Polytopes and Whitney-stratified spaces

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Kutztown University
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Platonic Solids

**Tetrahedron**
- 'Four sided'
- 4 faces
- 4 points
- 6 edges

**Octahedron**
- 'Eight sided'
- 8 faces
- 6 points
- 12 edges

**Hexahedron**
- 'Six sided'
- 6 faces
- 8 points
- 12 edges

**Icosahedron**
- 'Twenty sided'
- 20 faces
- 12 points
- 30 edges

**Dodecahedron**
- 'Twelve sided'
- 12 faces
- 20 points
- 30 edges

Reference: Natural Philosophy & Esoteric Geometry: The Cosmic Creations Collection joedubs.com
Athens, Greece

Thales (c. 417–369 BC)

There are 5 regular 3-dim'1 polytopes

Reference
Euclid's Elements
Book XIII
Prop. 13-17
(Digitized by Clay Math Institute)
Skip ahead \( \sim 2500 \text{ years} \)
def. A polytope is the convex hull of a finite # of vertices in \( \mathbb{R}^n \)

or

... is the bounded intersection of a finite # of closed half-spaces in \( \mathbb{R}^n \).
f-vector

$P$ n-dim polytope

The $f$-vector $(f_0, f_1, \ldots, f_{n-1})$ where

\[ f_i = \# \ i\text{-dim} \text{ faces} \text{ of } P \]

ex.

\[
\begin{array}{c}
\text{prism} \left( \begin{array}{c}
\text{hexagon}
\end{array} \right)
\end{array}
\]

$(f_0, f_1, f_2) = (12, 18, 8)$
Euler’s letter n° 149 to Goldbach, November 14th, 1750: reproduction of the third page (RGADA, f. 181, n. 1413, č. IV, fol. 270r)

In the paragraph numbered 6, Euler’s Polyhedron Formula appears for the first time; paragraph 11 gives the discrete form of the Gauss-Bonnet theorem (cf. n° 149, notes 5–8).
November 14 1750
Letter No. 149
Euler to Goldbach

\[ H + S = A + 2 \]

\( H \) = hedrae (facets)
\( S \) = anguli solidi (= solid angles = vertices),
\( A \) = arcus (= sharpness = edges).
Euler's relation  [1750 letter to Goldbach]

\[ f_0 - f_1 + f_2 = 2 \]

Proof by Descartes  [1596-1650]
using plane angles
Schläfli [1850s]
4-dim'1 polytopes

convex regular 4-polytope

5-cell (4-dim'1 simplex)
8-cell (4-dim'1 cube)
16-cell (hexadeca(chor)on)
24-cell
120-cell
600-cell

Schläfli symbol

\( r_{5,3,3,5} \)
Euler–Poincaré–Schläfli
\[ f_0 - f_1 + f_2 - \ldots + (-1)^{n-1} f_{n-1} = 1 - (-1)^n \]

The proofs

[\textit{Poincaré}, 1893] Introduced homology groups and algebraic topology

[\textit{Schläfli}, 1852] Assumed shellability of polytopes

[\textit{Bruggesser–Mani}, 1971] Polytopes are shellable
Question: Characterize $f$-vectors of $n$-dim polytopes.
\[ n = 2 \]

\[ (f_0, f_1) = (3, 3) \]

\[ (f_0, f_1) = (4, 4) \]

\[ f_0 = f_1 \quad (f_0 \geq 3) \]
[Steinitz 1906]

The $f$-vectors of 3-dim polytopes live in the integer cone.

\[ 2(f_0 - 4) \geq f_2 - 4 \]
\[ f_0 - 4 \leq 2(f_2 - 4) \]
Open \&: Characterize f-vectors of n-dim\'l polytopes for \( n \geq 4 \)

Known: Simplicial polytopes, and by duality, simple polytopes
\[ \Delta \] a simplicial complex of dim \( n \).

