Quantum Combinatorics

Margaret Readdy.

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Combinatorics & Graph Theory

Purdue.
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Let's count $[\approx 50,000 \text{ BC}^*]$

$\gamma_1 \cdots \gamma_n \in \mathcal{S}_n$, the symmetric group on an $n$ elt. set.

$$\sum_{\gamma \in \mathcal{S}_n} 1 = n! = n(n-1) \cdots 2 \cdot 1.$$ 

$$\sum_{5 \leq k \leq 1, \ldots, n \gamma} 1 = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}.$$ 

*Source: Wikipedia*
Let's \( q \)-count \( \lfloor 1700 \rfloor \) Euler.

\( q \)-analogue of \( n \in \mathbb{Z}^+ \)

\[ [n]_q = [n] = 1 + q + q^2 + \cdots + q^{n-1}, \]

\( q \) an indeterminate.

\[
\lim_{q \to 1} [n]_q = 1 + \frac{q^2 + q^4 + \cdots + q^n}{n} = n.
\]

\[ [n]! = [n] [n-1] \cdots [2] \cdot [1] \]

\* Theta functions

\[
\vartheta_3(a, b) = \sum_{n=-\infty}^{\infty} a^{n+b}.
\]

\( \left| ab \right| < 1. \)
inversion statistic on permutations

(see also Cramer 1750, Laplace 1772, Bézout 1764).

\[
\text{inv } (\pi) = | \{ (i, j) : \pi_i > \pi_j \text{ for } i < j \}\ |
\]

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( \text{inv } \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>0</td>
</tr>
<tr>
<td>132</td>
<td>1</td>
</tr>
<tr>
<td>213</td>
<td>1</td>
</tr>
<tr>
<td>231</td>
<td>2</td>
</tr>
<tr>
<td>312</td>
<td>2</td>
</tr>
<tr>
<td>321</td>
<td>3</td>
</tr>
</tbody>
</table>
Theorem: [MacMahon 1916].

$$\sum_{\pi \in S_n} q^{\text{inv} (\pi)} = [n]_q !$$

This is a combinatorial interpretation of $[n]_q !$. 
\[ \sum = 1 + 2q + 2q^2 + q^3 \]
\[ = 1 \cdot (1+q) \cdot (1+q+q^2) \]
\[ = \left[ 3 \right]_q \]
def. The **Gaussian polynomial** or **q-binomial**

\[
\binom{n}{{k}_q} = \frac{\binom{n}_q!}{\binom{{k}_q}_q! \binom{n-{{k}_q}}_q!}
\]

\[
= \frac{\binom{n}_q \binom{n-1}_q \cdots \binom{n-{{k}_q}+1}_q}{\binom{{k}_q}_q!}.
\]
ex. \( G(0^{n-w}, 1^w) \).

\( n=4, \quad w=2 \).

\[
\begin{array}{cccc}
0011 & 0101 & 0110 & 1001 \\
0 & 1 & 2 & 3 \\
1010 & 1100 \\
4 & 5 \\
\end{array}
\]
Theorem: [Marc Mahon 1916].

\[ \sum_{\tau \in \mathcal{C}(0^n, 1^n)} \tau^{\text{inv}} \nu = [n]. \]
Other combinatorial interpretations

1. \[ \binom{n}{k} = \# \text{\ # of } k\text{-dim } \subseteq \text{\ of an } n\text{-dim } \subseteq \text{\ over } \mathbb{F}_q. \]

ex. \[ \binom{a}{1} = 1+q. \]
when \( q = 3 \):

\[ y = 2x, \ y = vx \]
\[ y = 0, \ x = 0 \]

2. lattice path's. using \( n-k \) \( E \) 's \( \ell \) \( E \) 's \( N \) 's, \( (n-k, k) \) weighted by area under path.
Unimodality

A sequence \( \{a_0, a_1, \ldots, a_m\} \) is \textit{unimodal} if

\[
a_0 \leq a_1 \leq \ldots \leq a_j \geq \ldots \geq a_m
\]

for some \( 0 \leq j \leq m \).

ex. row of Pascal's triangle

\[
\begin{array}{c}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]
Unimodality of $[n_k]$ 

\[
[n_k] = q_0^{(n-k)} \cdot k^1 + \ldots + a_i q_i^1 + a_0.
\]

Theorem: [Sylvester 1878].
The coefficients of $[n_k]$ are unimodal, i.e.,

\[
a_0 \leq \ldots \leq a_j \geq \ldots \geq a_{(n-k)}^k.
\]

[O'Hara 1990] Gave the first combinatorial proof of this result.
O'Haral's proof

Fix $n$ and $\mathfrak{A}$. We weight an $\mathfrak{A}$-subet $S$ of $\{1, \ldots, n\}$ by

$$w^+(S) = \sum_{\mathfrak{S} \in S} s_\mathfrak{S}.$$  

Form a poset = partially ordered set.
\alpha. \quad \left[ \frac{4}{2} \right] = q^4 + q^3 + 2q^2 + q + 1.

