A Poset View of the Major Index

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Abstract

We introduce the Major MacMahon map from $\mathbb{Z} \langle a, b \rangle$ to $\mathbb{Z}[q]$, and show how this map commutes with the pyramid and bipyramid operators. When the Major MacMahon map is applied to the $ab$-index of a simplicial poset, it yields the $q$-analogue of $n!$ times the $h$-polynomial of the polytope. Applying the map to the Boolean algebra gives the distribution of the major index on the symmetric group, a seminal result due to MacMahon. Similarly, when applied to the cross-polytope we obtain the distribution of one of the major indexes on the signed permutations, due to Reiner.

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1 Introduction

One hundred and one years ago in 1913 Major Percy Alexander MacMahon [9] (see also his collected works [11]) introduced the major index of a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of the multiset $M = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\}$ of size $n$ to be the sum of the elements of its descent set, that is,

$$\text{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i.$$ 

He showed that the distribution of this permutation statistic is given by the $q$-analogue of the multinomial Gaussian coefficient, that is, the following identity holds:

$$\sum_{\pi} q^{\text{maj}(\pi)} = \frac{[n]!}{[\alpha_1]! \cdots [\alpha_k]!} = \binom{n}{\alpha},$$ (1.1)

where $\pi$ ranges over all permutations of the multiset $M$. Here $[n]! = [n] \cdot [n - 1] \cdots [1]$ denotes the $q$-analogue of $n!$, where $[n] = 1 + q + \cdots + q^{n-1}$.

Many properties of the descent set of a permutation $\pi$, that is, $\text{Des}(\pi) = \{ i : \pi_i > \pi_{i+1} \}$, have been studied by encoding the set by its $ab$-word; see for instance [6, 12]. For a multiset

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permutation $\pi \in \mathfrak{S}_M$ the \textit{ab}-word is given by $u(\pi) = u_1u_2 \cdots u_{n-1}$, where $u_i = b$ if $\pi_i > \pi_{i+1}$ and $u_i = a$ otherwise.

Inspired by this definition, we introduce the \textit{Major MacMahon map} $\Theta$ on the ring $\mathbb{Z}(a,b)$ of non-commutative polynomials in the variables $a$ and $b$ to $\mathbb{Z}[q]$, polynomials in the variable $q$, by

$$\Theta(w) = \prod_{i : u_i = b} q^i,$$

for a monomial $w = u_1u_2 \cdots u_n$ and extend $\Theta$ to all of $\mathbb{Z}(a,b)$ by linearity. In short, the map $\Theta$ sends each variable $a$ to 1 and the variables $b$ to $q$ to the power of its position, read from left to right. A Swedish example is $\Theta(abba) = q^5$.

2 Chain enumeration and products of posets

Let $P$ be a graded poset of rank $n + 1$ with minimal element $\hat{0}$, maximal element $\hat{1}$ and rank function $\rho$. Let the rank difference be defined by $\rho(x,y) = \rho(y) - \rho(x)$. The \textit{flag $f$-vector} entry $f_S$, for $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset $\{1,2,\ldots,n\}$, is the number of chains $c = \{0 = x_0 < x_1 < x_2 < \cdots < x_{k+1} = 1\}$ such that the rank of the element $x_i$ is $s_i$, that is, $\rho(x_i) = s_i$ for $1 \leq i \leq k$. The \textit{flag $h$-vector} is defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

For a subset $S$ of $\{1,2,\ldots,n\}$ define two \textit{ab}-polynomials of degree $n$ by $u_S = u_1u_2 \cdots u_n$ and $v_S = v_1v_2 \cdots v_n$ by

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S, \end{cases} \quad \text{and} \quad v_i = \begin{cases} a - b & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

The \textit{ab-index} of the poset $P$ is defined by the two equivalent expressions:

$$\Psi(P) = \sum_S f_S \cdot v_S = \sum_S h_S \cdot u_S,$$

where the two sums range over all subsets $S$ of $\{1,2,\ldots,n\}$. For more details on the \textit{ab-index} see [7] or the book [16, Section 3.17].

Recall that a graded poset $P$ is \textit{Eulerian} if every non-trivial interval has the same number of elements of even as odd rank. Equivalently, a poset is Eulerian if its Möbius function satisfies $\mu(x,y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in $P$. When the graded poset $P$ is Eulerian then the \textit{ab-index} $\Psi(P)$ can be written in terms of the non-commuting variables $c = a + b$ and $d = ab + ba$ and it is called the \textit{cd-index}; see [2]. For an $n$-dimensional convex polytope $P$ its face lattice $\mathcal{L}(P)$ is an Eulerian poset of rank $n + 1$. In this case we write $\Psi(P)$ for the \textit{ab-index} (\textit{cd-index}) instead of the cumbersome $\Psi(\mathcal{L}(P))$.

