Polytopes and Whitney-stratified Spaces

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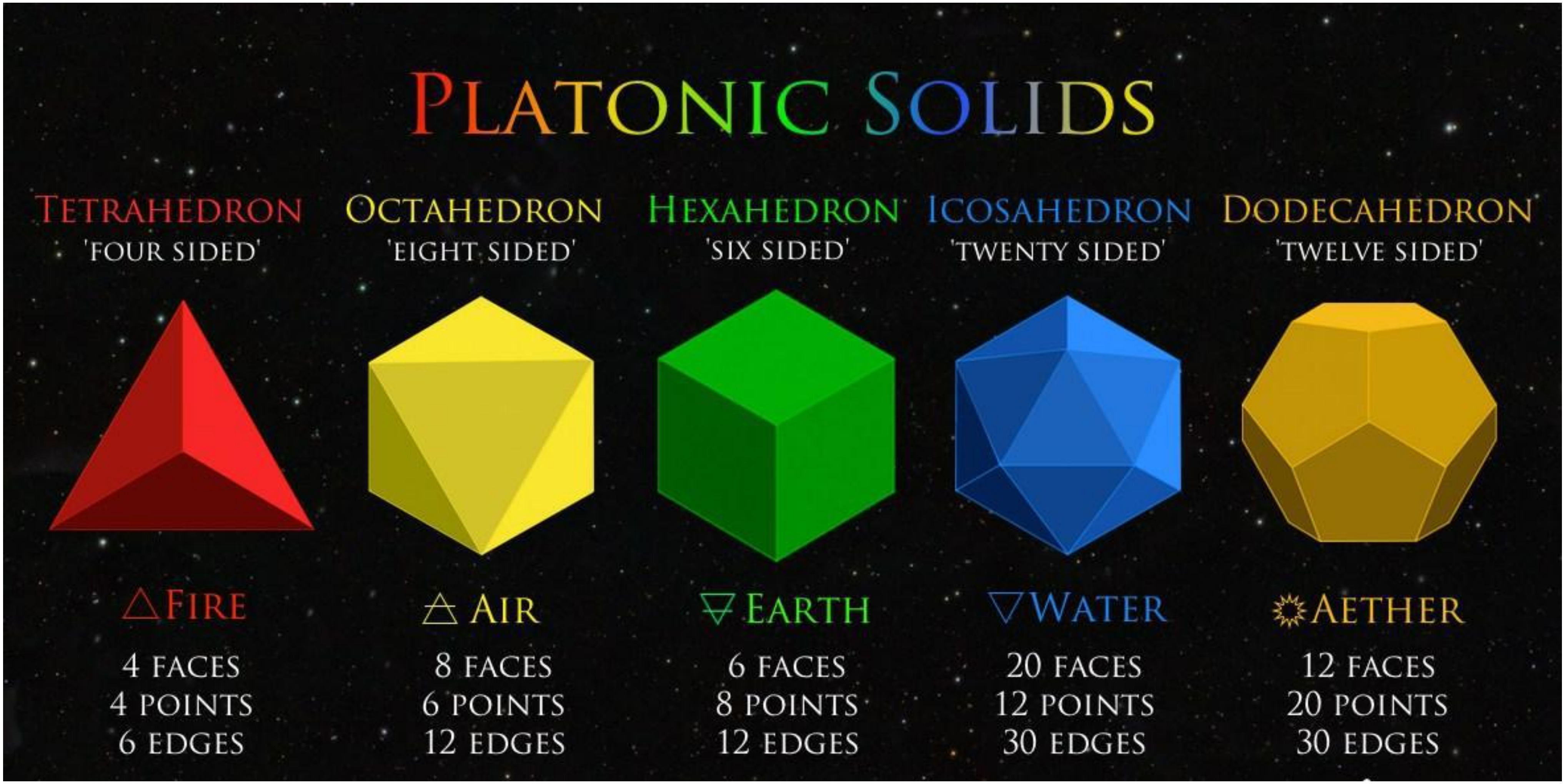
AWM Virtual Lecture Series
Kutztown University.

Thanks to the
Institute for Advanced Study
and the
Simons Foundation.



Reference: Natural Philosophy & Esoteric Geometry: The Cosmic Creations Collection joedubs.com

Athens, Greece

Thaetetus [~417-369 BC]

There are 5 regular 3-dim'l polytopes

<u>Reference</u>

Euclid's Elements

Book $\underline{\overline{XIII}}$

Prop. 13-17

(Digitized by Clay Math Institute).

