Euler flag enumeration of Whitney stratified spaces

Richard Ehrenborg, UK + IAS
Mark Goresky, IAS
Margaret Readdy, UK + IAS.
$P$ $n$-dim'1 polytope

The $f$-vector $(f_0, - , f_{n-1})$

$f_r = \# r$-dim'1 faces

[Steinitz 1906] Characterized $f$-vectors of $3$-dim'1 polytopes

Open 20: Characterize $f$-vectors of $n$-dim'1 polytopes, $n \geq 4$

[Stanley 1978; Billera-Lee 1980]

Done for simplicial polytopes
<table>
<thead>
<tr>
<th>$s$</th>
<th>$f_g$</th>
<th>$h_g$</th>
<th>$w_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>1</td>
<td>1</td>
<td>aaaa</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>11</td>
<td>baaa</td>
</tr>
<tr>
<td>1</td>
<td>18</td>
<td>17</td>
<td>abaa</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>7</td>
<td>aab</td>
</tr>
<tr>
<td>01</td>
<td>36</td>
<td>7</td>
<td>bbaa</td>
</tr>
<tr>
<td>02</td>
<td>36</td>
<td>17</td>
<td>bab</td>
</tr>
<tr>
<td>12</td>
<td>36</td>
<td>11</td>
<td>abbb</td>
</tr>
<tr>
<td>012</td>
<td>72</td>
<td>1</td>
<td>bbbb</td>
</tr>
</tbody>
</table>

$P$, n-dim'l polytope

The **flag f-vector** $f_s$

The **flag h-vector** $h_g$

$$h_g = \sum_{T \subseteq S} (-1)^{|S|-1} f_{T^*}$$

[Stanley] $h_g = h_{\theta}$. 
The ab-index

\[ \Xi(P) = \sum_{g} h_g \cdot w_g \]

ex. \[ \Xi(\text{aabb}) = 1 \text{ aavv} + 11 \text{ barv} + 17 \text{ abav} + 7 \text{ aab} + 7 \text{ bba} + 17 \text{ bab} + 11 \text{ abb} + 1 \text{ bbb} \]

\[ = (a+b)^3 + 10 (baa + aba + bav + abv) + 6 (aba + avb + bba + bab) \]

\[ = (a+b)^3 + 10 (ab + ba) (a+b) + 6 (a+b) (ab + ba) \]

\[ = c^3 + 10dc + 6cd, \]

where \( c = a+b, \ d = ab + ba \). The cd-index.
Theorem: \[\text{[Bayer-Klapper 1991; Stanley 1994]}\]
P polytope then \(\tau(P) \in \mathbb{Z} \langle c, d \rangle\).
P Eulerian poset then \(\tau(P) \in \mathbb{Z} \langle c, d \rangle\).

Eulerian: \(\mu(x,y) = (-1)^{\sigma(x,y)}\) for every interval \([x,y]\) in a graded poset \(P\).

Equivalently,
in each non-trivial interval \([x,y]\):
\[
\# \text{ elts of even rank} = \# \text{ elts of odd rank}.
\]
Some cd-history

1980's

1990's
[Bayer-Klapper]. $\Xi \geq 0$ for $\Delta$ (polytope), more generally, $S$-shellable face poset of regular CW-complex

[Stanley].

[Purtill] $n$-simplex, $n$-cube $\iff$ André + signed André permutations.

[Ehrenberg-R] Coalgebraic techniques

[Billera-Ehrenberg-R] Zonotopes span; OM / hyperplane arrangements

[Billera-Ehrenberg] $\Xi(n$-polytope) $\geq \Xi(n$-simplex)
cd-history (cont'd)

2000's

[Karu] \( \Gamma(Gorenstein^\ast \text{ posets}) \geq 0 \)

[Karu-Ehrenborg] \( \Gamma(Gorenstein^\ast \text{ lattices}) \geq \#(B_n) \)

[Ehrenborg-R.Slane] Arrangements of subtori

2010's

[Billera-Brenti] \( \Gamma(\text{Bruhat graphs}) \) via quasi-symmetric fns, Kazhdan-Lusztig thy.

[Ehrenborg-R.] \( \Gamma(\text{Balanced graphs}) \) via R-labelings.
Ex The n-gon \((n \geq 2)\).

\[\begin{array}{c|cccc}
S & f_S & h_S & w_S \\
\hline
\emptyset & 1 & 1 & a_S \\
0 & n & n-1 & b_S \\
1 & n & n-1 & a_S \\
01 & 2n & 1 & b_S.
\end{array}\]

\[\overline{w} (\n) = c^2 + (n-2) d.\]
ex. 1-gon

\[
\begin{array}{c|cc}
 s & f_3 & h_3 \\
\hline
\emptyset & 1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
01 & 1 & 0 \\
\hline
\end{array}
\]

\[
\hat{0} \rightarrow e \rightarrow \hat{1}
\]

Not Eulierian
Try again...

| s   | $\bar{f}_g$ | $\bar{h}_g = \sum_{T \in s} (-1)^{|s-T|} f_T$ |
|-----|-------------|---------------------------------------------|
| $\emptyset$ | 1           | 1                                           |
| 0    | 1           | 0                                           |
| 1    | 1           | 0                                           |
| 01   | 2           | 1                                           |

$\text{link}_e(v) = \cdot \cdot$

$\chi(\cdot \cdot) = 2,$

the Euler characteristic

$\chi(\mathbb{G}) = \alpha \omega + \beta \beta$

$= c^2 - d.$
Face poset

solid torus

\[ \eta_2 \subseteq \{ \cdot \cdot \} \]

\[ v_1 \quad e \quad v_2 \quad v_3 \]
Chain \( C = \{ \emptyset = x_0 < x_1 < \cdots < x_k = \hat{y} \} \)
in the face poset weighted by:
\[
\overline{\mathcal{G}}(c) = \chi(x_1) \cdot \chi(\text{link}_{x_2}(x_1)) \cdots \chi(\text{link}_{x_k}(x_{k-1}))
\]
\(\Rightarrow\) (cont'd).