There are also two products on posets that we will study. The first is the \textit{Cartesian product}, defined by $P \times Q = \{(x,y) : x \in P, y \in Q\}$ with the order relation $(x,y) \leq_{P \times Q} (z,w)$ if $x \leq_P z$
and \( y \leq Q \)). Note that the rank of the Cartesian product of two graded posets of ranks \( m \) and \( n \) is \( m + n \). As a special case we define \( \text{Pyr}(P) = P \times B_1 \), where \( B_1 \) is the Boolean algebra of rank 1.

The geometric reason is that this operation corresponds to the geometric operation of taking the pyramid of a polytope, that is, \( \mathcal{L}(\text{Pyr}(P)) = \text{Pyr}(\mathcal{L}(P)) \) for a polytope \( P \).

The second product is the \textit{dual diamond product}, defined by

\[
P \circ^* Q = (P - \{\hat{1}_P\}) \times (Q - \{\hat{1}_Q\}) \cup \{\hat{1}\}.
\]

The rank of the product \( P \circ^* Q \) is the sum of the ranks of \( P \) and \( Q \) minus one. This is the dual to the diamond product \( \diamond \) defined by removing the minimal elements of the posets, taking the Cartesian product and adjoining a new minimal element. The product \( \diamond \) behaves well with the quasi-symmetric functions of type \( B \). (See Sections 5 and 6.) However, we will dualize our presentation and keep working with the product \( \circ^* \).

Yet again, we have an important special case. We define \( \text{Bipyr}(P) = P \circ^* B_2 \). The geometric motivation is the connection to the bipyramid of a polytope, that is, \( \mathcal{L}(\text{Bipyr}(P)) = \text{Bipyr}(\mathcal{L}(P)) \) for a polytope \( P \).

### 3 Pyramids and bipyramids

Define on the ring \( \mathbb{Z}(a, b) \) of non-commutative polynomials in the variables \( a \) and \( b \) the two derivations \( G \) and \( D \) by

\[
G(1) = 0, \quad G(a) = ba, \quad G(b) = ab, \\
D(1) = 0, \quad D(a) = D(b) = ab + ba.
\]

Extend these two derivations to all of \( \mathbb{Z}(a, b) \) by linearity. The \textit{pyramid} and the \textit{bipyramid operators} are given by

\[
\text{Pyr}(w) = G(w) + w \cdot c \quad \text{and} \quad \text{Bipyr}(w) = D(w) + c \cdot w.
\]

These two operators are suitably named, since for a poset \( P \) we have

\[
\Psi(\text{Pyr}(P)) = \text{Pyr}(\Psi(P)) \quad \text{and} \quad \Psi(\text{Bipyr}(P)) = \text{Bipyr}(\Psi(P)).
\]

For further details, see [7].

**Theorem 3.1.** The Major MacMahon map \( \Theta \) commutes with right multiplication by \( c \), the derivation \( G \), the pyramid and the bipyramid operators as follows:

\[
\Theta(w \cdot c) = (1 + q^{n+1}) \cdot \Theta(w), \quad (3.1)
\]

\[
\Theta(G(w)) = q \cdot [n] \cdot \Theta(w), \quad (3.2)
\]

\[
\Theta(\text{Pyr}(w)) = [n + 2] \cdot \Theta(w), \quad (3.3)
\]

\[
\Theta(\text{Bipyr}(w)) = [2] \cdot [n + 1] \cdot \Theta(w), \quad (3.4)
\]

where \( w \) is a homogeneous \( ab \)-polynomial of degree \( n \).

**Proof.** It is enough to prove the four identities for an \( ab \)-monomial \( w \) of degree \( n \). Directly we have that \( \Theta(w \cdot a) = \Theta(w) \) and \( \Theta(w \cdot b) = q^{n+1} \cdot \Theta(w) \). Adding these two identities yields equation (3.1).
Assume that \( w \) consists of \( k \) \( b \)'s. We will label the \( n \) letters of \( w \) as follows: The \( k \) \( b \)'s are labeled 1 through \( k \) reading from right to left, whereas the \( n-k \) \( a \)'s are labeled \( k+1 \) through \( n \) reading left to right. As an example, the word \( w = \text{aababba} \) is written as \( w_4w_5w_3w_6w_2w_1w_7 \).

