$, is given by Lemma 2.4. The case when $i = 0$ is a straightforward calculation. By the universal coefficient theorem for homology, we have the short exact sequence

$$0 \longrightarrow H_i(\mathcal{C}_n) \otimes R \longrightarrow H_i(\mathcal{C}_n(R)) \longrightarrow H_{i-1}(\mathcal{C}_n) * R \longrightarrow 0, \tag{5.4}$$

where $*$ denotes the torsion product. Recall that $\mathbb{Z}_a \otimes \mathbb{Z}_b \cong \mathbb{Z}_a * \mathbb{Z}_b \cong \mathbb{Z}_{\gcd(a,b)}$ and $\mathbb{Z}_a \otimes \mathbb{Z} \cong \mathbb{Z}_a$. Apply (5.4) with $R = \mathbb{Z}_p$ where p is an odd prime number. We obtain $H_i(\mathcal{C}_n) \otimes \mathbb{Z}_p \cong 0$. Now $H_i(\mathcal{C}_n)$ is a finitely generated group. However, the condition $H_i(\mathcal{C}_n) \otimes \mathbb{Z}_p \cong 0$ implies that there are no generators of infinite order. Moreover, the order of every generator is relatively prime to p. Since this holds for all odd primes, we conclude that $H_i(\mathcal{C}_n)$ is a direct sum of finite cyclical groups whose orders are powers of 2.

Now apply (5.4) with $R = \mathbb{Z}_4$. We obtain the short exact sequence

$$0 \longrightarrow H_i(\mathcal{C}_n) \otimes \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow H_{i-1}(\mathcal{C}_n) * \mathbb{Z}_4 \longrightarrow 0.$$

Either $H_i(\mathcal{C}_n) \otimes \mathbb{Z}_4 \cong \mathbb{Z}_2$ or $H_i(\mathcal{C}_n) \otimes \mathbb{Z}_4 \cong 0$. In the first case we have $H_i(\mathcal{C}_n) \cong \mathbb{Z}_2$ and $H_{i-1}(\mathcal{C}_n) \cong 0$. In the second case $H_i(\mathcal{C}_n) \cong 0$ and $H_{i-1}(\mathcal{C}_n) \cong \mathbb{Z}_2$. Thus we have the two implications $H_{i-1}(\mathcal{C}_n) \cong 0 \Longrightarrow H_i(\mathcal{C}_n) \cong \mathbb{Z}_2$ and $H_{i-1}(\mathcal{C}_n) \cong \mathbb{Z}_2 \Longrightarrow H_i(\mathcal{C}_n) \cong 0$. Now by induction on i, where the base case is $i = 1$, the result follows. □

In order to get a better understanding of the homology at $W_{n,i}$ when i is even, we introduce a ring homomorphism λ from the tensor ring $T(\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle)$ to $\mathbb{Z}_2$ by $\lambda(1) = \lambda(\mathbf{c}) = \lambda(\otimes) = 1$ and $\lambda(\mathbf{d}) = 0$. Observe that λ restricts to the linear map $\lambda_{n,i} : W_{n,i} \longrightarrow \mathbb{Z}_2$.

Theorem 5.2 *Let i be a non-negative integer less than n. At $W_{n,i}$ in the Newtonian chain complex $\mathcal{C}_n$ we have*

$$\mathrm{im}(\partial_{i+1}) = \ker(\partial_i) \cap \ker(\lambda_{n,i}).$$

Proof: Observe that $\lambda_{n,i} \circ \partial_{i+1} = 0$, that is, $\mathrm{im}(\partial_{i+1}) \subseteq \ker(\lambda_{n,i})$. Hence when i is odd the statement is directly true since $\mathrm{im}(\partial_{i+1}) = \ker(\partial_i)$. Thus it remains to prove $\ker(\partial_i) \cap \ker(\lambda_{n,i}) \subseteq \mathrm{im}(\partial_{i+1})$ when i is even.