The \( h \)-vector

\[ \sum_{\varepsilon = 0}^{n+1} h_{\varepsilon} t^{n+1-\varepsilon} = \sum_{\varepsilon = -1}^{n} f_{\varepsilon} (t-1)^{n-\varepsilon} \]
\textbf{ex.}

\[ \vec{f} = (f_{-1}, f_0, f_1, f_2) = (1, 6, 12, 8) \]
\[ \vec{h} = (h_0, h_1, h_2, h_3) = (1, 3, 3, 1) \]

The \textit{h-vector} is:

1. \textit{[Kind-Kleinschmidt 1979]}
   Related to the shelling of the complex

2. \textit{[Stanley 1978]}
   Numerator of the Hilbert series of the Stanley-Reisner ring of $\Delta$
The \textit{g-theorem}

**Theorem:** [Stanley 1980; Billera–Lee 1981]

The vector \((h_0, \ldots, h_n)\) is the \(h\)-vector of an \(n\)-dimensional simplicial polytope if and only if

1. (Dehn–Sommerville eqns)
   \[ h_\sigma = h_{n-\sigma}, \quad \text{for } \sigma = 0, \ldots, \lfloor n/2 \rfloor \]

2. \((g_0, g_1, \ldots, g_{\lfloor n/2 \rfloor})\) is an M-sequence
   where \(g_0 = h_0 + g_\sigma = h_\sigma - h_{\sigma-1} \)
Upper + Lower Bound Theorems

Theorem: ① [Barnette 1973]
For P n-dim simplicial polytope w/ $f_0 = \omega$
\[ f_j (\text{Stack}(n, \omega)) \leq f_j (P) \text{ for } 0 \leq j < n + n \geq 4 \]

② [McMullen 1970]
For P n-dim polytope w/ $f_0 = \omega$
\[ f_j (P) \leq f_j (C(n, \omega)) \text{ for } 0 \leq j \leq n \]

③ [Stanley 1975]
Δ n-dim simplicial sphere w/ $f_0 = \omega$
\[ f_j (\Delta) \leq f_j (C(n, \omega)) \]
Recent work:

[ Murai - Novik 2015] Connected normal pseudomanifolds

[ Adiprasito - Sanyal 2016] Relative Stanley-Reisner theory + Minkowski sums

[ Zheng 2017, 2020] UBT + flag homology 5-manifolds

flag vectors

**P n-dim polytope**

For \( S = s_1 < \cdots < s_k \subseteq \{0, 1, \ldots, n-1 \} \) let

\[
f_S = \# \ F_1 \notin \cdots \notin F_k
\]

where \( \dim(F_i) = s_i \).

The **flag f-vector**

\[
(f_S)_{s \subseteq \{0, 1, \ldots, n-1 \}}
\]
The flag h-vector

\[ h_0 = \sum_{T \subseteq S} (-1)^{|T|} f_T \]

\[ h_\varnothing = h_0 \]

[Stanley 1979]

\[ h_\varnothing = h_\varnothing \]

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<table>
<thead>
<tr>
<th>S</th>
<th>fs</th>
<th>hs</th>
<th>ws</th>
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<tr>
<td>ø</td>
<td>1</td>
<td>1</td>
<td>aaaa</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>11</td>
<td>baaa</td>
</tr>
<tr>
<td>1</td>
<td>18</td>
<td>17</td>
<td>abaa</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>7</td>
<td>arab</td>
</tr>
<tr>
<td>01</td>
<td>36</td>
<td>7</td>
<td>bbaa</td>
</tr>
<tr>
<td>02</td>
<td>36</td>
<td>17</td>
<td>baba</td>
</tr>
<tr>
<td>12</td>
<td>36</td>
<td>11</td>
<td>abbb</td>
</tr>
<tr>
<td>012</td>
<td>72</td>
<td>1</td>
<td>bbbb</td>
</tr>
</tbody>
</table>

The \(ab\)-index

\[ \sum_{S \subseteq G} h_S \cdot ws \]
\[ \Phi (\varpi) = 1 a a a a + 11 b a a + 17 a b a + 7 a b b \\
+ 7 b b a + 17 b a b + 11 a b b + 1 b b b \]

\[ = (a+b)^3 + 6 a b a + 16 a b a + 6 a b a \\
+ 6 b b a + 16 b a b + 10 a b b \]

\[ = (a+b)^3 + 6 (a+b)(a b+b a)+ 10 (a b+b a) (a+b) \]

Let \( c = a+b \)

\[ d = a b+b a \]

Then

\[ \Phi (\varpi) = c^3 + 6 c d + 10 d c \]

The \textit{cd-index}
Theorem: \[ \text{[Bayer-Klapper 1991]} \]

- If \( P \) is polytopal then \( \varpi(P) \in \mathbb{Z} \times c, d \)
- If \( P \) is Eulerian then \( \varpi(P) \in \mathbb{Z} \times c, d \)

Eulerian poset: \( \mu((x,y)) = (-1)^{p(x,y)} \) for every interval \([x,y]\) in graded poset \( P \)

Equivalently, in every nontrivial interval \([x,y]\):

\[ \# \text{ edges of even rank} = \# \text{ edges of odd rank} \]
Some cd-history