The theorem is a consequence of the following claim. Applying the derivation \( G \) only to the letter \( w_i \) and then applying the Major MacMahon map yields \( q^i \cdot \Theta(w) \), that is,

\[
\Theta(u \cdot G(w_i) \cdot v) = q^i \cdot \Theta(u \cdot w_i \cdot v),
\]

(3.5) where \( w \) is factored as \( u \cdot w_i \cdot v \). To see this, first consider when \( 1 \leq i \leq k \). There are \( i \) \( b \)'s to the right of \( w_i \) including \( w_i \) itself. They each are shifted one step to the right when replacing \( w_i = b \) with \( G(b) = ab \) and hence we gain a factor of \( q^i \). The second case is when \( k+1 \leq i \leq n \). Then \( w_i \) is an \( a \) and is replaced by \( ba \) under the derivation \( G \). Assume that there are \( j \) \( b \)'s to the right of \( w_i \). When these \( j \) \( b \)'s are shifted one step to the right they contribute a factor of \( q^j \). We also create a new \( b \). It has \( i-k-1 \) \( a \)'s to the left and \( k-j \) \( b \)'s to the left. Hence the position of the new \( b \) is \( (i-k-1) + (k-j) + 1 = i-j \) and thus its contribution is \( q^{i-j} \). Again the factor is given by \( q^j \cdot q^{i-j} = q^i \), proving the claim. Now by summing over these \( n \) cases, identity (3.2) follows. Identity (3.3) is the sum of identities (3.1) and (3.2).

To prove identity (3.4), we use a different labeling of the monomial \( w \). This time label the \( k \) \( b \)'s with the subscripts 0 through \( k-1 \), rather than 1 through \( k \). That is, in our example \( w = \text{aababba} \) is now labeled as \( w_4w_5w_2w_6w_1w_0w_7 \). We claim that for \( w = u \cdot w_i \cdot v \) we have that

\[
\Theta(u \cdot D(w_i) \cdot v) = q^i \cdot [2] \cdot \Theta(w).
\]

The first case is \( 0 \leq i \leq k-1 \). Then \( w_i = b \) has \( i \) \( b \)'s to its right. Thus when replacing \( b \) with \( ba \) there are \( i \) \( b \)'s that are shifted one step, giving the factor \( q^i \). Similarly, when replacing \( w_i \) with \( ab \), there are \( i+1 \) \( b \)'s that are shifted one step, giving the factor \( q^{i+1} \). The sum of the two factors is \( q^i \cdot [2] \). The second case is \( k+1 \leq i \leq n \). It is as the second case above when replacing \( w_i \) with \( ba \), yielding the factor \( q^i \). When replacing \( w_i \) with \( ab \) there is one more shift, giving \( q^{i+1} \). Adding these two subcases completes the proof of the claim.

It is straightforward to observe that

\[
\Theta(c \cdot w) = q^k \cdot [2] \cdot \Theta(w).
\]

Calling this the case \( i = k \), the identity (3.4) follows by summing the \( n+1 \) cases \( 0 \leq i \leq n \).

Iterating equations (3.3) and (3.4) we obtain that the Major MacMahon map of the \( ab \)-index of the \( n \)-dimensional simplex \( \Delta_n \) and the \( n \)-dimensional cross-polytope \( C^*_n \).

**Corollary 3.2.** The \( n \)-dimensional simplex \( \Delta_n \) and the \( n \)-dimensional cross-polytope \( C^*_n \) satisfy

\[
\Theta(\Psi(\Delta_n)) = [n+1]!,
\]

\[
\Theta(\Psi(C^*_n)) = [2]^n \cdot [n]!.
\]

### 4 Simplicial posets

A graded poset \( P \) is **simplicial** if all of its lower order intervals are Boolean, that is, for all elements \( x < \bar{1} \) the interval \( [0, x] \) is isomorphic to the Boolean algebra \( B_{\rho(x)} \). It is well-known that all the
flag information of a simplicial poset of rank $n + 1$ is contained in the $f$-vector $(f_0, f_1, \ldots, f_n)$, where $f_0 = 1$ and $f_i = f_{i+1}$ for $1 \leq i \leq n$. The $h$-vector, equivalently, the $h$-polynomial $h(P) = h_0 + h_1 q + \cdots + h_n q^n$, is defined by the polynomial relation

$$h(q) = \sum_{i=0}^{n} f_i (q - 1)^{n-i}.$$ 

See for instance [19, Section 8.3]. The $h$-polynomial and the bipyramid operation commutes as follows

$$h(\text{Bipyr}(P)) = (1 + q) \cdot h(P).$$

We can now evaluate the Major MacMahon map on the $ab$-index of a simplicial poset.