Tensor product the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ with the chain complex $\mathcal{C}_n$ to obtain the following exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{C}_n \xrightarrow{\phi} \mathcal{C}_n \xrightarrow{\psi} \mathcal{C}_n(\mathbb{Z}_2) \longrightarrow 0.$$

By the zig-zag lemma and $H_{i+1}(\mathcal{C}_n) \cong H_{i-1}(\mathcal{C}_n) \cong 0$ we have the following exact sequence of homology groups:

$$0 \longrightarrow H_{i+1}(\mathcal{C}_n(\mathbb{Z}_2)) \xrightarrow{\partial_*} H_i(\mathcal{C}_n) \xrightarrow{\phi_*} H_i(\mathcal{C}_n) \xrightarrow{\psi_*} H_i(\mathcal{C}_n(\mathbb{Z}_2)) \longrightarrow 0.$$

Observe that all four groups are isomorphic to $\mathbb{Z}_2$. Since there is only one surjective map from $\mathbb{Z}_2$ to itself, we know that ψ_* is an isomorphism. (In fact, ∂_* is also an isomorphism and ϕ_* is the zero map.)

Let w be an element of $W_{n,i}$ such that $w \in \ker(\partial_i) \cap \ker(\lambda_{n,i})$. Since $\lambda_{n,i}$ counts modulo 2 the number of monomials in w that do not contain any $\mathbf{d}$'s, $\lambda_{n,i}(w) = 0$ implies that $\psi(w)$ is a sum of an even number of generators of the homology group $H_i(\mathcal{C}_n(\mathbb{Z}_2))$. However, since $H_i(\mathcal{C}_n(\mathbb{Z}_2)) \cong \mathbb{Z}_2$, it must be that $\psi(w)$ is mapped to the zero element in $H_i(\mathcal{C}_n(\mathbb{Z}_2))$. Since ψ_* is an isomorphism we know that w is mapped to the zero element in $H_i(\mathcal{C}_n)$. In other words, w belong to the image of ∂_{i+1}. $\square$

6 An application to Eulerian posets

As an application of the computation of the homology groups of $\mathcal{C}_n = \mathcal{C}_n(\mathbb{Z})$, we give a homological proof of the existence of the **cd**-index for Eulerian posets.

Let P be a graded posets of rank $n+1$ with rank function ρ. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define f_S to be the number of chains $\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}$ in the poset P such that $\rho(x_i) = s_i$. The 2^n values f_S constitute the *flag f-vector* of the poset. An equivalent vector is the *flag h-vector* which is defined by the two equivalent relations $h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T$ and $f_S = \sum_{T \subseteq S} h_T$. For S a subset of $\{1, \ldots, n\}$ define u_S to be the **ab**-monomial of degree n given by $u_S = u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* of the poset P is the **ab**-polynomial

$$\Psi(P) = \sum_S h_S \cdot u_S.$$

Observe that the **ab**-index is homogeneous of degree n.

A graded poset is *Eulerian* if every interval $[x, y] = \{z : x \leq z \leq y\}$, where $x < y$, contains the same number elements of even rank as elements of odd rank. This condition is equivalent to that the Möbius function $\mu(x, y)$ is given by $(-1)^{\rho(y)-\rho(x)}$. For Eulerian posets the following results holds. The original proof is due to Bayer and Klapper [6].

Theorem 6.1 *The* **ab**-*index of an Eulerian poset P can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. *In other words, for Eulerian posets the* **cd**-*index exists.*

Proof: We proceed by induction on the rank of the poset. In the case the rank of the poset is 1, the **ab**-index is 1 which lies in $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. Assume the result is true for Eulerian posets of rank at most n. Let P be an Eulerian poset of rank $n+1$ and let $\Psi(P)$ denote its **ab**-index. Using that the **ab**-index is a Newtonian coalgebra homomorphism [11, Proposition 3.1], we have

$$\Delta(\Psi(P)) = \sum_{\hat{0} < x < \hat{1}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}]).$$

By the induction hypothesis we obtain that $\Delta(\Psi(P)) \in W_{n,n-1}$. Since the Eulerian poset P consists of an even number of elements we have that $\lambda(\Delta(\Psi(P)) = 0$. Moreover, we have $\partial_{n-1}(\Delta(\Psi(P))) = 0$. Thus by Theorem 5.2 we have that there is an element $w \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ such that $\Delta(w) = \Delta(\Psi(P))$. However, since $H_n(\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle_n)$ is spanned by $(\mathbf{a} - \mathbf{b})^n$, we know that $\Psi(P) = w + \beta \cdot (\mathbf{a} - \mathbf{b})^n$ for some integer β.

