Juggling and q-analogues

Margaret Readdy
University of Kentucky

AWM Lecture Series

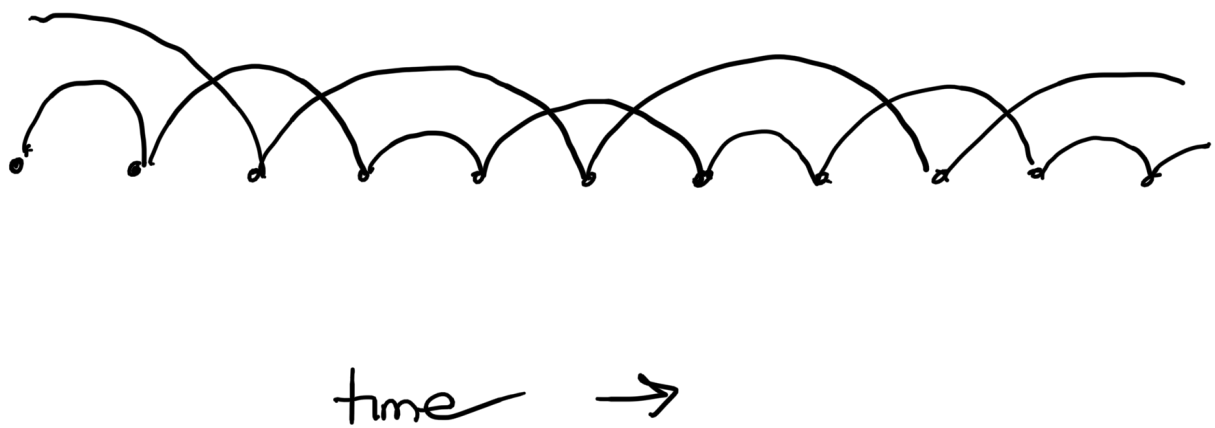
Juggling

Assume: ① One-handed juggler

② Juggler can catch + throw only one ball at a time

Known as simple juggling

ex. $(1, 2, 3)$



non-ex.

$(1,3,2)$.

Remark: A simple juggling pattern with period d and throw vector $\vec{a} = (a_1, \ldots, a_d)$ has

$$n = \frac{a_1 + \cdots + a_d}{d} \text{ balls}$$

ex. $(1,2,3)$

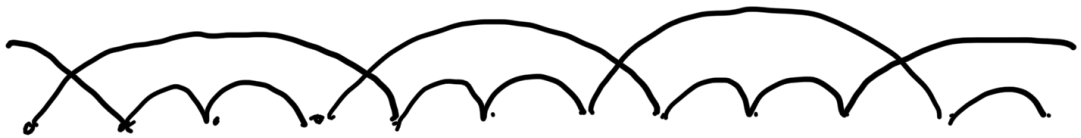
$$n = \frac{1+2+3}{3} = 2 \text{ balls}$$

Period 3

At most 2 balls

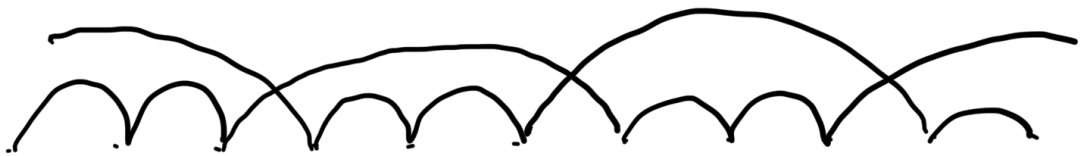
(1, 1, 1)

(4, 1, 1)



(1, 4, 1)

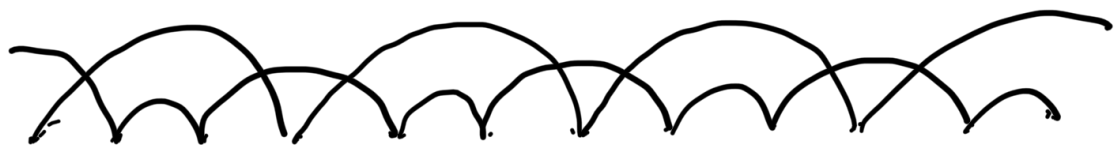
(1, 1, 4)



(1, 2, 3)

(2, 3, 1)

(3, 1, 2)



(2, 2, 2)