\begin{array}{c|c|c}
\text{wt} & \text{wt} - 3 \\
\hline
3, 4 & 7 & 4 \\
\hline
2, 4 & 6 & 3 \\
\hline
1, 4 & 5 & 2 \\
\hline
2, 3 & \phantom{0} & \phantom{0} \\
\hline
1, 3 & \phantom{0} & \phantom{0} \\
\hline
1, 2 & \phantom{0} & \phantom{0} \\
\hline
1, 1 & \phantom{0} & \phantom{0}
\end{array}
Construct a symmetric chain decomposition (SCD);

(Write \( P \) as a disjoint union of rank-symmetric saturated chains).

\[ \Rightarrow \text{ unimodal}. \]

\[ 9.14, \]
Theorem: [Park - Panova, 2013]

For $j, k \geq 8$ in $[jw/2]$, the coefficients satisfy:

\[ a_1 < \cdots < a_{\lfloor jw/2 \rfloor} = a_{\lfloor jw/2 \rfloor} > \cdots > a_{jw-1}. \]

Proof: Algebraic, uses combinatorics of Young tableaux, semigroup property of Kronecker coefficients of \( \mathfrak{g}_n \) representations.
Young's lattice and the poset of integer partitions.

\[ L(m,n) \]

Definition: For any partitions whose Ferrers diagram fits inside an \( m \times n \) rectangle.

Rank generating function of \( L(m,n) \) is:

\[ \sum_{\lambda \in L(m,n)} x^{\lambda} = \prod_{i=1}^{\min(m,n)} \frac{1}{1-x^i} \]

Open: Find a SCD for \( L(m,n) \).
[Stanton 1990].

For $\lambda = (8, 8, 4, 4)$:

1, 1, 2, 5, 6, 9, 11, 15, 17, 21, 23, 27, 28, 31, 30, 31, 27, 24, 18, 14, 8, 5, 4, 1

Open: Classify non-unimodal partitions.

[Stanton-Zanello 2015].

$F_\lambda$ unimodal for the shifted Ferrers diagram $\lambda = \langle n, n-1, n-2, n-3 \rangle$, $n \geq 4$.

Conjecture: [Stanton-Zanello].

$F_\lambda$ unimodal from shifted Ferrers diagrams with “minimal” $\lambda$ from arithmetic progressions.
Juggling + q-analogues

Assume: 1-handed juggler can catch and throw one ball at a time.

Theorem: \[ \text{[Buhler-Eisenbud-Graham-Wright]} \]

The number of juggling patterns of period \( d \) and at most \( n \) balls is \( n^d \).
n = 3, d = 3

Throw vector

(1, 1, 1)

(2, 2, 2)

(1, 3, 3)

(2, 3, 1)

(3, 1, 2)

(1, 1, 4)

(1, 4, 1)

(4, 1, 1).

\(2^3 = 8\).
\[(1,1,1)\]

\[(2,2,2)\]

\[(1,2,3)\]

\[(2,3,1)\]

\[(3,1,2)\]

\[(1,1,4)\]

\[(1,4,1)\]

\[(4,1,1)\]

\[\frac{q^3}{q^0} = 3q\]

\[\frac{q^1}{q^0} = 3q\]

\[\frac{(1+q)^3}{q^0} = [a]^3.\]
Theorem: [Ehrenbags - Reaiddyn].
The weight of juggling patterns of period $d$ and at most $n$ balls is $\lceil n \rceil^d$.

Proof. Cards for $\leq 3$ balls.

\[ q^0 \quad q^1 \quad q^2 \]
Application: Affine Weyl group $\tilde{A}_{d-1}$.

**Def.** [Lusztig]

$\tilde{A}_{d-1}$ is the group of bijections $\sigma: \mathbb{Z} \rightarrow \mathbb{Z}$ with composition satisfying

1. $\sigma(\tilde{z} + d) = \sigma(\tilde{z}) + d \quad \forall \tilde{z}$
2. $\sum_{\tilde{z}=1}^{d-1} (\sigma(\tilde{z}) - \tilde{z}) = 0$ "conservation of momentum"

Generated by the simple reflections:

$S_0$ 

$S_1$ 

$\vdots$ 

$S_{d-1}$
Theorem: [Bott]

The Poincaré series for $\tilde{A}_{d-1}$ is:

$$
\sum_{\sigma \in \tilde{A}_{d-1}} \varphi(\sigma) q^{|\sigma|} = \frac{1 - q^d}{(1 - q)^d}.
$$

ex.