If n is even then $(\mathbf{a}-\mathbf{b})^n = (\mathbf{c}^2 - 2\mathbf{d})^{n/2}$ and the result follows. Hence let us assume that n is odd. Observe that the coefficient of $\mathbf{a}^n$ in w is 1. By Philip Hall's theorem on the Möbius function we have that $[\mathbf{b}^n]\Psi(P) = h_{\{1,\ldots,n\}} = (-1)^{n+1} \cdot \mu(P) = 1$. Now consider the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ in $w + \beta \cdot (\mathbf{a}-\mathbf{b})^n$. By the symmetry of $\mathbf{c}$ and $\mathbf{d}$ in terms of the variables $\mathbf{a}$ and $\mathbf{b}$, we know that the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ in w are the same, say α. Hence comparing coefficients we obtain that $1 = \alpha + \beta$ and $1 = \alpha - \beta$. Thus $\beta = 0$ and we conclude that $\Psi(P)$ belongs to $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$, completing the induction. $\square$

As a remark, we could have proven the weaker statement that the coefficients of the **cd**-index have the form $r/2^s$ using Theorem 4.1. Namely, use the ring $R = \{r/2^s \;:\; r, s \in \mathbb{Z}\}$, that is, the integers localized at 2, in Theorem 4.1.

The first proof of the existence of the **cd**-index is due to Bayer and Klapper [6] using a shelling argument. Stanley [16] used Möbius inversion and the fact that $(\mathbf{a} + (-1)^n \cdot \mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^n$ is a **cd**-polynomial for all non-negative integers n. Two similar proofs are by Aguiar [2] and Billera and Liu [9], where Aguiar's proof takes place in a general Newtonian coalgebra setting. The proof by Ehrenborg [10] uses the Laplace pairing between $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ and the Billera-Liu flag algebra. Using the L-vector results of Bayer and Hetyei, a proof of the existence can be extracted from [4]. Another proof by Bayer and Hetyei appears in [5].

References

[1] M. Aguiar, Infinitesimal Hopf algebras, in: *New trends in Hopf algebra theory* (La Falda, 1999), pp. 1–29, *Contemp. Math.* **267**, American Mathematical Society, Providence, RI, 2000.

[2] M. Aguiar, Infinitesimal Hopf algebras and the cd-index of polytopes, *Discrete Comput. Geom.* **27** (2002), 3–28.

[3] M. Bayer and L. Billera, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.

[4] M. Bayer and G. Hetyei, Flag vectors of Eulerian partially ordered sets, *Europ. J. Comb.* **22** (2001), 5–26.

[5] M. Bayer and G. Hetyei, Generalizations of Eulerian partially ordered sets, flag numbers, and the Möbius function, *Discrete Math.* **256** (2002), 577-593.

[6] M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.

[7] L. J. Billera and R. Ehrenborg, Monotonicity properties of the **cd**-index for polytopes, *Math. Z.* **233** (2000), 421–441.

[8] L. J. Billera, R. Ehrenborg, and M. Readdy, The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A* **80** (1997), 79–105.

[9] L. J. Billera and N. Liu, Noncommutative enumeration in graded posets, *J. Algebraic Combin.* **12** (2000), 7–24.

[10] R. Ehrenborg, k-Eulerian posets, *Order* **18** (2001), 227–236.

[11] R. Ehrenborg and M. Readdy, Coproducts and the **cd**-index, *J. Algebraic Combin.* **8** (1998), 273–299.

[12] P. S. Hirschhorn and L. A. Raphael, Coalgebraic Foundation of the Method of Divided Differences, *Adv. Math.* **91** (1992), 75–135.

[13] S. A. Joni and G.-C. Rota, Coalgebras and Bialgebras in Combinatorics, *Stud. Appl. Math.* **61** (1979), 93–139.

[14] J. R. Munkres, "Elements of Algebraic Topology," Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[15] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[16] R. P. Stanley, Flag f-vectors and the cd-index, *Math. Z.* **216** (1994), 483–499.

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