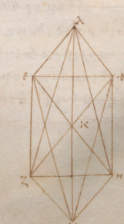


Image courtesy of the Clay Mathematics Institute

Skip ahead ~ 2500 years

def. A <u>polytope</u> is the convex hull of a finite # of vertices in $\mathbb{R}^n$

or

... is the bounded intersection of a finite # of closed half-spaces in $\mathbb{R}^n$.

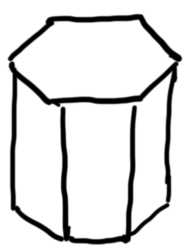
f-vector

P n-dim'l polytope

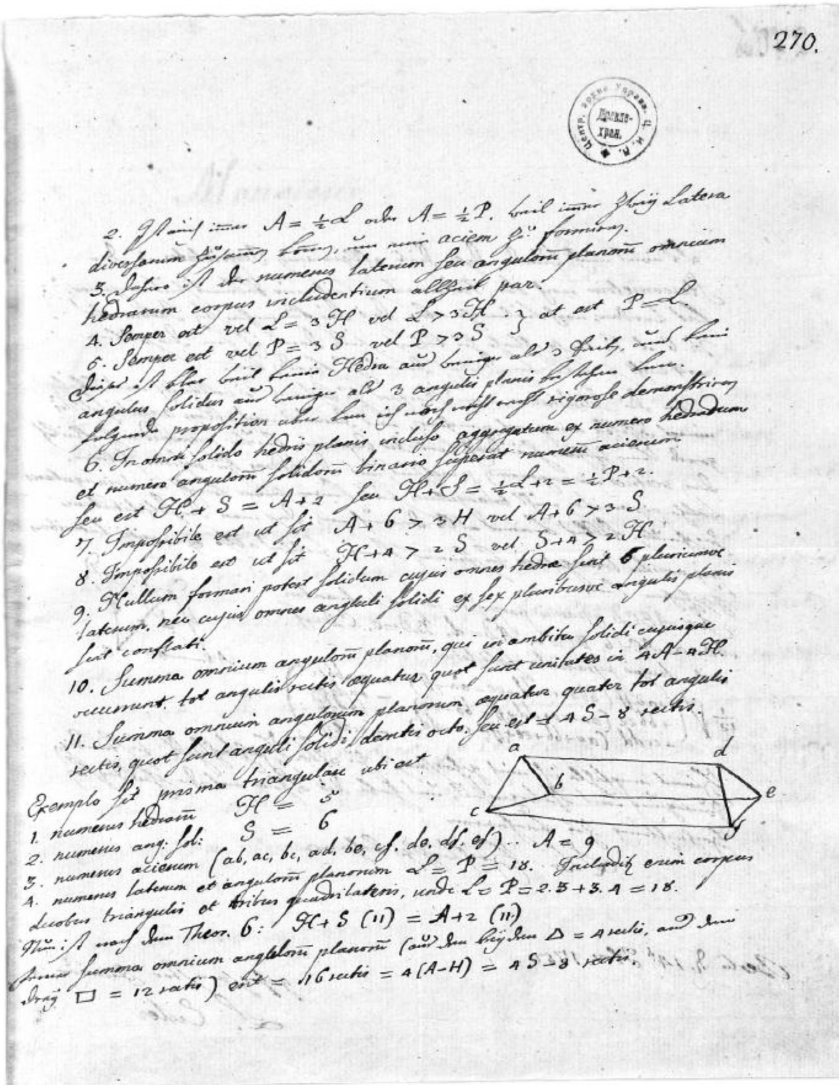
The f-vector $(f_0, f_1, \ldots, f_{n-1})$ where

f_i = # i-dim'l faces of P

ex.

 = Prism()

$$(f_0, f_1, f_2) = (12, 18, 8)$$



2. Gleich wie $A = \frac{1}{2}L$ oder $A = \frac{1}{2}P$, weil immer zwey Latera diversarum hedrarum zusammen kommen, um eine aciem zu formiren.
3. Daher ist der numerus laterum seu angulorum planorum omnium hedrarum corpus includentium allzeit par.
4. Semper est vel $L = 3H$ vel $L > 3H$ } et est $P = L$.
5. Semper est vel $P = 3S$ vel $P > 3S$ }
Dieses ist klar weil keine Hedra aus weniger als 3 Seiten, und kein angulus solidus aus weniger als 3 angulis planis bestehen kann. Folgende propositionen aber kann ich noch nicht recht rigorose demonstriren.
6. In omni solido hedris planis incluso aggregatum ex numero hedrarum et numero angulorum solidorum binario superat numerum acierum seu est $H + S = A + 2$ seu $H + S = \frac{1}{2}L + 2 = \frac{1}{2}P + 2$.
7. Impossibile est ut sit $A + 6 > 3H$ vel $A + 6 > 3S$
8. Impossibile est ut sit $H + 4 > 2S$ vel $S + 4 > 2H$
9. Nullum formari potest solidum cujus omnes hedrae sint 6 pluribusve lateribus, nec cujus omnes anguli solidi ex sex pluribusve angulis planis sint conflati.
10. Summa omnium angulorum planorum, qui in ambitu solidi cujusque occurrunt, tot angulis rectis aequatur quot sunt unitates in $4A - 4H$.
11. Summa omnium angulorum planorum aequatur quater tot angulis rectis, quot sunt anguli solidi, demtis octo, seu est $= 4S - 8$ rectis.
Exemplo sit prisma triangulare ubi est

