q-Stirling identities revisited

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NSA

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$S(n,k)$, Stirling numbers of the second kind

[Stirling 1730].
Expand z^n in terms of
falling factorial basis.

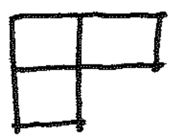
[Kaplansky - Riordan 1946]
$S(n,k) =$ # ways to place
$(n-k)$ non-attacking rooks
on a triangular board of
side length $n-1$

[Garsia - Remmel 1986]
K-R $\Longleftrightarrow$ set partitions of
$\{1, \ldots, n\}$ into k blocks

$$z^n = \sum_{k=0}^{n} S(n,k)\,(z)_k .$$

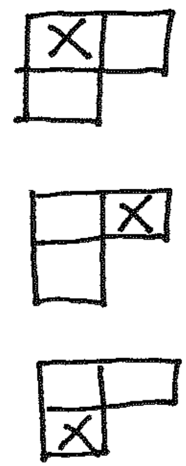
$$z^3 = 1 \cdot z(z-1)(z-2) + 3 \cdot z(z-1) + 1 \cdot z ,$$

$S(3,3)$



1/2/3

$S(3,2)$



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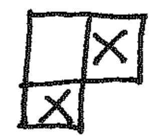

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Recurrence

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k),$$

$$1 \le k \le n.$$

Restricted growth words [Milne 1977].

Write π in standard form

$$\pi = B_1 / B_2 / \cdots / B_k$$

where $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$.

RG-word (restricted growth word)

$$w = w_1 \cdots w_n$$

where $w_i = j$ if $i \in B_j$.

ex. $\pi = 1347/258/69$.

$w = 121123123$

$R(n,k) \triangleq$ RG-words of length n with max'l entry k

$S_q[n,k]$, q-Stirling number of the second kind

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)}{}' \; wt(w)$$

where

$$wt(w) = q^{\sum_{i=1}^{n} (w_i - 1) - \binom{k}{2}}.$$

ex.

π	w	$wt(w)$
1/234	1222	q^2
134/12	1211	1
124/3	1121	1
123/4	1112	1
12/34	1122	q^1
13/24	1212	q^1
14/23	1221	q^1

$$\sum{}' = S_q[4,2] = q^2 + 3q + 3.$$

Easily

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q \, S_q[n-1,k], \quad 1 \le k < n$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$.

Other statistics which generate $S_q[n,k]$:

Milne [1982]

Wachs + White [1991].

