Combinatorial identities of Morel and extensions

Fall 2019 TLC

North Carolina State

Joint with Richard Ehrenborg + Sophie Morel.

Thanks to...

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ÉNS de Lyon

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Background.

[Morel 2011]. Computed intersection cohomology of Shimura varieties

Q: What is a Shimura variety?

Shimura varieties - a higher dim'l analogue of modular curves

Q: What is a modular curve?

A modular curve is the quotient

$$H / \Gamma$$

where

$H =$ upper half plane

$\Gamma =$ a congruence subgroup of the modular group of matrices in the special linear group $SL(2, \mathbb{Z})$.

$SL(2,\mathbb{Z})$: $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$

det = 1.

linear fractional transformations mapping H to H (H^* to H^*)

ex. $\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \text{ with } b,c \text{ even} \right\}.$

ex. The Klein quartic [1878].
(a quotient of the order 7
triangular tiling).

The Klein quartic

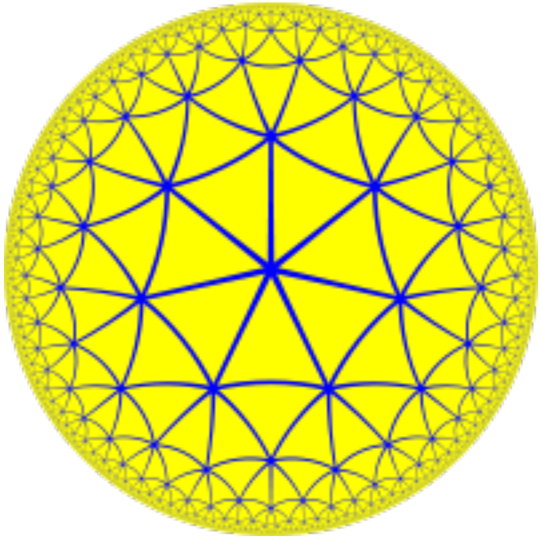


Image of the order 7 triangular tiling courtesy of Wikipedia, Tom Ruen

The Klein quartic

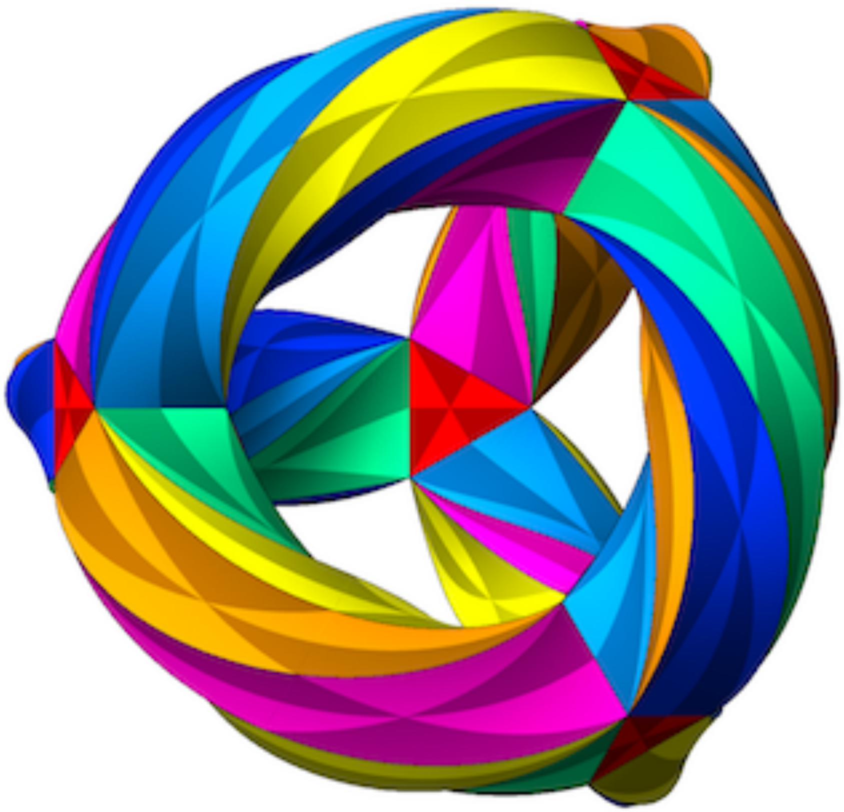


Image courtesy of Greg Egan

https://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuartic.html

The Klein quartic as a projective algebraic curve

$$x^3y + y^3z + z^3x = 0$$

Movie by Greg Egan:

https://www.gregegan.net/SCIENCE/KleinQuartic/KleinQuartic.html

Return to Shimura Varieties

[Shimura 1967] Generalized complex multiplication with Shimura varieties.

[Deligne 1971] Axiomatic framework.

[Langlands 1974]. Shimura varieties are good for Galois representations (see Fermat's Last Theorem)

Test conjectures.

"All zeta functions are automorphic" (part of Langlands program).

Fact: Shimura varieties are important in number theory.

[Goresky - MacPherson 1974] Intersection homology.

Is an analogue of homology to study singular spaces.

Local intersection cohomology used to prove nonnegativity of Kazhdan - Lusztig polynomials for Weyl groups.

Return to...

[Morel 2011] Computed intersection cohomology of Shimura varieties.

Combinatorial identities involving averaged discrete series characters of real reductive groups were naturally involved.

Combinatorics makes an entrance !!!