1. numerus hedrarum $H = 5$
2. numerus ang. sol: $S = 6$
3. numerus acierum (ab, ac, bc, ad, be, cf, de, df, ef) .. $A = 9$
4. numerus laterum et angulorum planorum $L = P = 18$. Includitur enim corpus duobus triangulis et tribus quadrilateris, unde $L = P = 2.3 + 3.4 = 18$.
Nun ist nach dem Theor. 6: $H + S\ (11) = A + 2\ (11)$ und die Summa omnium angulorum planorum (aus den beyden $\Delta = 4$ rectis, und den drey $\square = 12$ rectis) erit $= 16$ rectis $= 4(A-H) = 4S - 8$ rectis.

Euler's letter n° 149 to Goldbach, November 14th, 1750: reproduction of the third page (RGADA, f. 181, n. 1413, č. IV, fol. 270r)

In the paragraph numbered 6, Euler's Polyhedron Formula appears for the first time; paragraph 11 gives the discrete form of the Gauss-Bonnet theorem (cf. n° 149, notes 5–8).

November 14 1750

Letter No. 149

Euler to Goldbach

$$H + S = A + 2$$

H = hedrae (facets)

S = anguli solidi (= solid angles = vertices).

A = acies (= sharpness = edges).

Euler's relation [1750 letter to Goldbach]

$$f_0 - f_1 + f_2 = 2$$

Proof by Descartes [1596-1650]
using plane angles

Schläfli [1850's]

4-dim'l polytopes

convex regular 4-polytope		Schläfli symbol $\{p,q,r\}$
5-cell	(4-dim'l simplex)	$\{3,3,3\}$
8-cell	(4-dim'l cube)	$\{4,3,3\}$
16-cell	(hexadecachoron)	$\{3,3,4\}$
24-cell		$\{3,4,3\}$
120-cell		$\{5,3,3\}$
600-cell		$\{3,3,5\}$.

Euler - Poincaré - Schläfli

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} f_{n-1} = 1 - (-1)^n$$

The proofs

[Poincaré, 1893] Introduced homology groups and algebraic topology

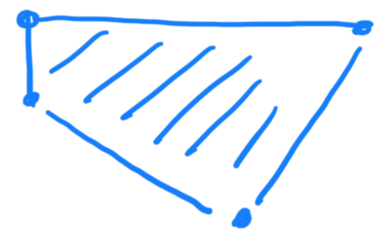
[Schläfli, 1852] Assumed shellability of polytopes.

[Bruggesser - Mani, 1971] Polytopes are shellable

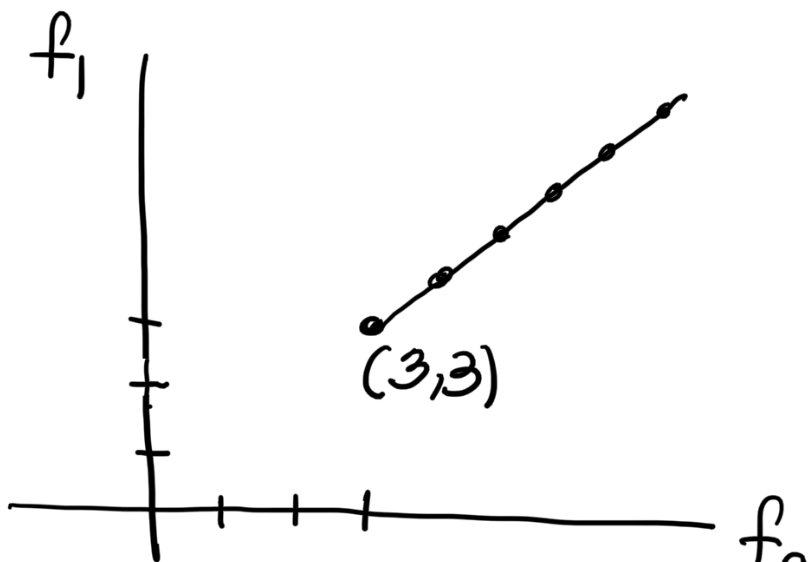
Question: Characterize f-vectors of n-dim'l polytopes