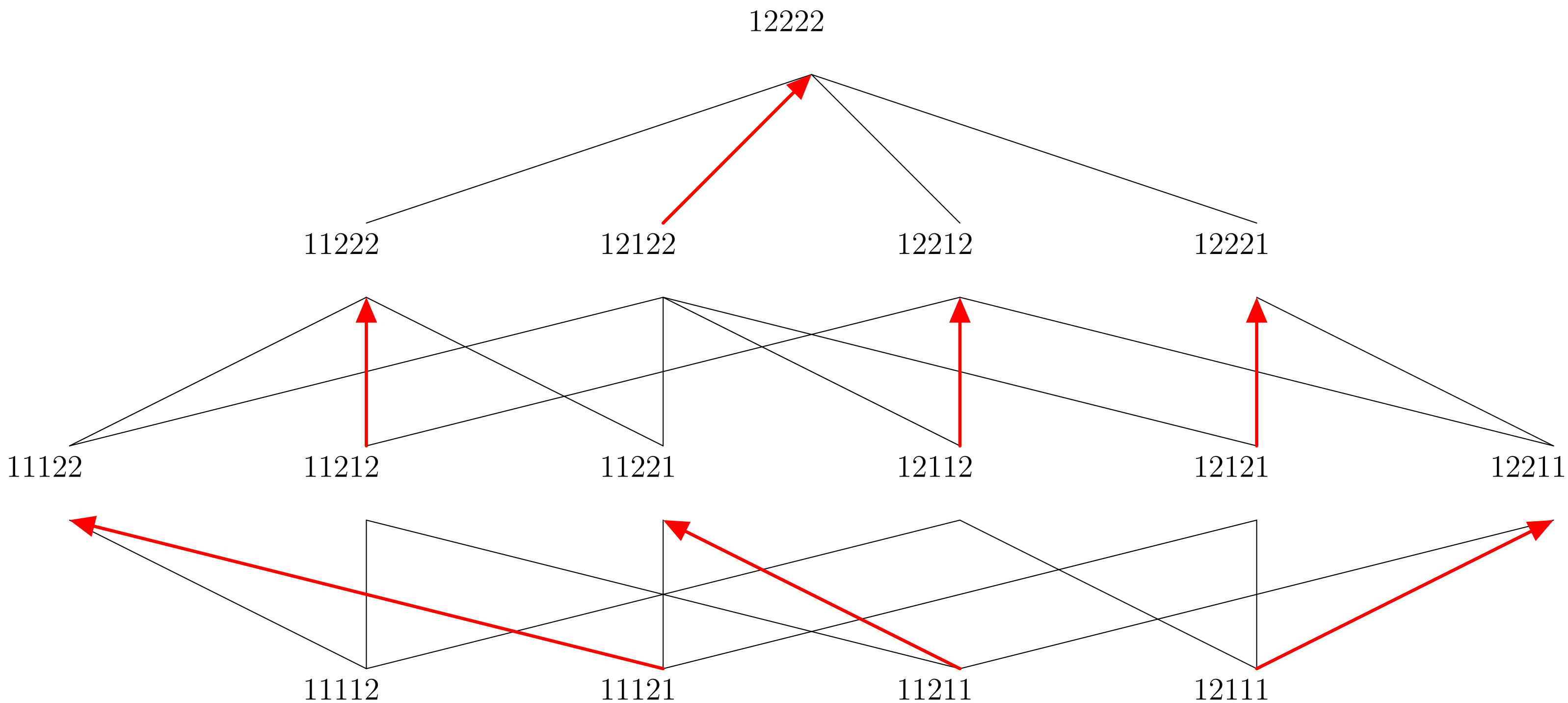
$\Pi(n,k)$, Stirling poset of the second kind

Elements are $\mathcal{B}(n,k)$

$v < w$ if $v = v_1 v_2 \cdots v_n$ and $w = v_1 v_2 \cdots (v_i + 1) \cdots v_n$.
for some index i.

$\Pi(n,k)$ is graded by degree of wt.

rank $= (n-k)(k-1)$.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[Cai - Readdy 2017].