[Bayer-Billera 1985] Ge
d' Dehn-Sommerville relations

[Bayer-Klapper 1991] Existence of \( \mathcal{F} \);
"cd-index is a basis for G-DDS"

[Purtill 1993] \( \exists (n\text{-simplex}) \iff \exists (n\text{-cube}) \)

[Stanley 1994] \( \exists \geq 0 \) for \( \alpha \) (polytope),
more generally,
for \( S \)-shellable face poset
of a regular CW-complex,
\[ \mathcal{E}\langle a, b \rangle = \text{polynomial algebra in noncommutative variables } a \text{ and } b \]

Product = concatenation

Coproduct
\[ \Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n \]

Extend to all of \( \mathcal{E}\langle a, b \rangle \) by linearity

\[ \text{ex. } \Delta(abbb) = 1 \otimes bbb + a \otimes b + ab \otimes 1 \]
A vector space with product \( \mu \) and coproduct \( \Delta \) \((V, \mu, \Delta)\) is a Newtonian coalgebra if it satisfies the Newtonian condition

\[
\Delta \circ \mu = (1 \otimes \mu) \circ (\Delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \Delta)
\]

In Sweedler notation

\[
\Delta(xy) = \sum x_i \otimes (\mu(\cdot y)) + \sum y \otimes (x \cdot y_{(1)}) \otimes y_{(2)}
\]
Theorem: [Ehrenberg-Readdy 1998]

\((\omega \langle a, b \rangle \cdot, \Delta)\) is a Newtonian coalgebra
[Ehrenborg–Hetyei, unpublished]

\( \mathcal{P} = \text{vector space of all types of graded posets with } \mathcal{O} \neq \mathcal{I} \text{ over a field } \mathbb{K} \)

Define a coproduct

\[ \Delta(\bar{P}) = \sum_{\underline{x} \in \bar{P}} [\hat{\mathcal{O}}, \underline{x}] \otimes [\underline{x}, \hat{\mathcal{I}}] \]

Define star product

\[ P \ast Q = \sum_{Q - r \hat{\mathcal{O}} \subseteq P} \sum_{P - r \hat{\mathcal{I}} \subseteq Q} \]

Theorem [Ehrenborg–Hetyei]

\((\mathcal{P}, \Delta, \ast)\) is a Newtonian coalgebra
Theorem: [Ehrenborg - Readdy 1998]

The ab-index is a Newtonian coalgebra homomorphism from $\mathfrak{p}$ to $\mathfrak{A}(a,b)$ with

$\mathbb{A}(\mathfrak{1}) = 1$

$\mathbb{A}(\mathfrak{P} \ast \mathfrak{Q}) = \mathbb{A}(\mathfrak{P}) \cdot \mathbb{A}(\mathfrak{Q})$

$\Delta(\mathbb{A}(\mathfrak{P})) = \sum_{\forall x \in \mathfrak{P}} \mathbb{A}([\hat{0}, x]) \otimes \mathbb{A}([x, \hat{1}])$

$0 < x < 1$
The miracle coproduct

\[ \Delta(c) = \Delta(a + b) = \Delta(a) + \Delta(b) \]
\[ = a \otimes 1 \]

\[ \Delta(d) = \Delta(ab + ba) = a \cdot \Delta(b) + \Delta(a) \cdot b \]
\[ + b \cdot \Delta(a) + \Delta(b) \cdot a \]
\[ = a \otimes b + 1 \otimes b \]
\[ + b \otimes 1 + 1 \otimes a \]
\[ = c \otimes 1 + 1 \otimes c \]
Corollary: [Ehrenberg-Readdy 1998]

The cd-index is a Newtonian coalgebra homomorphism from 
\( \mathcal{E} \) = linear space of graded Eulerian posets to 
\( \mathbb{W} < c, d > \) with

\[
\begin{align*}
\Pi(1) &= 1 \\
\Pi(P \star Q) &= \Pi(P) \cdot \Pi(Q) \\
\Delta(\Pi(P)) &= \sum_{\omega \in P} \Pi([0, \omega]) \otimes \Pi([\omega, 1]) \\
&\quad \hat{0} < \omega < \hat{1}
\end{align*}
\]
Implications (as sampling)

Inherent coalgebraic structure of cd-index allows:

1. Geometric operations

   \[ \text{Ex [E-R 1998]} \]
   \[ \text{Prism}(P) = \text{Prism}(P) \cdot c + D(\text{Prism}(P)) \]
   \[ \text{where } D(c) = 2d \text{ and } D(d) = cd + dc \]