[Ehrenborg - Jung 2013, Ehrenborg - Hedmark 2018].

Π_n^{ord}, the <u>ordered</u> <u>partition</u> <u>lattice</u>

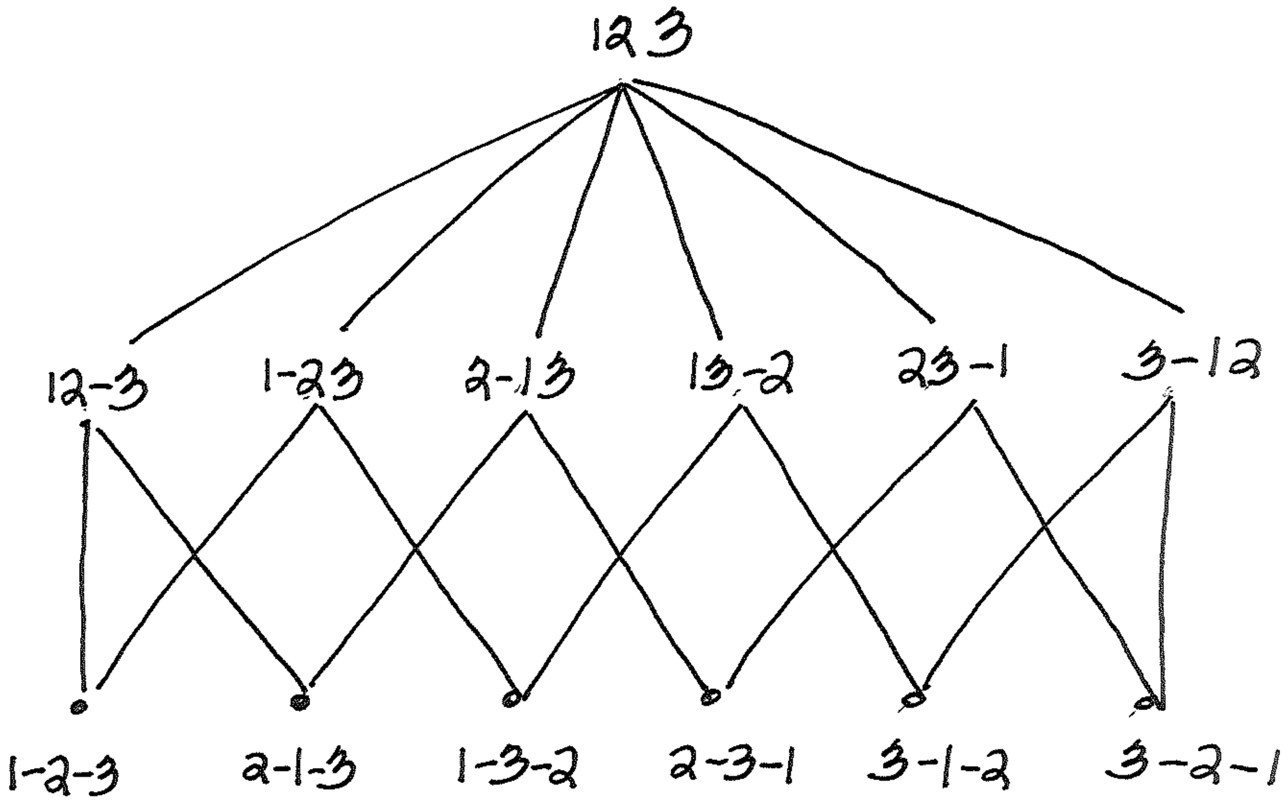
Elements: ordered partitions of $\{1, \ldots, n\}$:

$$(C_1, \ldots, C_k),$$

where C_i disjoint and $\dot{\bigcup_i} C_i = \{1, \ldots, n\}$

Partial order: Merge adjacent blocks.

ex. $\Pi_3^{\text{ord.}}$



$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$$

$$S \subseteq \{1, \ldots, n\}$$

$$\lambda_S = \sum_{i \in S} \lambda_i$$

def. $P(\lambda) = \{ \sigma = (C_1, \ldots, C_k) \in \Pi_n^{ord} :$

$$\sum_{i=1}^{j} \lambda_{C_i} > 0 \quad \cancel{\text{for}}\ j = 1, \ldots, k \}$$

ex. $\lambda = (5, 1, -4)$

$P(\lambda)$: 1-2-3

1-3-2

2-1-3

12-3

13-2.

1-23

2-13

123.

Lemme: [Morel]

$\lambda \in \mathbb{R}^n$. Then.

$$\sum_{\sigma \in P(\lambda)} (-1)^{|\sigma|} = \begin{cases} (-1)^n & \text{if } \lambda_1, \dots, \lambda_n > 0 \\ 0 & \text{otherwise.} \end{cases}$$

ex. (Verify).

$\lambda = (5, 1, -4)$

$\sigma \in P(\lambda)$	$(-1)^{\|\sigma\|}$
1-2-3	-1
1-3-2	-1
2-1-3	-1
12-3	$+1$
13-2	$+1$
1-23	$+1$
2-13	$+1$
123.	-1

$\sum = 0.$

Lemma: $P(\lambda)$ is an upper order ideal (filter) in the poset Π_n^{ord}.

