Juggling and $q$-analogues

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AWM Lecture Series
Juggling

Assume:
1. One-handed juggler
2. Juggler can catch and throw only one ball at a time

Known as simple juggling
ex.  \((1, 2, 3)\)
Remark: A simple juggling pattern with period $d$ and throw vector $\mathbf{a} = (a_1, \ldots, a_d)$ has $n = \frac{a_1 + \cdots + a_d}{d}$ balls.

ex. $(1, 2, 3)$

$$n = \frac{1 + 2 + 3}{3} = 2 \text{ balls}$$
Period 3
At most 2 balls

(1, 1, 1)

(4, 1, 1)

(1, 4, 1)

(1, 1, 4)
Theorem: [Buhler, Eisenbud, Graham, Wright]
The number of simple juggling patterns having period \( d \) and at most \( n \) balls is \( n^d \).
$q$-analogue

Look at crossings in a period

$(1,2,3)$

$2$ crossings

$q^2$
\[ M = 1 + 3q + 3q^2 + q^3 \]
\[ = (1+q)^2 \]
Theorem: [Ehrenberg - Reavaddy]
The sum of the weights of simple juggling patterns having period $d$ and at most $n$ balls is

$$[n]_d = (1 + q + \ldots + q^{n-1})^d.$$ 

Note: Setting $q = 1$ gives $n^d$ enumeration.
Proof

Juggling cards

$C_0$

$\theta$

$C_1$

$\theta'$

$C_2$

$\theta''$
period 5

\[ C_1 C_2 C_2 C_0 C_1 \text{ etc.} \]

\[ \vec{a} = (6, 3, 1, 2, 3) \]
Proof (cont’d)

Choose card for first throw.

Have \( n \) possible choices, each having weight \( q^0 = 1 \) or \( q^1 \) or \( q^2 \) or \( \ldots \) or \( q^{n-1} \).

Other throws have similar independent choices.

So,

\[
\underbrace{[n] \quad [n] \quad \ldots \quad [n]}_{d} = [n]^d.
\]

\[\Box\]
Multiplex Juggling

Now: One-handed juggler
Can catch & throw more than
one ball at a time

ex.  \( d = 2 \)
    \( a_1 = (4,1,1) \)
    \( x_5 = (0,0,1) \)
The \( q \)-binomial

Recall binomial theorem

\[
(ax + y)^n = \sum_{k=0}^{n} \binom{n}{k} a^k y^{n-k}
\]

\[
\binom{n}{k} = \frac{n!}{k! \cdots (n-k)!}
\]

where

\( n! = n(n-1) \cdots 2 \cdot 1 \).

\[\text{Pascal's Triangle:}\]

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
\end{array}
\]

Recurrence

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\]

\[\text{Ex. } (ax + y)^3 = ax^3 + 3ax^2y + 3axy^2 + y^3\]
The $q$-binomial
\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]! [n-k]!}
\]

ex. \[
\left[ \begin{array}{c} 4 \\ 2 \end{array} \right] = \frac{[4][3][2][1]}{[2][1][2][1]} = \frac{[4][3]}{[2]}
\]
\[
= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}
\]
\[
= (1+q^2)(1+q+q^2)
\]
\[
= q^4 + q^3 + 2q^2 + q + 1
\]
Facts:
1. \([\binom{n}{k}]\) is a polynomial in \(q\).
2. Recurrence

\[
\begin{align*}
\binom{n-1}{k-1} & \quad \binom{n}{k-1} \\
& \quad q^{n-k} \quad \binom{n}{k}
\end{align*}
\]

\[
\begin{align*}
\binom{n-1}{k} & \quad \binom{n}{k-1} \\
& \quad \binom{n}{k}
\end{align*}
\]
Binomial theorem

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

q-Binomial theorem

\[(qx+qy)^n = \sum_{k=0}^{n} \left[ \binom{n}{k} \right] q^{n-k} x^k y^{n-k}\]

where

\[y \cdot x = q^{\frac{1}{q}} \cdot xy\]
\[ a \chi (\chi + y)^3 = (\chi + y) (\chi + y) (\chi + y) \]
\[ = \chi^3 + \chi^2 y + \chi y \chi + y \chi \chi \]
\[ + y. y \chi + y \chi y + \chi y y + y^3 \]
\[ = \chi^3 + (\chi^2 y + q' \chi^2 y + q^2 \chi^2 y) \]
\[ + (q^2 \chi y y + q' \chi y y + \chi y y y) + y^3 \]
\[ = \left[ \frac{3}{3} \right] \chi^3 + \left[ \frac{3}{2} \right] \chi^2 y + \left[ \frac{3}{1} \right] \chi y^2 + \left[ \frac{3}{0} \right] y^3 \]
Back to multiplex juggling

Theorem: [Ehrenberg - Readdy]

The sum of the weight of multiplex juggling patterns with period \( d \), base vector \( \mathbf{x} \), and at most \( n \) balls is

\[
\begin{bmatrix}
    n \\
    \vdots \\
    n
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    a_1 \\
    \vdots \\
    a_{d-1}
\end{bmatrix}
\]

where

\[
a_j = \# \text{ of } x_i^j \neq 0, \quad \text{for } j = 0, \ldots, d-1
\]
Proof

Multiplex juggling cards

\[ \begin{align*}
q^0 & \quad q^1 & \quad q^2 & \quad q^3 & \quad q^4 \\
\end{align*} \]

Catch/throw 2 balls at time c
At most \( n = 4 \) balls

\[ \left[ \frac{4}{2} \right] = 1 + q + 2q^2 + q^3 + q^4 \]
Thank you!