$n = 2$

$(f_0, f_1) = (3, 3)$



$(f_0, f_1) = (4, 4).$



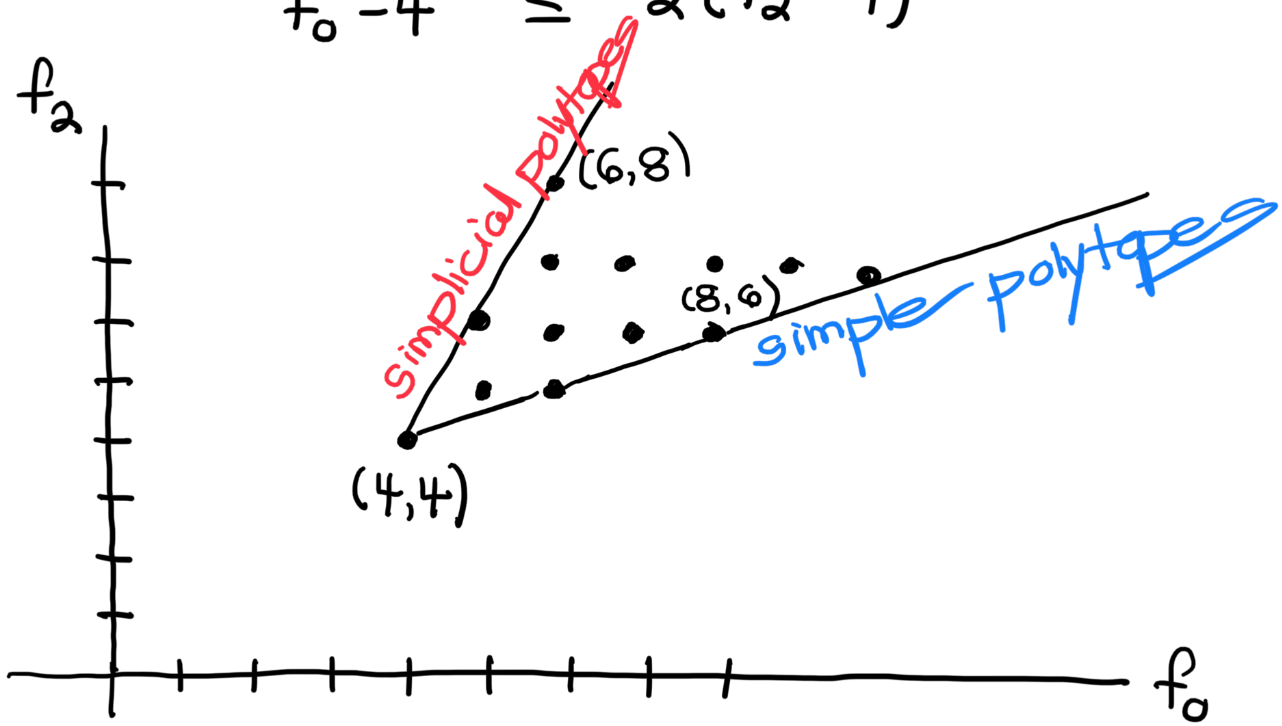
$f_0 = f_1 \quad (f_0 \geq 3).$

[Steinitz 1906]

The f-vectors of 3-dim'l polytopes live in the integer cone

$$2(f_0-4) \geq f_2-4$$

$$f_0-4 \leq 2(f_2-4)$$



Open 2: Characterize f-vectors of n-dim'l polytopes for $n \geq 4$

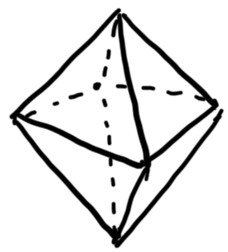
Known: Simplicial polytopes, and by duality, simple polytopes

Δ a simplicial complex of dim n.

The <u>h-vector</u>

$$\sum_{i=0}^{n+1} h_i t^{n+1-i} = \sum_{i=-1}^{n} f_i (t-1)^{n-i}$$

ex.



$$\vec{f} = (f_{-1}, f_0, f_1, f_2) = (1, 6, 12, 8)$$

$$\vec{h} = (h_0, h_1, h_2, h_3) = (1, 3, 3, 1).$$

The h-vector is:

① [Kind- Kleinschmidt 1979]
Related to the shelling of the complex

②. [Stanley 1978]
Numerator of the Hilbert series of the Stanley- Reisner ring of Δ

The g-theorem

Theorem: [Stanley 1980; Billera-Lee 1981]
The vector $(h_0, \dots, h_n)$ is the h-vector of an n-dim'l simplicial polytope if & only if