$\tilde{A}_1$

$\begin{array}{cccc}
\rightarrow & -2 & -1 & 0 \\
\rightarrow & 1 & 2 & 3
\end{array}$

$\tilde{A}_2$

$\begin{array}{cccc}
\rightarrow & -2 & -1 & 0 \\
\rightarrow & 1 & 2 & 3
\end{array}$

$\begin{array}{cccc}
\rightarrow & -2 & -1 & 0 \\
\rightarrow & 1 & 2 & 3
\end{array}$

$\begin{array}{cccc}
\rightarrow & -2 & -1 & 0 \\
\rightarrow & 1 & 2 & 3
\end{array}$

$$
\sum_{\sigma \in \tilde{A}_2} \varphi(\sigma) q^{|\sigma|} = \frac{1 - q}{(1 - q)^2}
$$

$$
= \frac{1 + q}{1 - q} = 1 + 2q + 2q^2 + 2q^3 + \cdots
$$
A Combinatorial Proof \cite{Ehrenberg-R}.

\[ P_n = \{ \sigma \in \hat{S}_{d-1} : n > \max \{\xi - \sigma(\xi)\} \}. \]

Add \( n \) to these \( \sigma \in P_n \).

Claim: Are juggling sequences with
\( (n-1), d - \ell(\sigma) \) crossings,
period \( d \) and exactly \( n \) balls.

Pf.

Nontrivial \( \square \)
ex. \( \tilde{A}_{g-1} \)

Add \( n = 2 \).

\( \exists (3, 1, 2) \).
Proof (cont'd)

\[ \sum_{\sigma \in \mathcal{P}_n} q^{(n-1)d - \ell(\sigma)} = [n]^d - [n-1]^d, \]

\[ \sum_{\sigma \in \mathcal{P}_n} \left( \frac{1}{q} \right)^{\ell(\sigma)} = \frac{[n]^d - [n-1]^d}{q^{(n-1)d}}, \]

\[ \sum_{\sigma \in \mathcal{P}_n} q^{\ell(\sigma)} = q^{(n-1)d} \left( (1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{n-1}})^d - (1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{n-2}})^d \right). \]

\[ = (q^{n-1} + q^{n-2} + \ldots + 1)^d - (q^{n-1} + \ldots + q)^d \]

\[ = [n]^d - (q^d \cdot [n-1])^d. \]
\[
\sum_{\sigma \in P_n} q^\ell = \left( \frac{1-q^n}{1-q} \right)^d - q^d \left( \frac{1-q^{n-1}}{1-q} \right)^d
\]

Now,
\[
\bigcup_{n \geq 1} P_n = \tilde{A}_{d-1}
\]

Let \( n \to \infty \),
\[
\sum_{\sigma \in \tilde{A}_{d-1}} q^\ell = \frac{1-q^d}{(1-q)^d}. \]
Cyclic sieving phenomenon. [Reiner–Stanton–White]

\( X \) finite set
\( C \) finite cyclic group acting on \( X \)
\( f(q) \) = polynomial in \( q \) w/ nonneg. \( \mathbb{Z} \)-coeffs.

def. \((X, C, f(q))\) exhibits CSP if for all \( g \in C \)

\[ |X^g| = f(\omega), \quad \omega \text{ an } n \text{th root of unity}, \quad n = |g|, \]

where \( X^g = \{ x \in X : gx = vx \} \).
Ex. a subset of \( \{1,2,3,4\} \).

\[ f(q) = \left[ \frac{1}{q^4} \right] = 1 + q + 2q^3 + q^5 + q^6. \]

\[ X: \begin{array}{ccc}
1 & 2 & 3 \\
4 & 2 & 4 \\
3 & 4 \\
\end{array} \]

\[ g = (1,2,3), \quad |g| = 3 \]

\( \omega \) a 3rd root of unity.

\[ f(\omega) = 1 + \omega + 2\omega^3 + \omega^5 + \omega^6 \\
= 2 + 2\omega + 2\omega^2 \\
= 0 \Rightarrow \text{No fixed points}. \]

\[ g = (1,2)(3), \quad |g| = 2. \]

\[ f(-1) = 1 - 1 + 2 - 1 + 1 \\
= 2. \]

Fixed points:

\[ 1 \quad 3 \quad 4. \]
$n \times n$ alternating sign matrices.

Entries are $0, \pm 1$.

Row and column sums are 1.

Nonzero entries alternate in sign.

ex. $n = 3$

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\quad \begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]
Theorem: [Zeilberger 1996]

The number of $n \times n$ alternating sign matrices is

$$\prod_{\psi=0}^{n-1} \frac{(3\psi+1)!}{(n+\psi)!}$$
The cyclic group of order 4, generated by rotation of $\pi/2$ on alternating sign matrices, exhibits a CSP with

$$X(q) = \prod_{\omega=0}^{n-1} \frac{[3\omega+0]!}{[n+\omega]!}$$

Open:

1. No linear algebra proof.
2. $X(q)$ is the generating function for descending plane partitions by weight.

$X(q)$ is not a statistic on ASMs.
Thank you!