1. Give a q-$(1+q)$-analogue of $S_q[n,k]$:

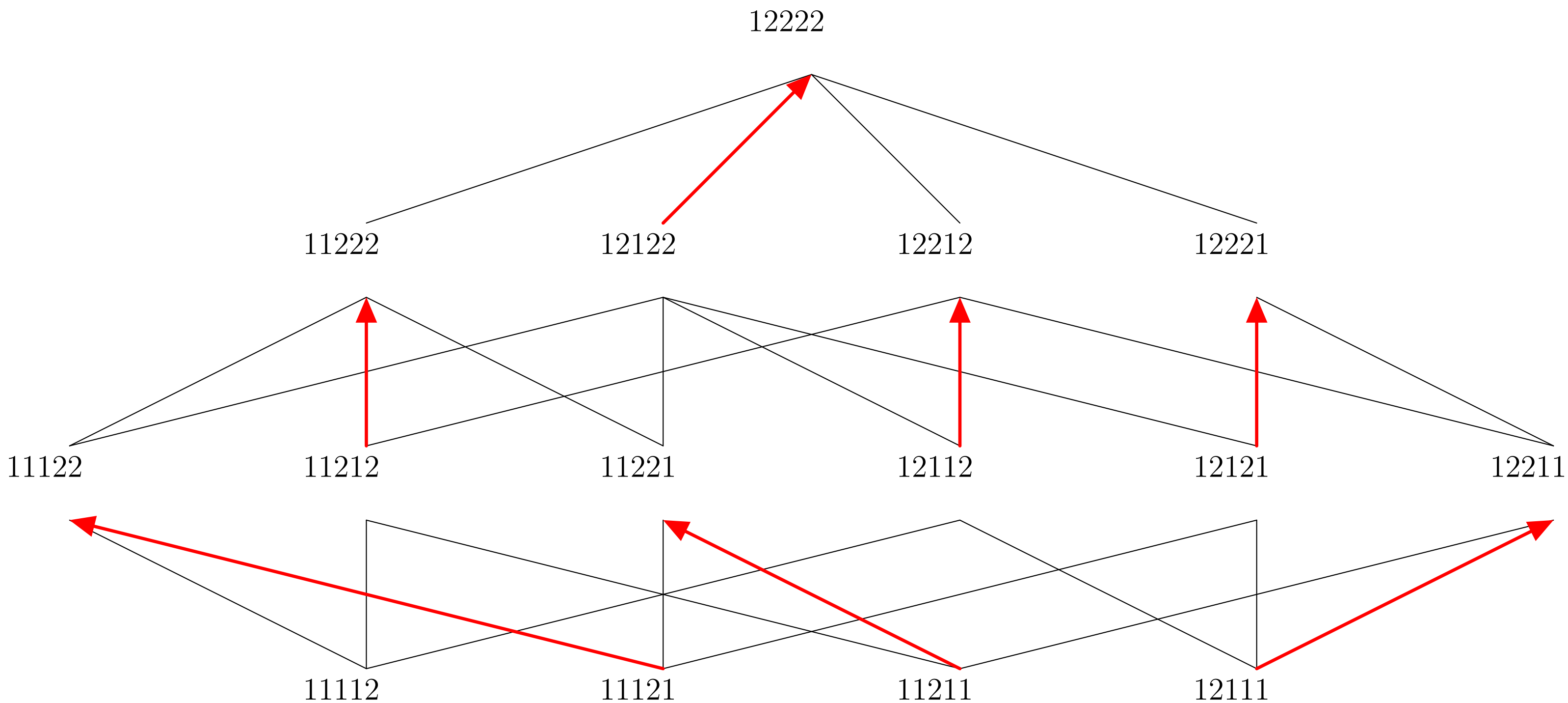
$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} (1+q)^{B(w)}$$

2. Poset decomposition into Boolean algebras.

$$\Pi(n,k) = \dot{\bigcup_{w \in \mathcal{A}(n,k)}} [w, \alpha(w)]$$

3. For the algebraic complex (C, ∂) supported by the poset $\Pi(n,k)$, a basis for the integer homology is given by the weakly increasing allowable RG-words. Furthermore,