① (Dehn-Sommerville eqns)
$h_i = h_{n-i}$, for $i = 0, \dots, \lfloor n/2 \rfloor$

②. $(g_0, g_1, \dots, g_{\lfloor n/2 \rfloor})$ is an M-sequence
where $g_0 = h_0$ & $g_i = h_i - h_{i-1}$

Upper & Lower Bound Theorems

Theorem: ① [Barnette 1973]

For P n-dim'l simplicial polytope w/ $f_0 = v$

$$f_j(\text{Stack}(n, v)) \leq f_j(P) \quad \text{for } 0 \leq j < n \ \& \ n \geq 4$$

② [McMullen 1970]

For P n-dim'l polytope w/ $f_0 = v$

$$f_j(P) \leq f_j(C(n, v)) \quad \text{for } 0 \leq j \leq n$$

③ [Stanley 1975]

Δ n-dim'l simplicial sphere w/ $f_0 = v$

$$f_j(\Delta) \leq f_j(C(n, v))$$

Recent work:

[Murai-Novik 2015] Connected normal pseudomanifolds

[Adripragito-Sanyal 2016] relative Stanley-Reisner theory + Minkowski sums

[Zheng 2017, 2020] UBT + flag homology 5-manifolds

[Nevo-Pineda-Villavicencio-Ugon, Yost 2020] LBT + UBT for almost simplicial polytopes.

flag vectors

P n-dim'l polytope

For $S = \{s_1 < \cdots < s_k\} \subseteq \{0, 1, \ldots, n-1\}$ let

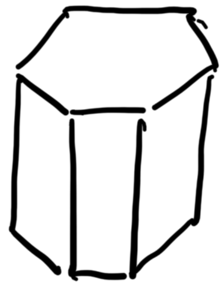
$$f_S = \# \quad F_1 \subsetneq \cdots \subsetneq F_k$$

where $\dim(F_i) = s_i$

The flag f-vector

$$(f_S)_{S \subseteq \{0,1,\ldots,n-1\}}$$

ex.



S	f_S	h_S	w_S
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb

The <u>flag h-vector</u>

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T$$

[Stanley 1979]

$$h_S = h_{\overline{S}}$$

The <u>ab-index</u>

$$\sum_S h_S \cdot w_S$$

$\Psi\Big($  $\Big) = 1aaa + 11baa + 17aba + 7aab$

$+ 7bba + 17bab + 11abb + 1bbb$

$= (a+b)^3 + 10baa + 16aba + 6aab$

$+ 6bba + 16bab + 10abb$

$= (a+b)^3 + 6(a+b)(ab+ba) + 10(ab+ba)(a+b)$

Let $c = a+b$

$d = ab+ba$

Then

$\Psi\Big($  $\Big) = c^3 + 6cd + 10dc$

The <u>cd-index</u>

Theorem: [Bayer-Klapper 1991]

P polytope then $\Phi(P) \in \mathbb{Z}\langle c,d\rangle$

P Eulerian poset then $\Phi(P) \in \mathbb{Z}\langle c,d\rangle$

Eulerian poset: $\mu(x,y) = (-1)^{\rho(x,y)}$ for every interval $[x,y]$ in graded poset P

Equivalently, in every nontrivial interval $[x,y]$:

$$\#\text{ elts of even rank} = \#\text{ elts of odd rank}.$$

Some cd-history

[Bayer-Billera 1985] Gen'd Dehn-Sommerville relations

[Bayer-Klapper 1991] Existence of Ψ; "cd-index is a basis for GDDS"

[Purtill 1993] Ψ(n-simplex) Ψ(n-cube) $\Longleftrightarrow$ André + signed André perms

[Stanley 1994] $\Psi \geq 0$ for $\mathcal{L}$(polytope), more generally, for S-shellable face poset of a regular CW-complex.

[Ehrenborg-Readdy 1998] Coalgebraic structure of $\mathcal{F}$

$k\langle a,b\rangle$ = polynomial algebra in noncommutative variables a and b

Product = concatenation

Coproduct

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n$$

Extend to all of $k\langle a,b\rangle$ by linearity

ex. $\Delta(abb) = 1 \otimes bb + a \otimes b + ab \otimes 1$

[Joni - Rota 1979]

V vector space with product μ + coproduct Δ

(V, μ, Δ) is a Newtonian coalgebra if it satisfies the Newtonian condition

$$\Delta \circ \mu = (1 \otimes \mu) \circ (\Delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \Delta)$$