2. New inequalities for flag vectors of polytopes

   \[ \text{[Ehrenborg 2005]} \]

3. Understand combinatorics of oriented matroids, arrangements of subspaces, and subtori

   \[ \text{[Billera- Ehrenborg- Readdy 1997]} \]
   \[ \text{[Ehrenborg- Readdy- Slone 2009]} \]
Inequalities

[Stanley 1994]  \( \Delta \geq 0 \) for \( S \)-shellable posets

[Billera-Heil 1999] The closure of the convex cone generated by flag f-vectors of posets is polyhedral

[Billera-Ehrenberg 2000]  \( \Delta (n \text{-dim}^3) \geq \Delta (n \text{-dim}^2) \)

[Karu 2006]  \( \Delta (\text{Gorenstein}^* \text{ posets}) \geq 0 \)

[Karu-Ehrenberg 2007]  \( \Delta (\text{Gorenstein}^* \text{ lattices}) \geq \Delta (B_n) \)

[Ehrenberg 2005] Techniques for lifting known inequalities (Stanley, toric, \text{k}alai convolution)  
New inequalities for polytopes of dim \( \geq 6 \)
### Whitney Stratifications

**Ex. The n-gon** (n \geq 2)

<table>
<thead>
<tr>
<th>\mathcal{S}</th>
<th>\mathcal{f}_S</th>
<th>\mathcal{h}_S</th>
<th>\mathcal{w}_S</th>
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</thead>
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<td>\emptyset</td>
<td>1</td>
<td>1</td>
<td>\omega</td>
</tr>
<tr>
<td>0</td>
<td>n</td>
<td>n-1</td>
<td>b(a)</td>
</tr>
<tr>
<td>1</td>
<td>n</td>
<td>n-1</td>
<td>a(b)</td>
</tr>
<tr>
<td>01</td>
<td>2n</td>
<td>1</td>
<td>b(b)</td>
</tr>
</tbody>
</table>

\[ \Xi(n\text{-gon}) = c^2 + (n-2)d \]
ex. 1-gon

\[ e \]

\[ \emptyset \]

\[ \emptyset \]

\[ 0 \]

\[ 1 \]

\[ 01 \]

\[ s \]

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<thead>
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<tr>
<td>\emptyset</td>
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<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>01</td>
<td>1</td>
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</tr>
</tbody>
</table>

\[ \hat{1} \]

\[ e \]

\[ \hat{0} \]

\[ \text{Not} \]

\[ \text{Eulerian} \]
ex. (Again ...) e

\[ \text{link}_e(v) = \bullet \bullet \]

\[ \chi(\bullet \bullet) = a, \text{ the Euler characteristic} \]

\[ \begin{array}{c|ccc}
 s & \bar{f}_s & \bar{h}_s = \sum_{T \subseteq S} (-1)^{|S-T|} \bar{f}_T \\
 \hline
 \emptyset & 1 & 1 \\
 0 & 1 & 0 \\
 1 & 1 & 0 \\
 01 & 2 & 1 \\
\end{array} \]

\[ \bar{\epsilon}(1\text{-gon}) = a a + b b \]

\[ = (a + b)^2 - (ab + ba) \]

\[ = c^2 - d \]
Face poset

\[ \mathcal{P} - \{\emptyset\} \]

\[ v_1 - v_2 - v_3 \]

solid torus
Chain $c = \hat{c} \Rightarrow x_0 < x_1 < \ldots < x_n = \hat{y}$
in the face poset weighted by

$$\overline{\zeta}(c) = \chi(x_0) \cdot \chi(\text{link}_{x_0}(x_1)) \ldots \chi(\text{link}_{x_n}(x_{n-1}))$$
\( \exists \) (cont'd).

\begin{tabular}{|c|cccc|}
\hline
\( s \) & \( f_s \) & \( h_s \) & \( 3dC \) & \(-2cd\) \\
\hline
\( \emptyset \) & 0 & 0 & 0 & 0 \\
0 & 3 & 3 & 3 & 0 \\
1 & 1 & 1 & 3 & -2 \\
2 & -2 & -2 & 0 & -2 \\
01 & 2 & -2 & 0 & -2 \\
02 & 2 & 1 & 3 & -2 \\
12 & 2 & 3 & 3 & 0 \\
012 & 4 & 0 & 0 & 0 \\
\hline
\end{tabular}

\( \exists (\bigcirc) = 3dC - 2cd \)
These are examples of Whitney stratifications.