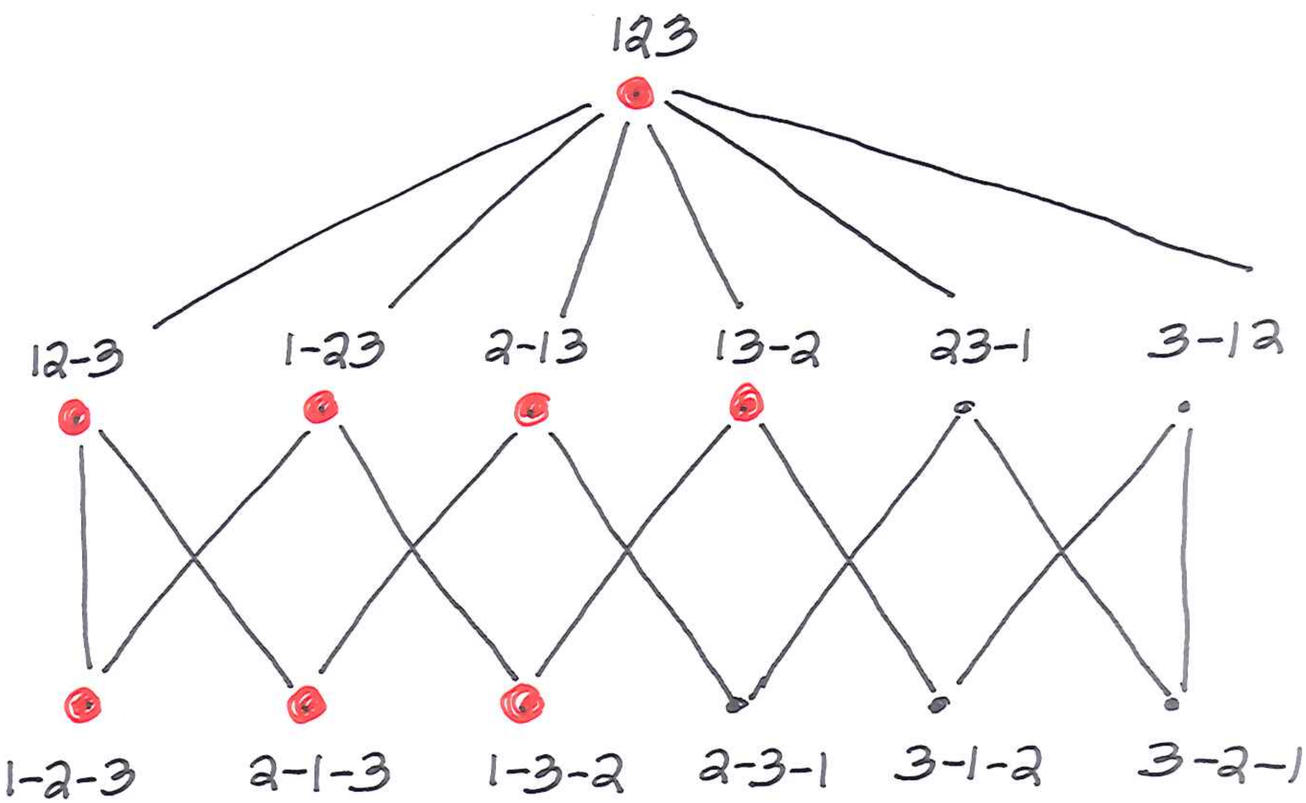
<u>Proof</u>:

Going up in the poset corresponds to merging two adjacent blocks in Π_n^{ord}.

The positivity condition $\sum_{i=1}^{j} \lambda_{C_i} > 0$ for $j = 1, \ldots, k$.

One less inequality to check. $\square$

ex. $\Pi_3^{\text{ord.}}$



● = element in $P(\lambda)$
for $\lambda = (5, 1, -4)$.

Lemma 1: $\sigma = (C_1, \ldots, C_k) \in P(\lambda)$ an ordered partition

Say $|C_j| > 1$.

Let $a \in C_j$ with λ_a a max value in C_j.

Let

$$\sigma' = (C_1, \ldots, C_{j-1}, \{a\}, C_j - \{a\}, \ldots, C_k).$$

Then.

① $\sigma' < \sigma$

②. $\sigma' \in P(\lambda)$.

<u>Proof</u>

① Easy.

②. ET Show $\displaystyle\sum_{i=1}^{j-1} \lambda_{C_i} + \lambda_a > 0.$

If $\lambda_a \geq 0$, done.

If $\lambda_a < 0 \Rightarrow$ all λ values in block C_j are negative.

So $\displaystyle\sum_1^{j-1} \lambda_{C_i} + \lambda_a > \sum_1^{j-1} \lambda_{C_i} + \lambda_{C_j} > 0$ ▨

def. $A(\lambda) = \{\tau \in \mathfrak{S}_n : \sum_{i=1}^{j} \lambda_{\tau_i} > 0 \text{ for } j = 1, \ldots, n\}$.

Lemma 2: $P(\lambda)$ is generated by $A(\lambda)$.
(Given $\sigma \in P(\lambda)$, $\exists\, \tau \in A(\lambda)$ with $\tau \le \sigma$ in Π_n^{ord}).