Theorem: [Buhler, Eigenbud, Graham, Wright]

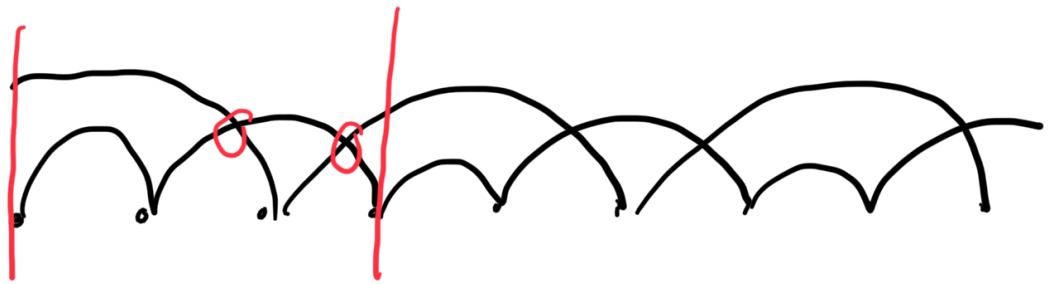
The number of simple juggling patterns having period d and at most n balls is

$$n^d.$$

q-analogue

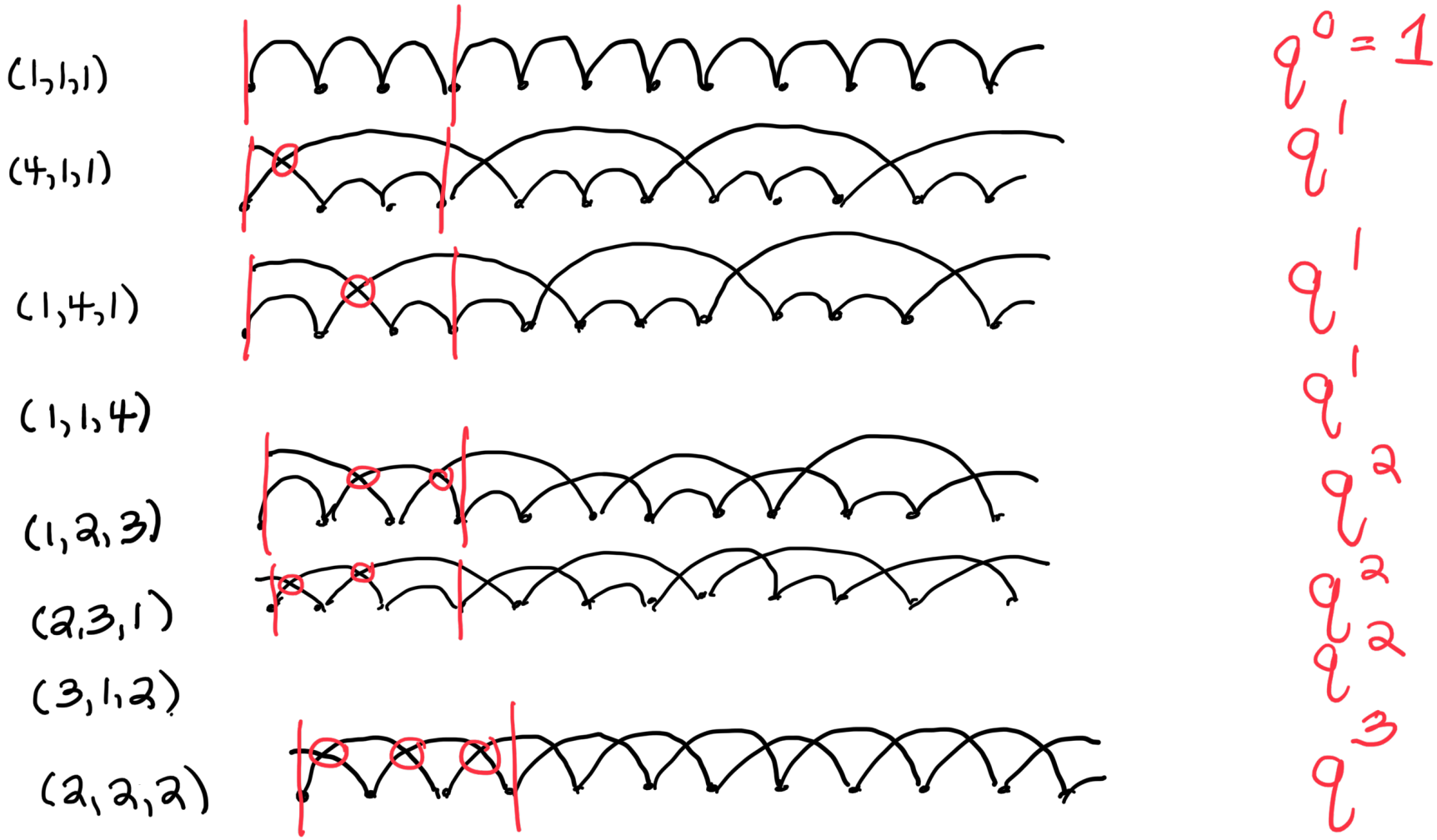
Look at crossings in a period

(1,2,3)



q^2

2 crossings


(1,1,1)
(4,1,1)