In Sweedler notation

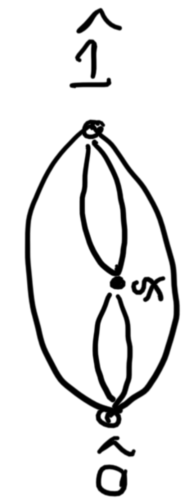
$$\Delta(xy) = \sum_{x} x_{(1)} \otimes (x_{(2)} \cdot y) + \sum_{y} (x \cdot y_{(1)}) \otimes y_{(2)}$$

Theorem: [Ehrenborg-Readdy 1998]

$(\mathbb{Z}\langle a,b\rangle, \cdot, \Delta)$ is a Newtonian coalgebra

[Ehrenborg–Hetyei, unpublished]

$\mathcal{Q}$ = vector space of all types of graded posets with $\hat{0} \neq \hat{1}$ over a field $\Bbbk$

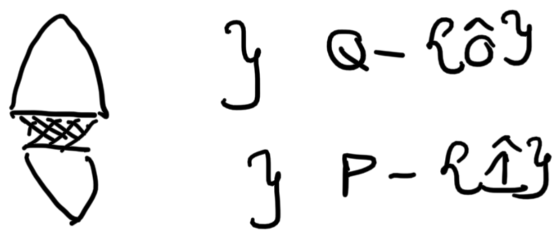


Define a coproduct

$$\Delta(\overline{P}) = \sum_{\substack{x \in P \\ \hat{0} < x < P}} [\hat{0}, x] \otimes [x, \hat{1}]$$

Define star product

$P * Q =$



Theorem: [Ehrenborg–Hetyei]

$(\mathcal{Q}, \Delta, *)$ is a Newtonian coalgebra

Theorem: [Ehrenborg - Readdy 1998]

The ab-index is a Newtonian coalgebra homomorphism from Φ to $k\langle a,b\rangle$ with

$\Psi($  $) = 1$

$$\Psi(P * Q) = \Psi(P)\cdot\Psi(Q)$$

$$\Delta(\Psi(P)) = \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x]) \otimes \Psi([x,\hat{1}])$$

The miracle coproduct

$$\Delta(c) = \Delta(a+b) = \Delta(a) + \Delta(b)$$
$$= 2 \cdot 1 \otimes 1$$

$$\Delta(d) = \Delta(ab+ba) = a \cdot \Delta(b) + \Delta(a) \cdot b$$
$$+ b\,\Delta(a) + \Delta(b) \cdot a$$
$$= a \otimes 1 + 1 \otimes b$$
$$+ b \otimes 1 + 1 \otimes a$$
$$= c \otimes 1 + 1 \otimes c$$

Corollary: [Ehrenborg-Readdy 1998]

The cd-index is a Newtonian coalgebra homomorphism from $\mathcal{E}$ = linear space of graded Eulerian posets to $\mathbb{Z}\langle c,d\rangle$ with

$$\Psi(\mathbf{:}) = 1$$

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q)$$

$$\Delta(\Psi(P)) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}])$$

Implications (a sampling)

Inherent coalgebraic structure of cd-index allows:

1. Geometric operations

   ex [E-R 1998]

   $\Psi(\text{Prism}(P)) = \Psi(P)\cdot c + D(\Psi(P))$

   where $D(c) = 2d$ and $D(d) = cd + dc$

2. New inequalities for flag vectors of polytopes

   [Ehrenborg 2005]

3. Understand combinatorics of oriented matroids, arrangements of subspaces + subtori

   [Billera - Ehrenborg - Readdy 1997]

   [Ehrenborg - Readdy - Slone 2009]

Inequalities

[Stanley 1994] $\Psi \geq 0$ for S-shellable posets

[Billera-Hetyei 1999] The closure of the convex cone generated by flag f-vectors of posets is polyhedral

[Billera-Ehrenborg 2000] $\Psi\left(\begin{array}{c}\text{n-dim'l}\\ \text{polytope}\end{array}\right) \geq \Psi\left(\begin{array}{c}\text{n-dim'l}\\ \text{simplex}\end{array}\right)$