Proof
Iterate Lemma 1. $\boxed{/\!/}$

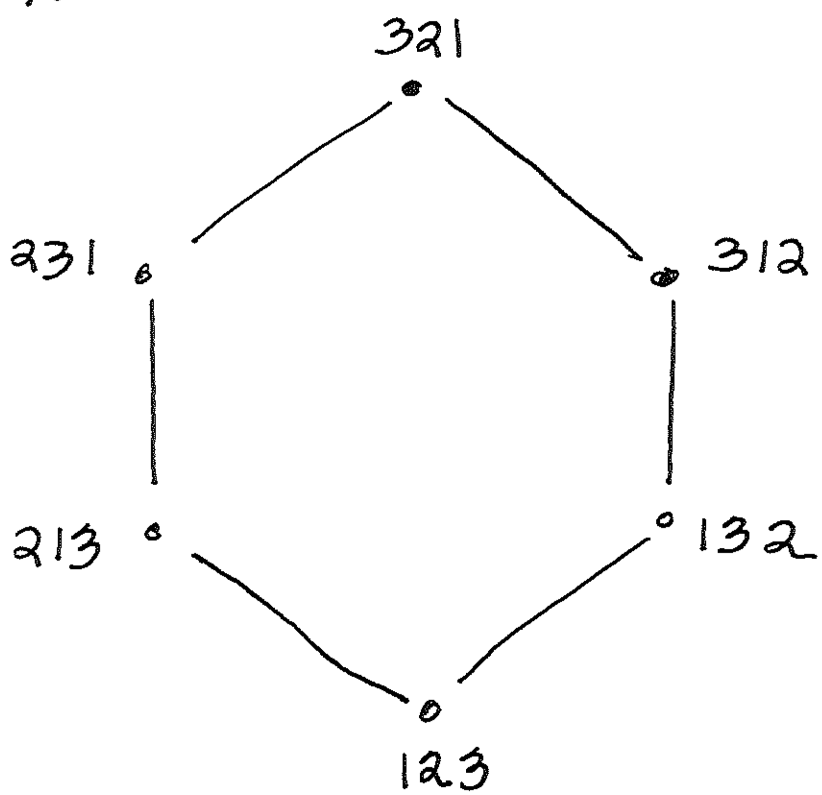
Remarks: Lemma 1 $\Rightarrow$ intervals $[\sigma_1, \sigma_2]$ in $P(\lambda)$ are $\cong B_{|\sigma_1| - |\sigma_2|}$

Lemma 2 $\Rightarrow$ $P(\lambda)$ is generated by $A(\lambda)$.

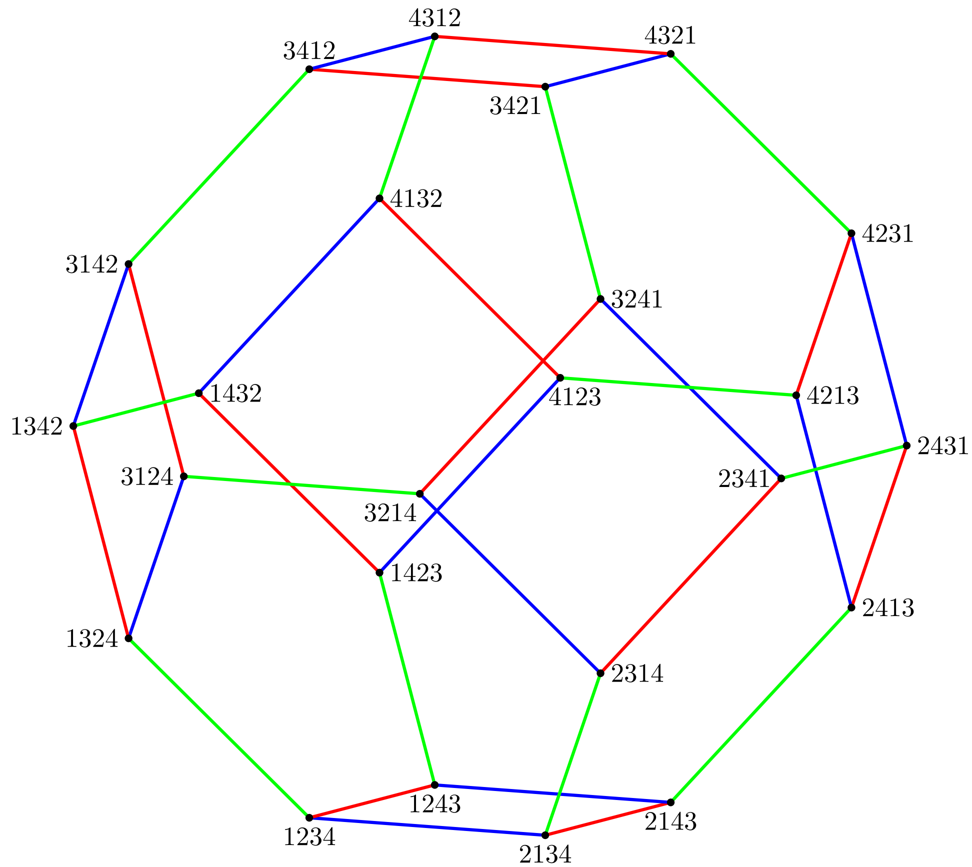
The permutahedron

$$\mathrm{Perm}(n) = \operatorname{conv}\{ (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \ \text{ where } \ \tau = \tau_1 \cdots \tau_n \in \mathfrak{S}_n \}.$$

ex. Perm (3).