$$\sum_{i \geq 0} \dim(H_i(C, \partial; \mathbb{Z})) \cdot q^i = \begin{bmatrix} n-1-\lfloor k/2 \rfloor \\ \lfloor k-1/2 \rfloor \end{bmatrix}_{q^2}.$$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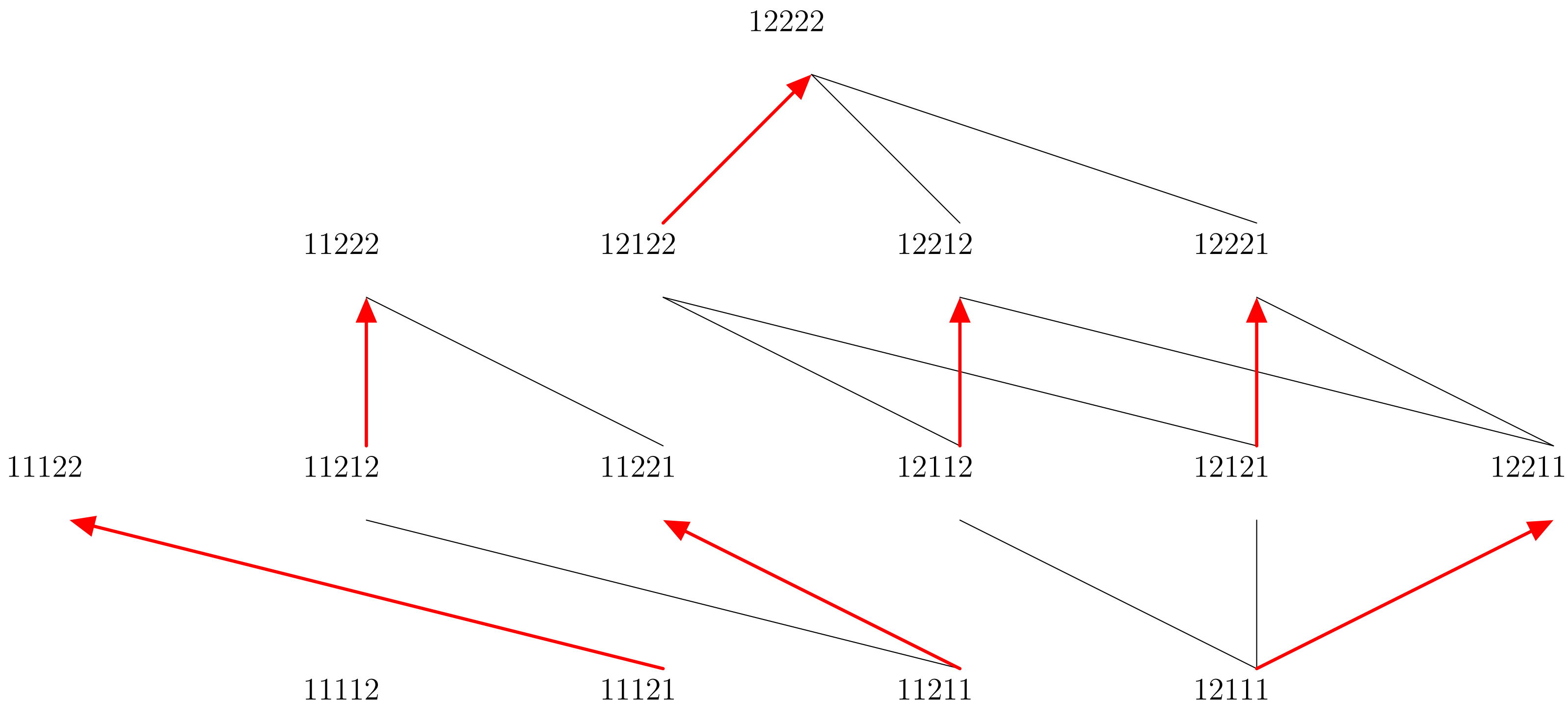