**Theorem 4.1.** For a simplicial poset $P$ of rank $n + 1$ the following identity holds:

$$\Theta(\Psi(P)) = [n]! \cdot h(P). \quad (4.1)$$

**Proof.** Let $B_n \cup \{\hat{1}\}$ denote the Boolean algebra $B_n$ with a new maximal element added. Note that $B_n \cup \{\hat{1}\}$ is indeed a simplicial poset and its $h$-polynomial is 1. Furthermore, equation (4.1) holds for $B_n \cup \{\hat{1}\}$ since

$$\Theta(\Psi(B_n \cup \{\hat{1}\})) = \Theta(\Psi(B_n)) = [n]! = [n]! \cdot h(B_n \cup \{\hat{1}\}).$$

Also, if (4.1) holds for a poset $P$ then it also holds for $\text{Bipyr}(P)$, since we have

$$\Theta(\Psi(\text{Bipyr}(P))) = [2] \cdot [n + 1] \cdot \Theta(\Psi(P)) = [2] \cdot [n + 1] \cdot [n]! \cdot h(P) = [n + 1]! \cdot h(\text{Bipyr}(P)).$$

Observe that both sides of (4.1) are linear in the $h$-polynomial. Hence to prove it for any simplicial poset $P$ it is enough to prove it for a basis of the span of all simplicial posets of rank $n+1$. Such a basis is given by the posets

$$B_n = \left\{ \text{Bipyr}^i(B_n \cup \{\hat{1}\}) \right\}_{0 \leq i \leq n}.$$ 

This is a basis since the polynomials $h(\text{Bipyr}^i(B_n \cup \{\hat{1}\})) = (1 + q)^i$, for $0 \leq i \leq n$, are a basis for polynomials of degree at most $n$.

Finally, since every element in the basis is built up by iterating bipyramids of the posets $B_n \cup \{\hat{1}\}$, the theorem holds for all simplicial posets. \hfill \Box

Observe that the poset $\text{Bipyr}^i(B_{n-i} \cup \{\hat{1}\})$ is the face lattice of the simplicial complex consisting of the $2^i$ facets of the $n$-dimensional cross-polytope in the cone $x_1, \ldots, x_{n-i} \geq 0$.

For an Eulerian simplicial poset $P$, the $h$-vector is symmetric, that is, $h_i = h_{n-i}$. In other words, the $h$-polynomial is palindromic. Stanley [15] introduced the simplicial shelling components, that is, the $cd$-polynomials $\Phi_{n,i}$ such that the $cd$-index of an Eulerian simplicial poset $P$ of rank $n + 1$ is given by

$$\Psi(P) = \sum_{i=0}^{n} h_i \cdot \Phi_{n,i}. \quad (4.2)$$

These $cd$-polynomials satisfies the recursion $\Phi_{n,0} = \Psi(B_n) \cdot \mathbf{c}$ and $\Phi_{n,i} = G(\Phi_{n-1,i-1})$; see [7, Section 8]. The Major MacMahon map of these polynomials is described by the next result.
Corollary 4.2. The Major MacMahon map of the simplicial shelling components is given by
\[ \Theta(\Phi_{n,i}) = q^i \cdot [2(n-i)] \cdot [n-1]! \].

Proof. When \( i = 0 \) we have \( \Theta(\Phi_{n,0}) = \Theta(\Psi(B_n) \cdot c) = (1 + q^n) \cdot [n]! = [2n] \cdot [n-1]! \). Also when \( i \geq 1 \) we obtain \( \Theta(\Phi_{n,i}) = \Theta(G(\Phi_{n-1,i-1})) = q^i \cdot [2(n-i)] \cdot [n-1]! \).

We end with the following observation.

Theorem 4.3. For an Eulerian poset \( P \) of rank \( n + 1 \), the polynomial \([2]^\lceil n/2 \rceil\) divides \( \Theta(\Psi(P)) \).

Proof. It is enough to show this result for a \( cd \)-monomial \( w \) of degree \( n \). A \( c \) in an odd position \( i \) of \( w \) yields a factor of \( 1 + q^i \). A \( d \) that covers an odd position \( i \) of \( w \) yields either \( q^i - 1 + q^i \) or \( q^i + q^{i+1} \). Each of these polynomials contributes a factor of \( 1 + q \). The result follows since there are \( \lceil n/2 \rceil \) odd positions.

5 The Cartesian product of posets

We now study how the Major MacMahon map behaves under the Cartesian product. Recall that for a poset \( P \) the \( ab \)-index \( \Psi(P) \) encodes the flag \( f \)-vector information of the poset \( P \). There is another encoding of this information as a quasi-symmetric function. For further information about quasi-symmetric functions, see [17, Section 7.19].