Perm(4)



Courtesy of Richard Ehrenborg

Facts :

1. $\dim(\mathrm{Perm}(n)) = n-1$

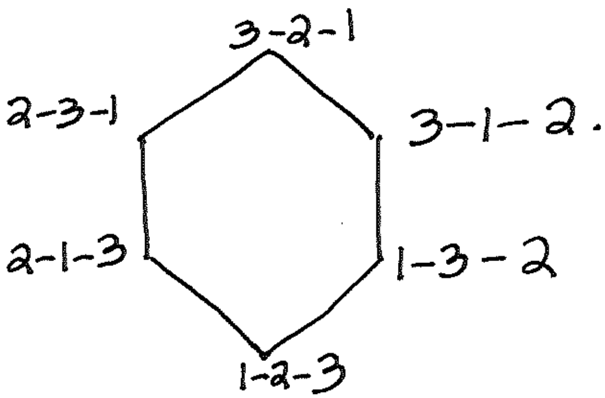
2. Is a simple polytope

3. 1-skeleton is

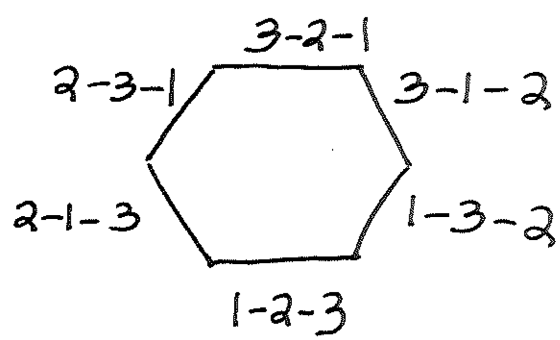
4. $\Pi_n^{\mathrm{ord}} \cong \mathcal{L}(\mathrm{Perm}(n))$

5. The dual polytope $(\mathrm{Perm}(n))^*$ is a simplicial polytope.

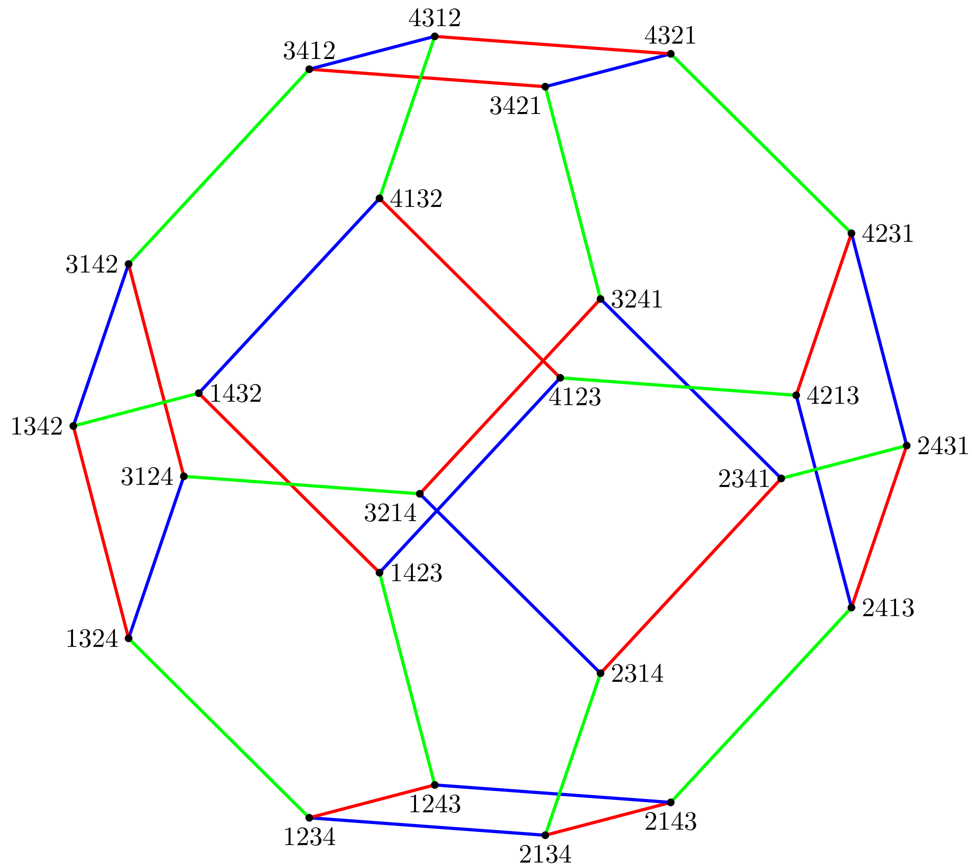
Perm (3)



$\partial(\mathrm{Perm}(3)^*)$



Perm(4)