[Karu 2006] $\Psi(\text{Gorenstein* posets}) \geq 0$

[Karu-Ehrenborg 2007] $\Psi(\text{Gorenstein* lattices}) \geq \Psi(B_n)$

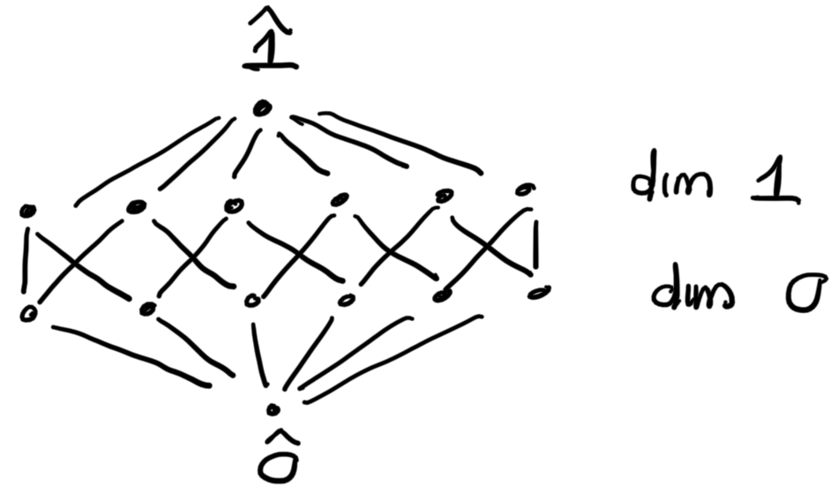
[Ehrenborg 2005] Techniques for lifting known inequalities (Stanley, toric g, Kalai convolution)

New inequalities for polytopes of dim ≥ 6

Whitney stratifications

ex. The n-gon $(n \geq 2)$

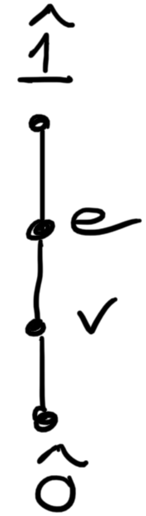
S	f_S	h_S	w_S
$\emptyset$	1	1	aa
0	n	$n-1$	ba
1	n	$n-1$	ab
01	$2n$	1	bb



$$\Psi(n\text{-gon}) = c^2 + (n-2)d$$

ex. 1-gon

S	f_S	h_S
$\emptyset$	1	1
0	1	0
1	1	0
01	1	0



NOT Eulerian

ex. (Again ...)

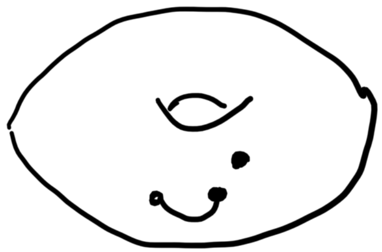
$\mathrm{link}_e(v) = \bullet \quad \bullet$

$\chi(\bullet \quad \bullet) = 2$, the Euler characteristic

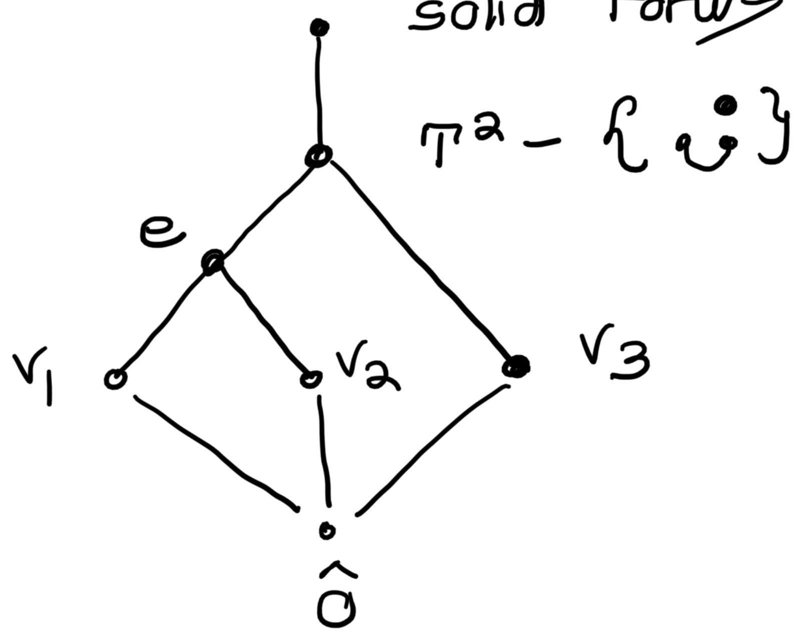
S	$\bar{f}_S$	$\bar{h}_S = \sum_{T \subseteq S} (-1)^{\lvert S-T \rvert} \bar{f}_T$
$\emptyset$	1	1
0	1	0
1	1	0
01	2	1

$$\Psi(\text{1-gon}) = aa + bb$$
$$= (a+b)^2 - (ab+ba)$$
$$= c^2 - d$$

ex.



Face poset



Chain $c = \{\hat{0} = x_0 < x_1 < \dots < x_k = \hat{1}\}$
in the face poset weighted by

$$\bar{\xi}(c) = \chi(x_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_k}(x_{k-1}))$$

ex. (cont'd).