A composition \( \alpha \) of \( n \) is a list of positive integers \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) such that \( \alpha_1 + \alpha_2 + \cdots + \alpha_k = n \). Let \( \text{Comp}(n) \) denote the set of compositions of \( n \). There are three natural bijections between \( ab \)-monomials \( u \) of degree \( n \), subsets \( S \) of the set \( \{1, 2, \ldots, n\} \) and compositions of \( n + 1 \). Given a composition \( \alpha \in \text{Comp}_{n+1} \) we have the subset \( S_\alpha \), the \( ab \)-monomial \( u_\alpha \) and the \( ab \)-polynomial \( v_\alpha \) defined by

\[
S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\},
\]
\[
u_\alpha = a^{\alpha_1-1} \cdot b \cdot a^{\alpha_2-1} \cdot b \cdots b \cdot a^{\alpha_k-1},
\]
\[
v_\alpha = (a - b)^{\alpha_1-1} \cdot b \cdot (a - b)^{\alpha_2-1} \cdot b \cdots b \cdot (a - b)^{\alpha_k-1}.
\]

For \( S \) a subset of \( \{1, 2, \ldots, n\} \) let \( co(S) \) denote associated composition.

The monomial quasi-symmetric function \( M_\alpha \) is defined as the sum

\[
M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} t_{i_1}^{\alpha_1} \cdot t_{i_2}^{\alpha_2} \cdots t_{i_k}^{\alpha_k}.
\]

A second basis is given by the fundamental quasi-symmetric function \( L_\alpha \) defined as

\[
L_\alpha = \sum_{S_\alpha \subseteq T \subseteq \{1, 2, \ldots, n\}} M_{co(T)}.
\]

Following [8] define an injective linear map \( \gamma : \mathbb{Z}(a, b) \rightarrow \text{QSym} \) by

\[
\gamma(v_\alpha) = M_\alpha,
\]

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for a composition $\alpha$ of $n \geq 1$. The image of $\gamma$ is all quasi-symmetric functions without constant term. Moreover, the image of the ab-monomial $u_\alpha$ under $\gamma$ is the fundamental quasi-symmetric function $L_\alpha$, that is,

$$\gamma(u_\alpha) = L_\alpha.$$ 

Another way to encode the flag vectors of a poset $P$ is by the quasi-symmetric function of the poset. It is quickly defined as $F(P) = \gamma(\Psi(P))$. A more poset-oriented definition is the following limit of sums over multichains

$$F(P) = \lim_{k \to \infty} \sum_{\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_k = \hat{1}} t^{x_0,x_1} \cdot t^{x_1,x_2} \cdots t^{x_{k-1},x_k}.$$ 

For more on the quasi-symmetric function of a poset, see [5].

The stable principal specialization of a quasi-symmetric function is the substitution $ps(f) = f(1, q, q^2, \ldots)$. Note that this is a homeomorphism, that is, $ps(f \cdot g) = ps(f) \cdot ps(g)$.

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ let $\alpha^*$ denote the reverse composition, that is, $\alpha^* = (\alpha_k, \ldots, \alpha_2, \alpha_1)$. This involution extends to an anti-automorphism on $QSym$ by $M^*_\alpha \mapsto M^*_{\alpha^*}$. Define $ps^*$ by the relation $ps^*(f) = ps(f^*)$. Informally speaking, this corresponds to the substitution $ps^*(f) = f(\ldots, q^2, q, 1)$.

**Theorem 5.1.** For a homogeneous ab-polynomial $w$ of degree $n - 1$ the Major MacMahon map is given by

$$\Theta(w) = (1 - q)^n \cdot [n]! \cdot ps^*(\gamma(w)).$$

(5.1)

For a poset $P$ of rank $n$ this identity is

$$\Theta(\Psi(P)) = (1 - q)^n \cdot [n]! \cdot ps^*(F(P)).$$

(5.2)

**Proof.** It is enough to prove identity (5.1) for an ab-monomial $w$ of degree $n - 1$. Let $\alpha$ be the composition of $n$ corresponding to the reverse monomial $w^*$. Furthermore, let $e(\alpha)$ be the sum $\sum_{i \in S_\alpha} (n - i)$. Note that $e(\alpha)$ is in fact the sum $\sum_{i \in S} i$, where $S$ is the subset associated with the ab-monomial $w$. That is, we have $q^{e(\alpha)} = \Theta(w)$. Equation (5.1) follows from Lemma 7.19.10 in [17]. By applying the first identity to $\Psi(P)$, we obtain identity (5.2).

Since the quasi-symmetric function is multiplicative under the Cartesian product, we have the next result.