4312
3412
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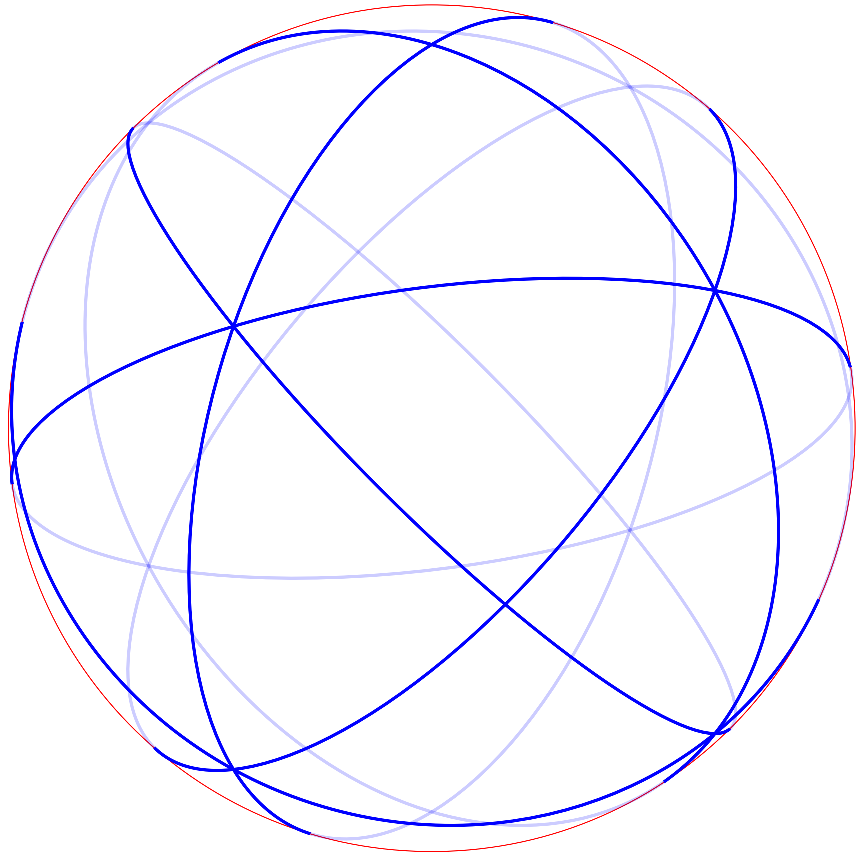
1234
2134

The dual Perm(4)*



Courtesy of Alex Happ

Weak Bruhat order on $\mathfrak{S}_n$

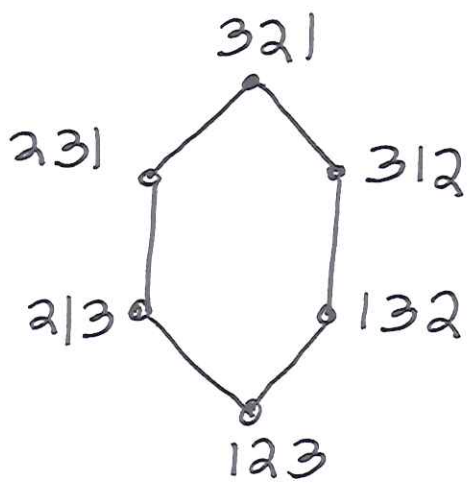
$s_i = (i, i+1)$, $\quad i = 1, \dots, n-1 \quad$ adjacent transpositions

$\sigma \in \mathfrak{S}_n$

$\mathrm{inv}(\sigma) = |\{(i,j) : \sigma_i > \sigma_j \text{ for } i<j\}|$.

$\sigma \lessdot \sigma s_i$ if $\mathrm{inv}(\sigma)+1 = \mathrm{inv}(\sigma s_i)$

ex. $\mathfrak{S}_3$.



graded by inv(·).

Δ pure simplicial complex of dim d

is <u>shellable</u> if

① $\dim \Delta = 0$ <u>or</u>

②. $\exists$ ordering $F_1, \ldots, F_s$ of facets

s.t.

$$\overline{F_j} \cap (\overline{F_1} \cup \cdots \cup \overline{F_{j-1}})$$

is a pure simplicial complex of

dimension $d-1$ for $j = 1, \ldots, s$.

[Bruggesser - Mani 1971].

∂(polytope) is shellable.

Proof idea: "rocket ship."

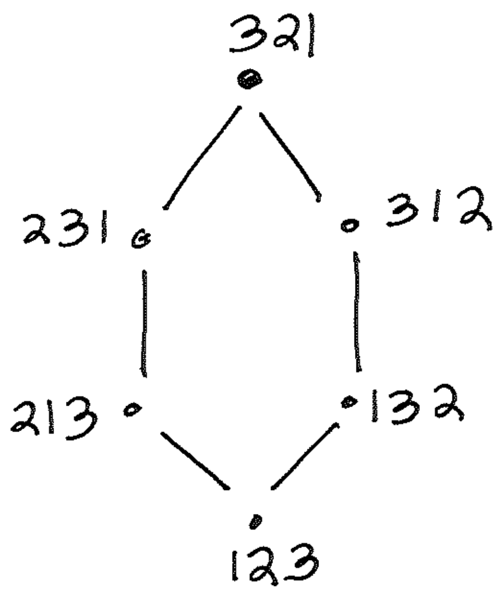
Sommerville's 1929 "proof" of Euler-Poincaré-Schläfli:

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d$$ for P a d-dim'l polytope

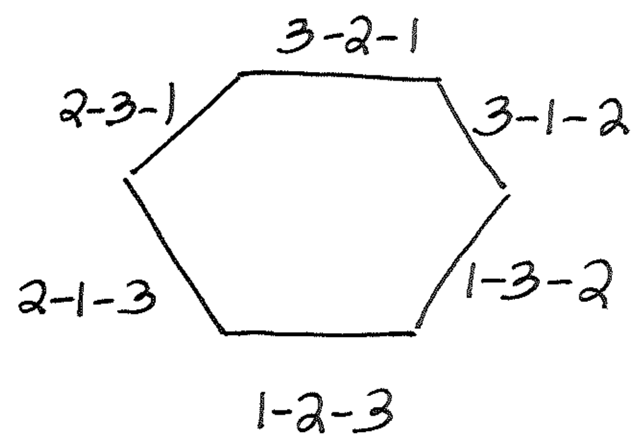
assumed shellability.