(1,4,1)
(1,1,4)
(1,2,3)
(2,3,1)
(3,1,2)
(2,2,2)
$q^0 = 1$
q^1
q^1
q^1
q^2
q^2
q^2
q^3

$$\Sigma = 1 + 3q + 3q^2 + q^3$$

$$= (1+q)^2$$

Theorem: [Ehrenborg - Readdy]

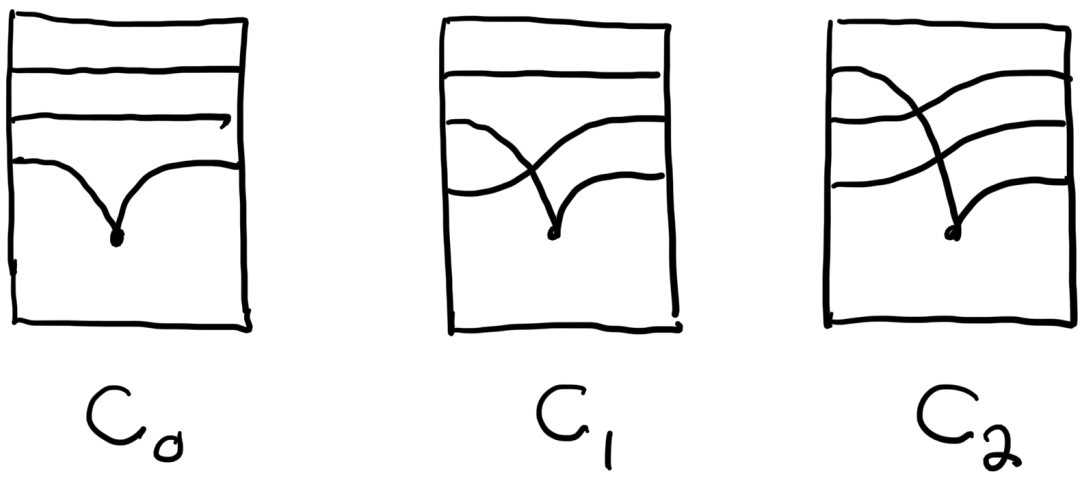
The sum of the weights of simple juggling patterns having period d and at most n balls is