**Theorem 5.2.** For two posets $P$ and $Q$ of ranks $m$, respectively $n$, the following identity holds:

$$\Theta(\Psi(P \times Q)) = \left[ \frac{m+n}{n} \right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).$$

(5.3)
Proof. The proof is a direct verification as follows:

\[
\Theta(\Psi(P \times Q)) = (1 - q)^{m+n} \cdot [m+n]! \cdot \text{ps}(F(P^* \times Q^*)) \\
= \left[\frac{m+n}{m}\right] \cdot (1 - q)^{m+n} \cdot [m]! \cdot [n]! \cdot \text{ps}(F(P^*)) \cdot \text{ps}(F(Q^*)) \\
= \left[\frac{m+n}{m}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).
\]

\[
F_{B^*}(P) = \sum_{\hat{0} \leq x < \hat{1}} F([\hat{0}, x]) \cdot s^{\rho(x, \overline{1})-1}. \tag{6.1}
\]

6 The dual diamond product

Define the quasi-symmetric function of type $B^*$ of a poset $P$ to be the expression

\[ F_{B^*}(P) = \sum_{\hat{0} \leq x < \hat{1}} F([\hat{0}, x]) \cdot s^{|x, \overline{1}|}. \]

This is an element of the algebra $\text{QSym} \otimes \mathbb{Z}[s]$ which we view as the quasi-symmetric functions of type $B^*$. We view $\text{QSym}_{B^*}$ as an subalgebra of $\mathbb{Z}[t_1, t_2, \ldots; s]$, which is quasi-symmetric in the variables $t_1, t_2, \ldots$. For instance, a basis for $\text{QSym}_{B^*}$ is given by a $M_{\alpha} \cdot s^i$ where $\alpha$ ranges over all compositions and $i$ over all non-negative integers.

Furthermore, the type $B^*$ quasi-symmetric function $F_{B^*}$ is multiplicative respect to the product $\circ$, that is, $F_{B^*}(P \circ Q) = F_{B^*}(P) \cdot F_{B^*}(Q)$; see [8, Theorem 13.3].

Let $f$ be a homogeneous quasi-symmetric function such that $f \cdot s^j$ is a quasi-symmetric function of type $B^*$. We define the stable principal specialization of the quasi-symmetric function $f \cdot s^j$ of type $B^*$ to be $\text{ps}_{B^*}(f \cdot s^j) = q^{\deg(f)} \cdot \text{ps}^*(f)$. This is the substitution $s = 1$, $t_k = q$, $t_{k-1} = q^2$, $\ldots$ as $k$ tends to infinity, since $f(\ldots, q^3, q^2, q) = q^{\deg(f)} \cdot f(\ldots, q^2, q, 1)$. Especially, for a poset $P$ we have

\[ \text{ps}_{B^*}(F_{B^*}(P)) = \sum_{\hat{0} \leq x < \hat{1}} q^{|x|} \cdot \text{ps}^*(F([\hat{0}, x])). \tag{6.1} \]

Theorem 6.1. For a poset $P$ of rank $n + 1$ the relationship between the Major MacMahon map and the stable principal specialization of type $B^*$ is given by

\[ \Theta(\Psi(P)) = (1 - q)^n \cdot [n]! \cdot \text{ps}_{B^*}(F_{B^*}(P^*)). \tag{6.2} \]

Especially, for a homogeneous $ab$-polynomial $w$ of degree $n$ the Major MacMahon map is given by

\[ \Theta(w) = (1 - q)^n \cdot [n]! \cdot \text{ps}_{B^*}(\gamma_{B^*}(w^*)). \tag{6.3} \]
Proof. For the poset $P$ we have

$$\begin{align*}
\text{ps}^*(F(P)) &= \lim_{k \to \infty} \sum_{\emptyset = x_0 \preceq x_1 \preceq \cdots \preceq x_k = \emptyset} (q^{k-1})^{\rho(x_0, x_1)} \cdots (q^2)^{\rho(x_{k-3}, x_{k-2})} \cdot q^{\rho(x_{k-2}, x_{k-1})} \cdot 1^{\rho(x_{k-1}, x_k)} \\
&= \lim_{k \to \infty} \sum_{\emptyset = x_0 \preceq x_1 \preceq \cdots \preceq x_k = \emptyset} (q^{k-1})^{\rho(x_0, x_1)} \cdots (q^2)^{\rho(x_{k-3}, x_{k-2})} \cdot q^{\rho(x_{k-2}, x_{k-1})} \\
&= \lim_{k \to \infty} \sum_{\emptyset = x_0 \preceq x_1 \preceq \cdots \preceq x_k = \emptyset} q^{\rho(x_{k-1})} \cdot (q^{k-2})^{\rho(x_0, x_1)} \cdots q^{\rho(x_{k-3}, x_{k-2})} \cdot 1^{\rho(x_{k-2}, x_{k-1})} \\
&= \sum_{\emptyset \preceq x \preceq \emptyset} q^{\rho(x)} \cdot \text{ps}^*(F([\emptyset, x])) \\
&= \sum_{\emptyset \preceq x \preceq \emptyset} q^{\rho(x)} \cdot \text{ps}^*(F([\emptyset, x])) + q^{n+1} \cdot \text{ps}^*(F(P)).
\end{align*}$$