[Björner 1984] Any linear extension of the weak Bruhat order is a shelling order of $\partial(\mathrm{Perm}(n)^*)$.

ex. weak Bruhat order on $\mathfrak{S}_3$:



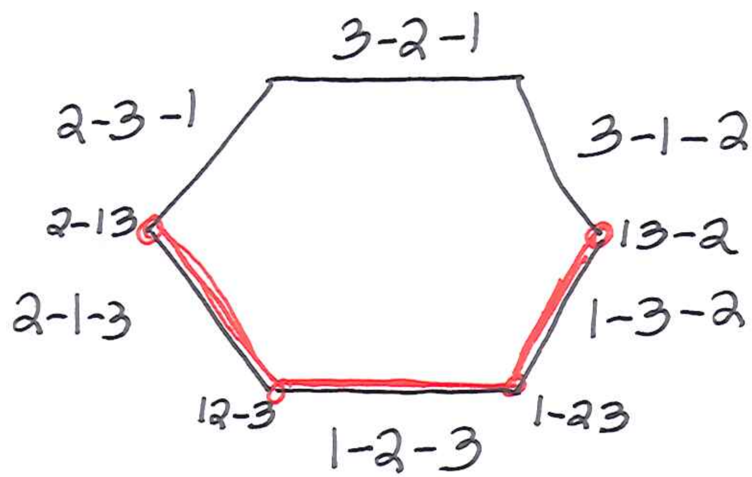
$\partial(\mathrm{Perm}(3)^*)$.



Let

$$\Sigma(\lambda) = (P(\lambda))^* \quad \text{be the weighted complex.}$$

ex. (original).

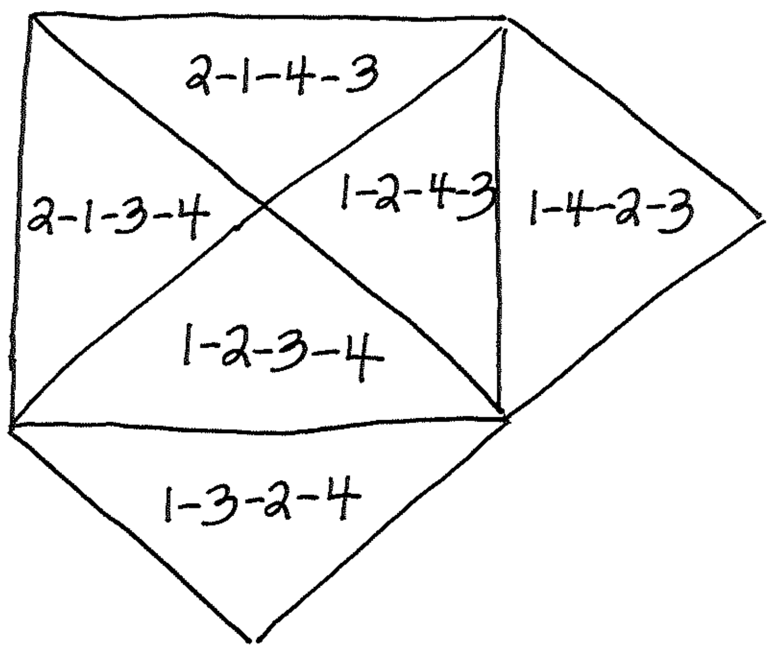


$\lambda = (5, 1, -4)$.

$(P(\lambda))^*$ in red.

ex. $\lambda = (5, 1, -2, -3)$.

$\Sigma'(\lambda)$:



Theorem: [Ehrenborg - Morel - Readdy]

Let $\lambda \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$.

and $\lambda_1 + \cdots + \lambda_n > 0$.

Then $\Sigma(\lambda)$ is a partial shelling (start of a shelling) of $\partial(\mathrm{Perm}(n)^*)$.

Hence $\Sigma(\lambda)$ is shellable.

Furthermore $\Sigma(\lambda)$ is homeomorphic to a sphere or ball via.

$$\Sigma(\lambda) \cong \begin{cases} \mathbb{S}^{n-2} & \text{if } \lambda_n > 0 \\ \mathbb{B}^{n-2} & \text{if } \lambda_n < 0. \end{cases}$$

Proof

The only homology facet is

$$\tau_0 \triangleq n \cdots 2\,1$$

and $\tau_0 \in \partial(\mathrm{Perm}(n))^*$ if

$\lambda_n > 0$ $\blacksquare$

Corollary: [Morel]

$\lambda \in \mathbb{R}^n$. Then

$$\sum_{\sigma \in P(\lambda)} (-1)^{|\sigma|} = \begin{cases} (-1)^n & \text{if } \lambda_1, \dots, \lambda_n > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof

LHS is Euler characteristic of $\Sigma(\lambda)$.

Is either a sphere or ball. $\blacksquare$

More identities...

$G = SL_n(\mathbb{R})$

$Sp_{2n}(\mathbb{R}) = \{ g \in GL_{2n}(\mathbb{R}) : \ g^T J g = J \}$,

where

$$J = \begin{bmatrix} & & & & 1 \\ & 0 & & \iddots & \\ & & -1 \ 1 & & \\ & \iddots & & 0 & \\ -1 & & & & \end{bmatrix}$$

$\rho : G \to GL_N(\mathbb{C})$ irreducible representation

Harish character

$$\Theta_\rho(x) = \operatorname{Tr} \rho(x), \qquad x \in G$$

Question: How to calculate.