S	$\bar{f}_S$	$\bar{h}_S$	$3dc$	$-2cd$
$\emptyset$	0	0	0	0
0	3	3	3	0
1	1	1	3	-2
2	-2	-2	0	-2
01	2	-2	0	-2
02	2	1	3	-2
12	2	3	3	0
012	4	0	0	0

$\Phi\Big($  $\Big) = 3dc - 2cd$

These are examples of
<u>Whitney</u> <u>stratifications</u>

Subdivide space into strata

$$W = \dot{\bigcup_{X \in P}} X$$

Condition of the frontier:

$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_P Y$ in the face poset P

Whitney conditions A and B:

No fractal behavior

No infinite wiggling ex. $x \cdot \sin\left(\frac{1}{x}\right)$

$\Rightarrow$ The links are well-behaved.

The Fine Print

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_{\mathcal{P}} Y.$$

This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

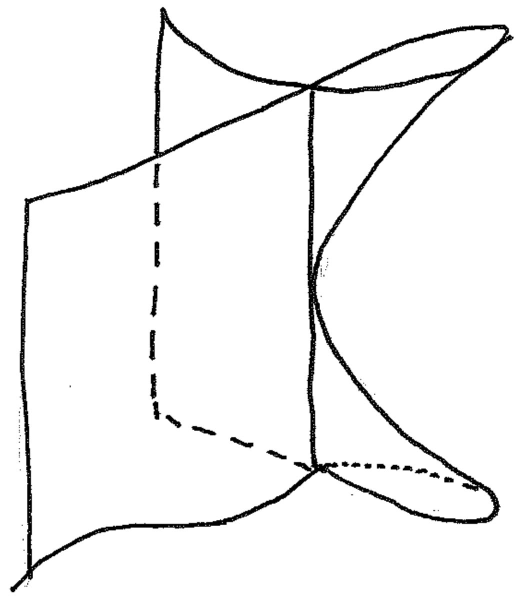
hold.

Examples of Whitney stratifications:

- convex polytopes
- regular cell complexes
- real or complex algebraic sets
- analytic sets
- semi-analytic sets
- quotients of smooth manifolds by compact group actions

Whitney stratifications: Have applications to robotics + computational geometry.

ex. The Whitney cusp.



Whitney stratifications (their face posets) are examples of ...

Quasi-graded posets

def. A quasi-graded poset $(P, \rho, \bar{\zeta})$ consists of

i. P finite poset with $\hat{0}$ + $\hat{1}$ (not necessarily graded)

ii. $\rho : P \to \mathbb{N}$ order-preserving ($x < y \Rightarrow \rho(x) < \rho(y)$)

iii. $\bar{\zeta} \in I(P)$, the incidence algebra of P. The weighted zeta function satisfies $\bar{\zeta}(x,x) = 1 \quad \forall x \in P$.

def. $(P, \rho, \bar{\xi})$ <u>Eulerian</u> if

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \bar{\xi}(x,y) \cdot \bar{\xi}(y,z) = \delta_{x,z}$$

Remark: $\bar{\xi} = \xi$ gives the <u>classical Eulerian condition</u>

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} = \delta_{x,z}$$

Define the ab-index of $(P, \rho, \bar{\bar{\xi}})$

$$\Psi(P, \rho, \bar{\bar{\xi}}) = \sum_{S} \bar{h}_S \cdot w_S$$

with

$$\bar{\bar{\xi}}(c) = \bar{\bar{\xi}}(x_0, x_1) \cdot \bar{\bar{\xi}}(x_1, x_2) \cdots \bar{\bar{\xi}}(x_{k-1}, x_k)$$

for a chain c: $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$

Theorem: [Ehrenborg - Goresky - Readdy 2015].

Let $(P, \rho, \bar{\xi})$ be an Eulerian quasi-graded poset.

Then

1. The flag $\bar{f}$-vector satisfies the gen'd Dehn-Sommerville relations.

2. $\Psi(P, \rho, \bar{\xi}) \in \mathbb{Z}\langle c,d \rangle$.

3. Alexander duality generalizes to Eulerian quasi-graded posets

Theorem: [Ehrenborg - Goresky - Readdy 2015]

M manifold with a Whitney stratified boundary.

Then the face poset is quasi-graded and Eulerian, with

$$\rho(x) = \dim(x) + 1.$$

$$\bar{\zeta}(x,y) = \chi(\mathrm{link}_y(x))$$

Current work + remarks (a sampling...)

① Find a combinatorial interpretation for the cd-coefficients

①′ Note the cd-coefficients of whitney-stratified manifolds can be negative

② Find linear inequalities that hold among entries of cd-index of a whitney-stratified manifold

③ Find new minimization inequalities

④ [E-R-G] Kalai's convolution for polytopes continues to hold for manifolds

⑤ Extend Ehrenborg's lifting technique to stratified manifolds

Thank you!