Rearranging terms yields

$$\begin{align*}
\sum_{\emptyset \preceq x \preceq \emptyset} q^{\rho(x)} \cdot \text{ps}^*(F([\emptyset, x])) &= (1 - q^{n+1}) \cdot \text{ps}^*(F(P)) \\
&= (1 - q^{n+1}) \cdot \text{ps}(F^*) \\
&= (1 - q^{n+1}) \cdot \frac{\Theta(\Psi(P))}{(1 - q)^{n+1} \cdot [n + 1]!} \\
&= \frac{\Theta(\Psi(P))}{(1 - q)^n \cdot [n]!}.
\end{align*}$$

Combining the last identity with (6.1) yields the desired result. \hfill \qed

Theorem 6.2. For two posets $P$ and $Q$ of ranks $m + 1$, respectively $n + 1$, the identity holds:

$$\Theta(\Psi(P \circ^* Q)) = \left[\frac{m + n}{n}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \quad (6.4)$$

Proof. The proof is a direct verification as follows:

$$\begin{align*}
\Theta(\Psi(P \circ^* Q)) &= (1 - q)^{m+n} \cdot [m + n]! \cdot \text{ps}_{B^*}(F_{B^*}(P^* \circ^* Q^*)) \\
&= \left[\frac{m + n}{m}\right] \cdot (1 - q)^{m+n} \cdot [m]! \cdot [n]! \cdot \text{ps}_{B^*}(F_{B^*}(P^*)) \cdot \text{ps}_{B^*}(F_{B^*}(Q^*)) \\
&= \left[\frac{m + n}{m}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \quad \square
\end{align*}$$

7 Permutations

One connection between permutations and posets is via the concept of $R$-labelings. For more details, see [16, Section 3.14]. Let $\mathcal{E}(P)$ be the set of all cover relations of $P$, that is, $\mathcal{E}(P) = \{(x, y) \in P^2 : x < y\}$. A graded poset $P$ has an $R$-labeling if there is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where
\( \Lambda \) is a linearly ordered set, such that in every interval \([x, y]\) in \( P \) there is a unique maximal chain \( c = \{x = x_0 < x_1 < \cdots < x_k = y\} \) such that \( \lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k) \).

For a maximal chain \( c \) in the poset \( P \) of rank \( n \), let \( \lambda(c) \) denote the list \((\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))\). The Jordan–Hölder set of \( P \), denoted by \( JH(P) \), is the set of all the lists \( \lambda(c) \) where \( c \) ranges over all maximal chains of \( P \). The descent set of a list of labels \( \lambda(c) \) is the set of positions where there are descents in the list. Similarly, we define the descent word of \( \lambda(c) \) to be \( u_{\lambda(c)} = u_1u_2\cdots u_{n-1} \) where \( u_i = b \) if \( \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1}) \) and \( u_i = a \) otherwise.

The bridge between posets and permutations is given by the next result.

**Theorem 7.1.** For an \( R \)-labeling \( \lambda \) of a graded poset \( P \) we have that

\[
\Psi(P) = \sum_{c} u_{\lambda(c)},
\]

where the sum is over the Jordan–Hölder set \( JH(P) \).

This a reformulation of a result of Björner and Stanley [3, Theorem 2.7]. The reformulation can be found in [6, Lemma 3.1].

As a corollary we obtain MacMahon’s classical result on the major index on a multiset; see [9]. For a composition \( \alpha \) of \( n \) let \( \mathcal{S}_\alpha \) denote all the permutations of the multiset \( \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\} \).