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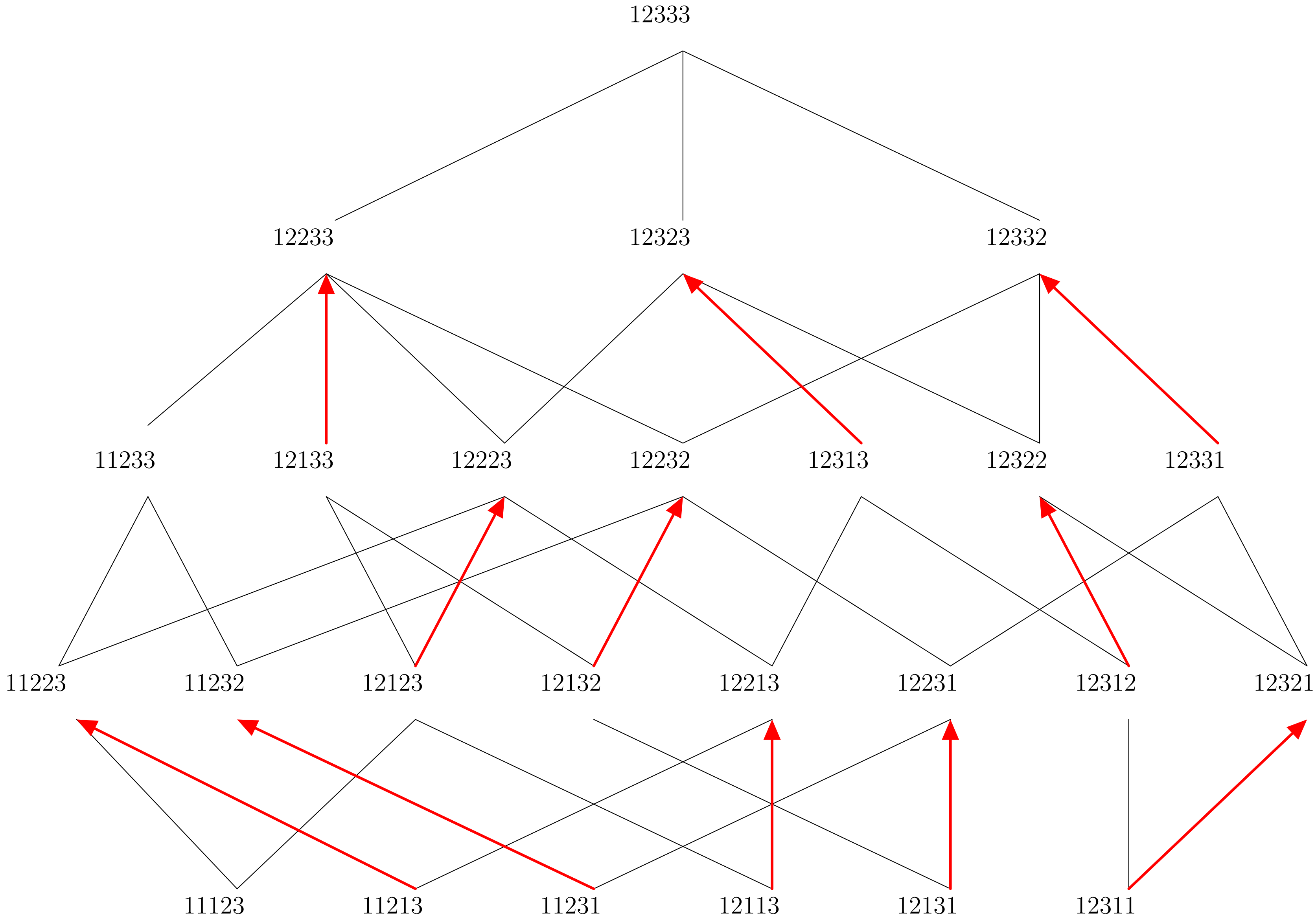
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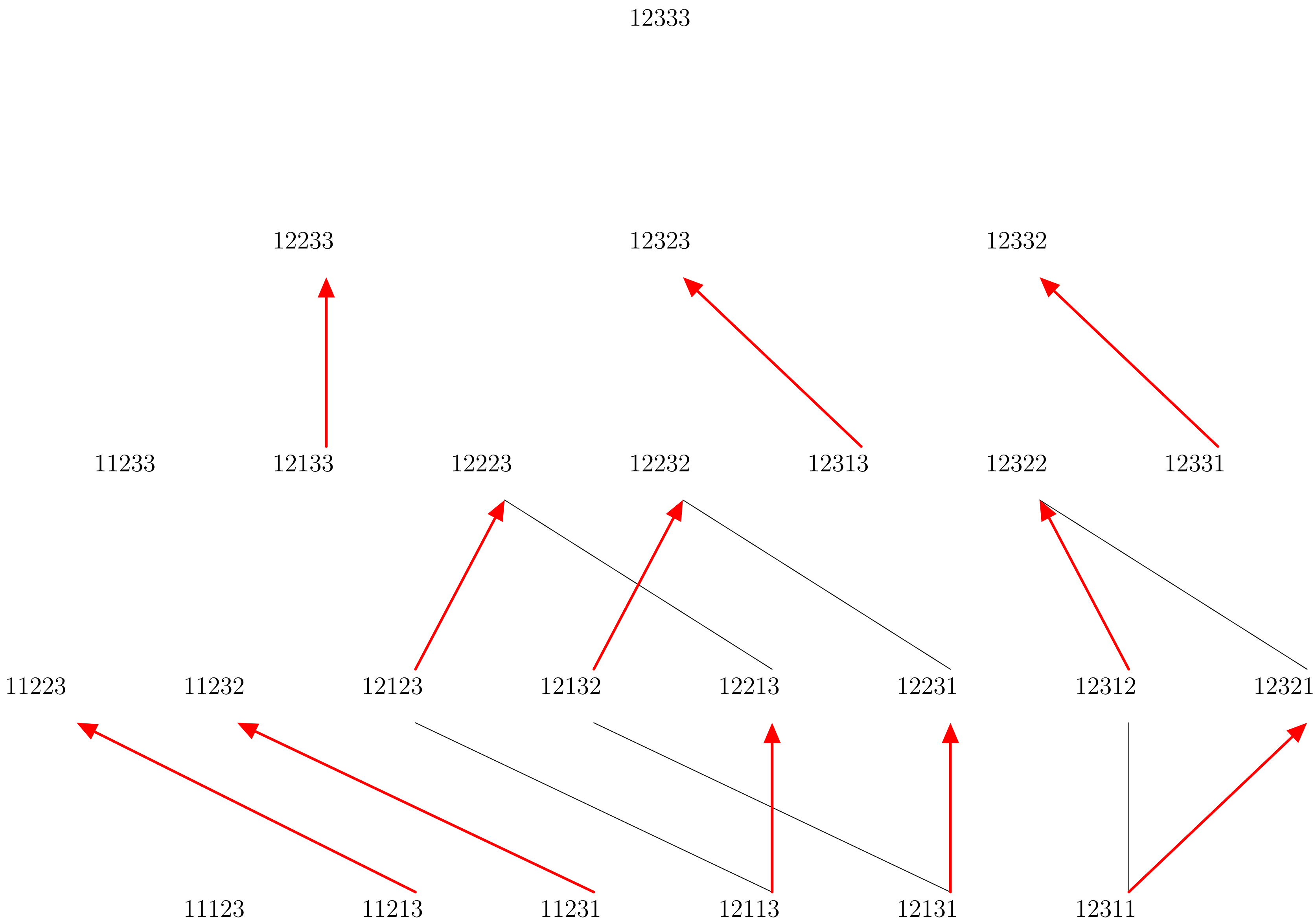
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Return to RG-words.