$$[n]^d = (1+q+\cdots+q^{n-1})^d.$$

Note: Setting $q=1$ gives n^d enumeration

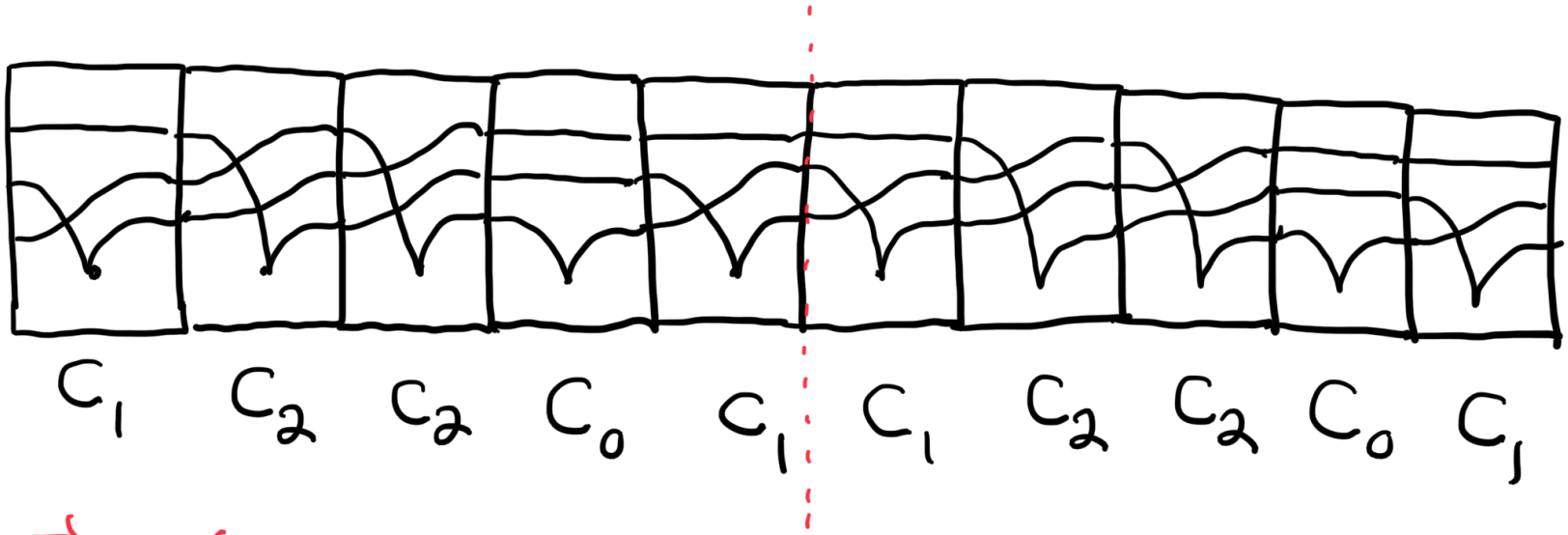
<u>Proof</u>

Juggling cards



ex. period 5

$c_1\ c_2\ c_2\ c_0\ c_1$ etc.



$\vec{a} = (6, 3, 1, 2, 3)$.

<u>Proof</u> (cont'd)

Choose card for first throw.

Have n possible choices, each having weight

$$q^0 = 1 \text{ or } q^1 \text{ or } q^2 \text{ or } \cdots \text{ or } q^{n-1}$$

Other throws have similar indep. choices

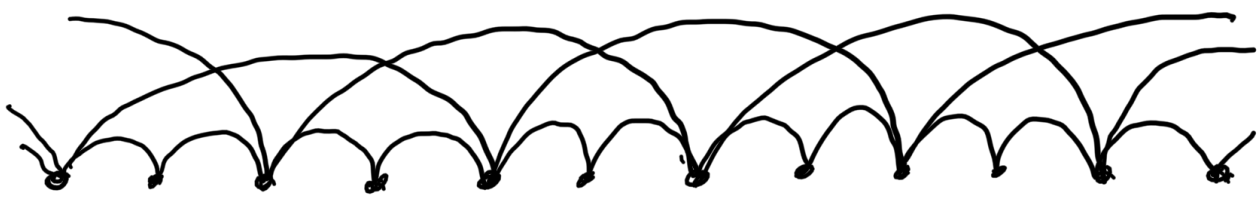
So

$$\underbrace{[n]\,[n]\,\cdots\,[n]}_{d} = [n]^d. \qquad \blacksquare$$

Multiplex Juggling

Now: One-handed juggler
Can catch + throw more than one ball at a time

ex $d=2$
$\vec{a} = (4,1,1)$
$\vec{x} = (0,0,1)$



The q-binomial

Recall binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$ the binomial coefficient.

where

$$n! = n(n-1)\cdots 2\cdot 1.$$

Pascal's Triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

Recurrence

$$\binom{n-1}{k-1} \qquad \binom{n-1}{k}$$

$$\binom{n}{k}$$

ex. $(x+y)^3$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

The q-binomial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}$$

ex. $$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4][3][2][1]}{[2][1][2][1]} = \frac{[4]\,[3]}{[2]}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}$$

$$= (1+q^2)(1+q+q^2)$$

$$= q^4 + q^3 + 2q^2 + q + 1$$

Facts:

① $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q.