**Corollary 7.2 (MacMahon).** For a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of \( n \) the following identity holds:

\[
\sum_{\pi \in \mathcal{S}_\alpha} q^{maj(\pi)} = \frac{[n]!}{[\alpha_1]! \cdots [\alpha_k]!}.
\]

**Proof.** Let \( P_i \) denote the chain of rank \( \alpha_i \) for \( i = 1, \ldots, k \). Furthermore, label all the cover relations in \( P_i \) with \( i \). Let \( L \) denote the distributive lattice \( P_1 \times P_2 \times \cdots \times P_k \). Furthermore, let \( L \) inherit an \( R \)-labeling from its factors, that is, if \( x = (x_1, x_2, \ldots, x_k) < (y_1, y_2, \ldots, y_k) = y \) let the label \( \lambda(x, y) \) be the unique coordinate \( i \) such that \( x_i < y_i \). Observe that the Jordan–Hölder set of \( L \) is \( \mathcal{S}_\alpha \). Direct computation yields \( \Psi(P_i) = a^{\alpha_i-1} \), so the Major MacMahon map is \( \Theta(\Psi(P_i)) = 1 \). Iterating Theorem 5.2 evaluates the Major MacMahon map on \( L \):

\[
\sum_{\pi \in \mathcal{S}_\alpha} q^{maj(\pi)} = \Theta \left( \sum_{\pi \in \mathcal{S}_\alpha} u(\pi) \right) = \Theta(\Psi(L)) = \left[ \frac{n}{\alpha} \right].
\]

For a vector \( r = (r_1, r_2, \ldots, r_n) \) of positive integers let an \( r \)-signed permutation be a list \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1}) = ((j_1, \pi_1), (j_2, \pi_2), \ldots, (j_n, \pi_n), 0) \) such that \( \pi_1 \pi_2 \cdots \pi_n \) is a permutation in the symmetric group \( S_n \) and the sign \( j_i \) is from the set \( S_{\pi_i} = \{-1\} \cup \{2, \ldots, r_{\pi_i}\} \). On the set of labels \( \Lambda = \{(j, i) : 1 \leq i \leq n, j \in S_i\} \cup \{0\} \) we use the lexicographic order with the extra condition that \( 0 < (j, i) \) if and only if \( 0 < j \). Denote the set of \( r \)-signed permutations by \( \mathcal{S}_n^r \). The descent set of an \( r \)-signed permutation \( \sigma \) is the set \( Des(\sigma) = \{ i : \sigma_i > \sigma_{i+1}\} \) and the major index is defined as \( maj(\sigma) = \sum_{i \in Des(\sigma)} i \). Similar to Corollary 7.2, we have the following result.
Corollary 7.3. The distribution of the major index for $r$-signed permutations is given by
\[ \sum_{\sigma \in \mathcal{S}_n^r} q^{\text{maj}(\sigma)} = [n]! \cdot \prod_{i=1}^{n} (1 + (r_i - 1) \cdot q). \]

Proof. The proof is the same as Corollary 7.2 except we replace the chains with the posets $P_i$ in Figure 1. Note that $\Psi(P_i) = a + (r_i - 1) \cdot b$. Let $L$ be the lattice $L = P_1 \circ \cdots \circ P_n$. Let $L$ inherit the labels of the cover relations from its factors with the extra condition that the cover relations attached to the maximal element receive the label 0. This is an $R$-labeling and the labels of the maximal chains are exactly the $r$-signed permutations.

For signed permutations, that is, $r = (2, 2, \ldots, 2)$, the above result follows from an identity due to Reiner [13, Equation (5)].

8 Concluding remarks

We suggest the following $q, t$-extension of the Major MacMahon map $\Theta$. Define $\Theta^{q,t} : \mathbb{Z}(a, b) \rightarrow \mathbb{Z}[q, t]$ by
\[ \Theta^{q,t}(w) = \Theta(w) \cdot w_{a=1, b=q} = \prod_{i : u_i = b} q^i \cdot t, \] (8.1)
for an $ab$-monomial $w = u_1 u_2 \cdots u_n$. Applying this map to the $ab$-index of the Boolean algebra yields one of the four types of $q$-Eulerian polynomials:
\[ \Theta^{q,t}(\Psi(B_n)) = A_n^{\text{maj}, \text{des}}(q, t) = \sum_{\pi \in \mathcal{S}_n} q^{\text{maj}(\pi)} t^{\text{des}(\pi)}. \]

The following identity has been attributed to Carlitz [4], but goes back to MacMahon [10, Volume 2, Chapter IV, §462],
\[ \sum_{k \geq 0} [k + 1]^n \cdot t^k = A_n^{\text{maj}, \text{des}}(q, t) \prod_{j=0}^{n} (1 - t \cdot q^j). \] (8.2)
For recent work on the $q$-Eulerian polynomials, see Shareshian and Wachs [14]. It is natural to ask if there is a poset approach to identity (8.2).

There are several different ways to extend the major index to signed permutations. Two of our favorites are [1, 18].
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References