Goal: use RG-words to give combinatorial proofs of q-Stirling identities.

Work with Y. Cai & R. Ehrenborg.

Recurrence structured identities

[Mercer 1990, de Médicis - Leroux 1993]

$$S_q[n+1, k+1] = \sum_{m=k}^{n} \binom{n}{m} q^{m-k} S_q[m, k].$$

[Cai - Ehrenborg - R 2017]

$$q^{n-m} S_q[n, m] = \sum_{k=m}^{n} (-1)^{n-k} \binom{n}{k} S_q[k+1, m+1]$$

[de Médicis - Leroux 1993].

$$S_q[n+1, k+1] = \sum_{j=k}^{n} [k+1]_q^{n-j} \cdot S_q[j, k].$$

$$(n-k) S_q[n, k] = \sum_{j=1}^{n-k} S_q[n-j, k] \left([1]_q^j + [2]_q^j + \dots + [k]_q^j \right).$$

Gould's generating function

[Gould 1961].

$$\sum_{n \geq k} S_q[n,k]\, t^n = \frac{t^k}{\prod_{i=1}^{k} (1 - [i]_q t)}.$$

Gould: Analytic proof.

Ernst: Orthogonality of q-Stirling numbers of first + second kind

Wachs + White: p,q-version (without proof).

Cai - Ehrenborg - R: RG-word proof of Gould's result.

Poset proof of Carlitz's identity

$\mathbb{P}^n$ = set of words of length n with entries in $\mathbb{P}$.

$\mathbb{P}^n$ is a lattice via $v_1 \cdots v_n \le w_1 \cdots w_n \iff v_i \le w_i \;\; \forall i$.

Factor $v \in Q(n,k)$:

$$v = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdot 3 \cdot \cdots \cdot u_{k-1}\, k \cdot u_k .$$

where $u_i \in [1, i]^*$.

For $m \ge n$ define

$$\omega_m(v) = m \cdot u_1 \cdot m \cdot u_2 \cdot m \cdot \cdots \cdot u_{k-1} \cdot m \cdot u_k .$$

Claim: $[v, \omega_m(v)]$ in $\mathbb{P}^n$

$$\cong ?$$

$$C_m \times C_{m-1} \times \cdots \times C_{m-k+1}$$

Theorem: The n-fold Cartesian product of the m-chain has decomposition

$$[1,m]^n = \dot{\bigcup_{0 \le k \le \min(m,n)}} \dot{\bigcup_{v \in \mathcal{Q}(n,k)}} [v, \omega_m(v))$$

<u>Proof</u>

Define $f : \mathbb{P}^n \to \mathbb{P}^n$ via

$f(w) = w$ if w is an RG-word.