Subdivide space into strata:
\[ W = \bigcup_{x \in P} X \]

Condition of the frontier:
\[ X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_Y Y \text{ in the face poset } P \]

Whitney conditions A and B:
- No fractal behavior: \( wx, vx: \sin \left( \frac{1}{x} \right) \)
- No infinite wiggling: \( wx, vx \)

\implies The links are well-behaved.
The Fine Print

**Definition** Let $W$ be a closed subset of a smooth manifold $M$, and suppose $W$ can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of $W$ satisfying the condition of the frontier:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq p Y.$$

This implies the closure of each stratum is a union of strata. We say $W$ is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of $M$ (not necessarily connected).
2. If $X \leq_p Y$ then Whitney’s conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to $x$. Also assume that (with respect to some local coordinate system on the manifold $M$) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line $\ell$ and the tangent planes $T_{y_i} Y$ converge to some limiting plane $\tau$. Then the inclusions

   (A) $T_x X \subseteq \tau$ and (B) $\ell \subseteq \tau$

hold.
Examples of Whitney stratifications:

- convex polytopes
- regular cell complexes
- real or complex algebraic sets
- analytic sets
- semi-analytic sets
- quotients of smooth manifolds by compact group actions

Whitney stratifications: Have applications to robotics and computational geometry.
The Whitney curve.

Whitney stratifications (their face posets) are examples of ...
Quasi-graded posets

def. A quasi-graded poset \((P, \mathcal{J}, \Xi)\) consists of

i. \(P\) finite poset with \(\hat{0} + \hat{1}\)
   (not necessarily graded)

ii. \(\mathcal{J}: P \to \mathbb{N}\) order-preserving
   \((x < y \implies \mathcal{J}(x) < \mathcal{J}(y))\)

iii. \(\Xi \in I(P)\), the incidence algebra of \(P\)
    The weighted zeta function satisfies
    \(\Xi(x, x) = 1\) \(\forall x \in P\).
\( \text{def. } (P, \rho, \tau) \) Eulerian if

\[
\sum_{\alpha \leq \gamma \leq \zeta} (-1)^{\rho(\alpha, \gamma)} \tau(\alpha, \gamma) \cdot \tau(\gamma, \zeta) = S_{\alpha, \zeta}
\]

Remark: \( \tau = \tau \) gives the classical Eulerian condition

\[
\sum_{\alpha \leq \gamma \leq \zeta} (-1)^{\rho(\alpha, \gamma)} = S_{\alpha, \zeta}
\]
Define the ab-index of $(P, \mu, \overline{\xi})$

$$\overline{a} (P, \mu, \overline{\xi}) = \sum_{g} \overline{h}_{g} \cdot \mu_{g}$$

with

$$\overline{\xi} (c) = \overline{\xi} (\underline{x}_0, \underline{x}_1) \cdot \overline{\xi} (\underline{x}_1, \underline{x}_2) \cdots \overline{\xi} (\underline{x}_{\kappa-1}, \underline{x}_{\kappa})$$

for a chain $c$: $\underline{0} = \underline{x}_0 < \underline{x}_1 < \cdots < \underline{x}_{\kappa} = \overline{1}$
Theorem: [Ehrenborg-Gore-Keedoo 2015]

Let \((P, \mu, \tau)\) be an Ewelenian quasi-graded poset.

Then

1. The flag \(\overline{f}\)-vectors satisfy the genus Dehn-Sommerville relations.

2. \(\Delta (P, \mu, \tau) \in \mathbb{Z} \langle c, d \rangle\).

3. Alexander duality generalizes to Ewelenian quasi-graded posets.
Theorem: [Ehrenborg, Goresky, Readdy 2015]

Let $M$ be a manifold with a Whitney stratification boundary.

Then the face poset is quasi-graded and Eulerian, with

$$\nu(x) = \dim(x) + 1.$$  

$$\zeta(x,y) = \chi(\text{link}_y(x))$$

where $\nu(x)$ and $\zeta(x,y)$ are functions that describe the structure of the poset.
Current work remarks (or sampling...)

1. Find a combinatorial interpretation for the cd-coefficients

1. Note the cd-coefficients of Whitney-stratified manifolds can be negative

2. Find linear inequalities that hold among entries of cd-index of a Whitney-stratified manifold

3. Find new minimization inequalities

4. [E-R-G] Kalai’s convolution for polytopes continues to hold for manifolds

5. Extend Ehrenborg’s lifting technique to stratified manifolds
Thank you!