② Recurrence

$$\begin{tikzcd} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \arrow[dr, "q^{n-k}"'] & & \begin{bmatrix} n-1 \\ k \end{bmatrix} \arrow[dl] \\ & \begin{bmatrix} n \\ k \end{bmatrix}. & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \arrow[dr] & & \begin{bmatrix} n-1 \\ k \end{bmatrix} \arrow[dl, "q^{k}"] \\ & \begin{bmatrix} n \\ k \end{bmatrix} & \end{tikzcd}$$

Binomial theorem

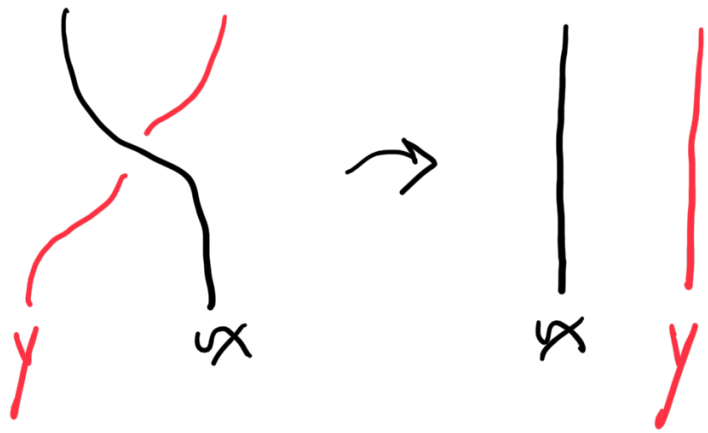
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

q-Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

where

$$y \cdot x = q^{1} \cdot xy$$



ex. $(x+y)^3 = (x+y)(x+y)(x+y)$

$$= x^3 + x^2y + xyx + yxx + y\cdot yx + yxy + xyy + y^3$$

$$= x^3 + (x^2y + q^1 \cdot x^2y + q^2 x^2y) + (q^2 xyy + q^1 xyy + xyy) + y^3$$

$$= \begin{bmatrix}3\\3\end{bmatrix} x^3 + \begin{bmatrix}3\\2\end{bmatrix} x^2y + \begin{bmatrix}3\\1\end{bmatrix} xy^2 + \begin{bmatrix}3\\0\end{bmatrix} y^3$$

Back to multiplex juggling

Theorem: [Ehrenborg - Readdy]

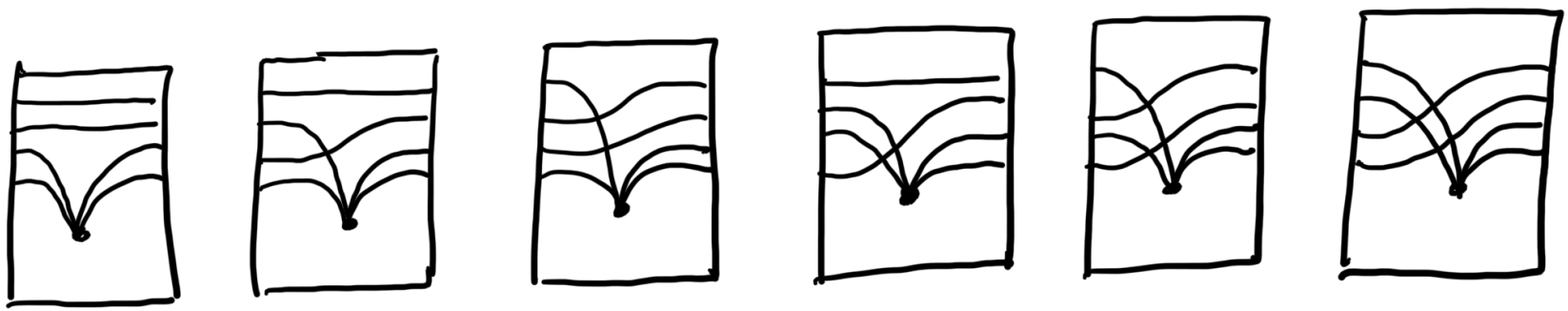
The sum of the weight of multiplex juggling patterns with period d, base vector $\vec{\alpha}$ and at most n balls is

$$\begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}$$

where $\alpha_j = \#\{ i : x_i = j \}$ for $j = 0, \ldots, d-1$

Proof

Multiplex juggling cards



q^0 q^1 q^2 q^2 q^3 q^4

Catch/throw 2 balls at time i

At most $n=4$ balls

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4 \qquad \blacksquare$$

Thank you!