Otherwise if i = smallest index s.t. w fails to be an RG-word, i.e., $\max(0, w_1, \ldots, w_{i-1}) + 1 < w_i$, let

$f(w)$ = replace ith entry by $\max(0, w_1, \ldots, w_{i-1}) + 1$.

Note: In poset $f(w) \le w$.

Claim: ① $f^{n+1}(w) = f^n(w)$

② . $f^n(w)$ is an RG-word.

Define $\varphi : \mathbb{P}^n \to \dot{\bigcup}_{0 \le k \le n} \mathcal{R}(n,k)$

$$\varphi(w_1 \cdots w_n) = f^n(w)$$

Claim: ① φ is a surjection

② . $\varphi(w) \le w$ in $\mathbb{P}^n$.

Let $v \in \mathcal{R}(n,k)$ with $v = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots u_{k-1}\, k\, u_k$,

$u_i \in [1,i]^*$

The fiber

$$\varphi^{-1}(v) = \{ j_1 \cdot u_1 \cdot j_2 \cdot u_2 \cdot \cdots \cdot u_{k-1}\, j_k \cdot u_k : i \le j_i \text{ for } i = 1, \dots, k \}$$

$$\cong \mathbb{P}^k.$$

Restrict

$$\varphi^{-1}(v) \cap [1, m]^n = [v, \omega_m(v)].$$

Then take disjoint union over all RG-words v. $\boxed{/\!/}$

[Carlitz 1948].

$$[m]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [k]_q! \begin{bmatrix} m \\ k \end{bmatrix}_q .$$

Proof [C-E-R].

LHS = rank-generating function of $[1,m]^n$.

RHS: rank-generating fn. of $[v, \omega_m(v)]$ is

$$q^{\binom{k}{2}} \, wt(v) \, [m]_q [m-1]_q \cdots [m-k+1]_q ,$$

Then sum over all RG-words. $\boxed{}$

The q- Frobenius identity

[Garsia - Remmel 1986]

$$\sum_{m \geq 0} [m]_q^n \, x^m = \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} \cdot S_q[n,k]\, [k]_q!\, x^k}{(1-x)(1-qx)\cdots(1-q^k x)}.$$

Determinantal identity

[Ehrenborg 2003].

Let $n, s \in \mathbb{Z}^+ \cup \{0\}$. Then

$$\det\left(S_q[s+i+j,\, s+j]\right)_{0 \le i,j \le n}$$

$$= [s]_q^{0}\, [s+1]_q^{1} \cdots [s+n]_q^{n}.$$

New two-parameter identity.

[Cai-Ehrenborg-R 2017].

Let $n, r, s \in \mathbb{Z}^+ \cup \{0\}$ with $s < r \leq n$.

Then,

$$\sum_{k=r}^{n} (-q^s)^{k-r} \cdot ([k-s-1]_q)_{k-r} \cdot S_q[n,k]$$

$$= \sum_{i=r-1}^{n-1} S_q[i, r-1] \cdot [s]_q^{n-i-1}.$$

Corollary: [Mercier 1990].

$$\sum_{k=1}^{n} (-1)^{k} \cdot [k-1]_q ! \; S_q[n,k] = 0$$

$$\sum_{k=2}^{n} (-1)^{k} q^{k-2} [k-2]_q \; S_q[n,k] = n-1.$$

Proof [C-E-R].

Take $(r,s) = (1,0)$ and

$(r,s) = (2,1)$, respectively $\square$

Recall

$$S_q[n,k] = h_{n-k}([1]_q, [2]_q, \ldots, [k]_q).$$

Have a symmetric function perspective

Theorem: [C-E-R].

$$\sum_{i=0}^{n-r} h_i(x_1, x_2, \ldots, x_{r-1}) \cdot x_s^{\,n-r-i}$$

$$= \sum_{k=r}^{n} (x_s - x_r) \cdot (x_s - x_{r+1}) \cdots (x_s - x_{k-1}) \cdot h_{n-k}(x_1, \ldots, x_k).$$

Thank you!