Character formula

$$\Theta_\lambda(\gamma) = \sum_{w \in W}' n(r, w) \; w(\lambda) \; \gamma \; \Delta_w(y)^{-1}$$

$n(r,w)$ are averaged discrete series constants.

Formulas for $n(r,w)$ due to

① Herb

② Goresky - Kottwitz - MacPherson.
Looked at the double quotient

$$\Gamma \backslash G / K$$

Γ = arithmetic subgroup

ex. $G = SL_2$ gives modular curves
$G = Sp_{2n}(\mathbb{R})$ an algebraic variety over $\mathbb{Q}$ (a Siegel modular variety).

Look at cohomology + action of $Gal(\overline{\mathbb{Q}})/\mathbb{Q}$)

Can realize Langland's correspondence + test conjectures

[Morel]: Identified two virtual representations of the real points of a maximal torus T of the general symplectic group

(Morel related work of Arthur + Goresky-Harder-Kottwitz-MacPherson).

Combinatorics to the rescue!!!

Theorem

$\parallel$

Lemme : [Morel].

For every $\lambda \in \mathbb{R}^n$

$$S(\lambda) = (-1)^n \, T(\lambda)$$

where

$$S(\lambda) = \sum_{\sigma \in \Sigma(\lambda)} (-1)^{|\sigma|} \cdot (-1)^{g(\sigma)}$$

and.

$$T(\lambda) = \sum_{p \in M_n} (-1)^p \cdot c(p, \lambda).$$

More details

The map $g: \Sigma(\lambda) \to \mathfrak{S}_n$:

Take $\sigma \in \Sigma(\lambda)$

Order elements in each block in <u>decreasing</u> order.

Record as a permutation reading left to right.

$M_n =$ set of all max'l matchings on $\{1, \ldots, n\}$

For $p \in M_n$, two edges $\{a, c\}$ + $\{b, d\}$ <u>cross</u>

if $a < b < c < d$.

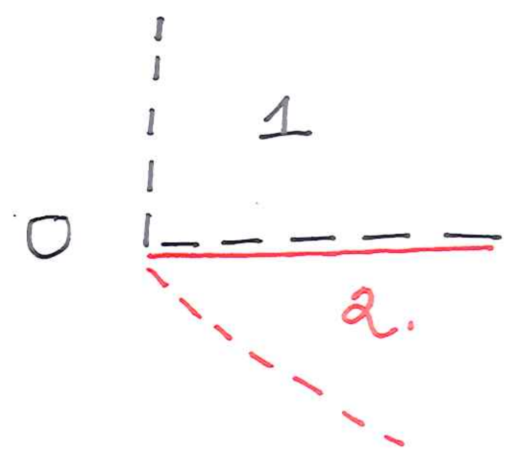
def. $(-1)^p = (-1)^{\text{cross}(p)}$.

$$\begin{cases} 1, & \text{if } n \text{ even} \\ (-1)^{i-1}, & \text{if } n \text{ odd } + i \text{ is the unique isolated vertex in max'l matching } p. \end{cases}$$

Define $c_1 : \mathbb{R} \to \mathbb{R}$, $\quad c_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$c_1(a) = \mathbb{1}_{\{a > 0\}}.$$

$$c_2(a) = \begin{cases} 1, & \text{if } a, b > 0 \\ 2, & \text{if } a > -b \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$



Hence if $e = \{i, j\}$ with $i < j$

$$c(e, \lambda) = c_2(\lambda_i, \lambda_j).$$

Set

$$c(p, \lambda) = \prod_{j=1}^{\lfloor n/2 \rfloor} c(e_j, \lambda) \cdot \begin{cases} 1 & \text{if } n \text{ even} \\ c_1(\lambda_i) & \text{if } n \text{ odd.} \end{cases}$$

Proof of Lemma

(Inspired by Herb's work on discrete series characters).

①. Prove for λ with $\lambda_1 \geq \cdots \geq \lambda_n$. (base case)

②. Given $\lambda \in \mathbb{R}^n$ let

$$\mu = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+2}, \dots, \lambda_n) \in \mathbb{R}^{n-2}.$$

Prop:

$$S(\lambda) + S(s_i \lambda) = -2 \, \mathbb{1}_{\lambda_i + \lambda_{i+1} > 0} \cdot S(\mu)$$

$$T(\lambda) + T(s_i \lambda) = +2 \, \mathbb{1}_{\lambda_i + \lambda_{i+1} > 0} \cdot T(\mu).$$

Pf

Induct on n

Assume Thm true for $n-2$

$\{s_1, \ldots, s_{n-1}\}$ generate $\mathfrak{S}_n$.

Thm true for $\lambda \iff \exists\, \tau \in \mathfrak{S}_n$ s.t.
Thm true for $\tau \cdot \lambda$.

Can find $\tau \in \mathfrak{S}_n$ s.t. $\lambda_{\tau(1)} > \cdots > \lambda_{\tau(n)}$

Done by ① $\blacksquare$

Current Research

Today: The case $\mathfrak{S}_n$ (type A Coxeter group).

Natural question: Extend to other root systems

Setting: Goresky - Kottwitz - MacPherson + Herb gave 2 different formulations for Harish-Chandra's character formula for stable discrete series of real reductive groups.

Identities we obtain in this setting have no representation theory explanation ... yet...

Thank you!