<table>
<thead>
<tr>
<th>s</th>
<th>( \bar{f}_s )</th>
<th>( \bar{h}_s )</th>
<th>( 3dc )</th>
<th>( -2cd )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<tr>
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<td>4</td>
<td>0</td>
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</table>
These are examples of Whitney stratifications.

Subdivide space into strata:

\[ W = \bigcup_{x \in P} x \]

Condition of the frontier:

\[ x \cap \overline{y} \neq \emptyset \iff x \subseteq \overline{y} \iff x \leq y \text{ in the face poset } P. \]

Whitney conditions A + B:
- No fractal behavior
- No infinite wiggling \( \text{ex. } x \cdot \sin \left( \frac{1}{x} \right) \)

\[ \Rightarrow \text{ The links are well-defined.} \]
Whitney stratifications exist for:
real or complex algebraic sets
analytic sets
semi-analytic sets
quotients of smooth manifolds by compact group actions.
**The Fine Print**

**Definition** Let \( W \) be a closed subset of a smooth manifold \( M \), and suppose \( W \) can be written as a locally finite disjoint union

\[
W = \bigcup_{X \in \mathcal{P}} X
\]

where \( \mathcal{P} \) is a poset. Furthermore, suppose each \( X \in \mathcal{P} \) is a locally closed subset of \( W \) satisfying the condition of the frontier:

\[
X \cap \bar{Y} \neq \emptyset \iff X \subseteq \bar{Y} \iff X \leq_{\mathcal{P}} Y.
\]

This implies the closure of each stratum is a union of strata. We say \( W \) is a Whitney stratification if

1. Each \( X \in \mathcal{P} \) is a locally closed smooth submanifold of \( M \) (not necessarily connected).
2. If \( X <_{\mathcal{P}} Y \) then Whitney's conditions (A) and (B) hold: Suppose \( y_i \in Y \) is a sequence of points converging to some \( x \in X \) and that \( x_i \in X \) converges to \( x \). Also assume that (with respect to some local coordinate system on the manifold \( M \)) the secant lines \( \ell_i = \overline{x_i y_i} \) converge to some limiting line \( \ell \) and the tangent planes \( T_{y_i} Y \) converge to some limiting plane \( \tau \). Then the inclusions

\[
\text{(A) } T_x X \subseteq \tau \quad \text{ and } \quad \text{(B) } \ell \subseteq \tau
\]

hold.
ux. The Whitney map.

Whitney stratifications (their face posets) are examples of...
A quasi-graded poset \((P, \leq, \bar{z})\) consists of

c. \(P\) finite poset with \(\hat{0} + \hat{1}\) (not necessarily graded)

c. \(\rho: P \to \mathbb{N}\) order-preserving \((x < y \Rightarrow \rho(x) < \rho(y))\)

c. \(\bar{z} \in \mathcal{F}(P)\), the weighted zeta function satisfying \(\bar{z}(x, x) = 1\) \(\forall x \in P\).
def. \((P, \rho, \bar{\zeta})\) is \underline{Eulerian} if

\[
\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \bar{\zeta}(x,y) \cdot \bar{\zeta}(y,z) = S_{x,z}.
\]

Remark: \(\bar{\zeta} = \zeta\) gives the \underline{classical Eulerian condition}

\[
\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} = S_{x,z}.
\]
Define

\[ \Pi(P, \rho, \bar{z}) = \sum_{g} h_g \cdot \omega_g \]

with

\[ \bar{z}(c) = \bar{z}(x_0, x_1) \cdot \bar{z}(x_1, x_2) \cdots \bar{z}(x_{\omega-1}, x_\omega) \]

for a chain

\[ c: \hat{0} = x_0 < x_1 < \cdots < x_\omega = \hat{1}. \]
Theorem: \((P, \rho, \overline{\zeta})\) an Eulerian quasi-graded poset.

Then
\[ \overline{\zeta}(P, \rho, \overline{\zeta}) \in \mathbb{Z} < c, d >. \]

Theorem: \(\mathcal{M}\) manifold with \(\mathcal{W}\) Whitney stratified boundary.

Then the face poset is quasi-graded + Eulerian, where
\[ \nu(x) = \dim(x) + 1. \]

\[ \overline{\zeta}(x, y) = \chi(\text{link}_y(x)). \]
Open 2 and Work in progress:

1. Inequalities:
   Kalai convolution still works
   What about Ehrenberg's lifting technique?

2. \((P, \rho, \xi)^{\text{Eulerian}}\) quasi-graded poset \(\Rightarrow\) Find W Whitney stratified space.

   [Purtill] \(n\)-simplex and \(n\)-cube
   [Karu] operators on sheaves of v.s.
4. Stanley-Reisner ring for barycentric subdivision of a stratified space? What should the Cohen-Macaulay property be?

5. Non-linear inequalities?

6. Inequalities for $\Pi$ (manifold arrangements)?

7. Muci's gen'd Tutte polynomial for spherical + toric arrangements. Develop a similar polynomial for manifold arrangements (or for some natural subclass).
Happy Birthday,

Bruce.

\[ k \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c^3 + de - 2cd \]

\[ H \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A \text{ birthday cake without candles} \]

\[ \ldots \text{since who is counting anyway...} \]