q-Combinatorics: A new view

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Discrete Math Day at WPI.

Thanks to the Simons Foundation.

Let's count [~ 50,000 BC*]

$$\sum_{\pi \in \mathfrak{S}_n} 1 = n!$$

$$\sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = k}} 1 = \binom{n}{k}.$$

* Source: Wikipedia

Let's q-count [1700's Euler*].

q-analogue of $n \in \mathbb{Z}^+$

$$[n]_q = [n] = 1 + q + \cdots + q^{n-1},$$

q an indeterminate.

$$\lim_{q \to 1} [n]_q = \underbrace{1 + \cdots + 1}_{n} = n.$$

$$[n]! = [n][n-1] \cdots [2][1].$$

* Theta functions

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}}, \quad |ab| < 1$$


LEONHARD EULER
1707–1783
e-k+f=2
130
HELVETIA
ANGELO BOOG
2007

Combinatorial interpretation

[MacMahon 1916]

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = [n]!,$$

where

$$\mathrm{inv}(\pi) = \{ (i,j) : \quad i<j \text{ and } \pi_i > \pi_j \}.$$

for $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$.

Gaussian polynomial. (the q-binomial)

$n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[k]!\,[n-k]!} & 0 \le k \le n \\ 0 & k < 0 \text{ or } k > n. \end{cases}$$

Comb'l interpretation.

$$\sum_{w \in \mathfrak{S}(1^k, 0^{n-k})} q^{\operatorname{inv} w} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

[MacMahon 1916]

ex. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

π	inv π
0011	0
0101	1
0110	2
1001	2
1010	3
1100	4.

$$\sum_{\pi \in \mathfrak{S}\{1^2, 0^2\}} q^{\text{inv}\, \pi} = q^4 + q^3 + 2q^2 + q + 1.$$

Check $$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{[4][3]}{[2]} = \frac{(1+q)(1+q^2)(1+q+q^2)}{(1+q)}.$$

The negative q-binomial [Fu-Reiner-Stanton-Thiem, 2012]

def.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' \triangleq (-1)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q}$$

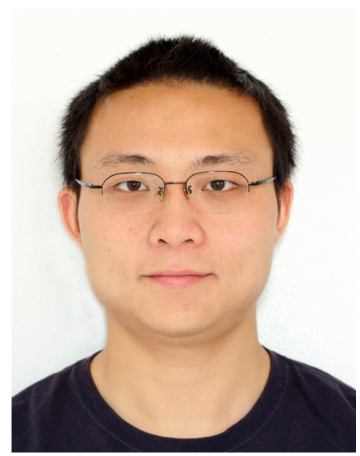
ex.

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q' = q^4 - q^3 + 2q^2 - q + 1 .$$

Dennis Stanton

Vic Reiner

Nathaniel Thiem



Shishuo Fu

Theorem: [Fu – Reiner – Stanton – Thiem].

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q = \sum_{w \in \Omega(n,k)'} wt(w)$$

$$= \sum_{w \in \Omega(n,k)'} q^{a(w)} (q-1)^{p(w)}$$

where $\Omega(n,k)'$ is a certain subset of $\mathfrak{S}\{1^k, 0^{n-k}\}$,

$p(w)$ = number of $\underline{10}$ pairs in w

$a(w) = inv(w) - p(w)$.

Corollary: [F-R-S-T]

The q-binomial can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega(n,k)'} q^{a(w)} (1+q)^{p(w)}.$$

def. Given $w = w_1 \cdots w_n \in \mathfrak{S}\{1^k, 0^{n-k}\}$, pair

i. $n = 1$. Leave letter unpaired.

ii. $n \geq 2$ + k odd: Pair $\underline{w_1 w_2}$

Repeat on $w_3 \cdots w_n$

iii. $n \geq 2$ + k even: Pair w_1.

Repeat on $w_2 \cdots w_n$.

ex. $\underline{01}\ \underline{1}\ \underline{00}\ \underline{10}\ \underline{1}\ \underline{01}$

$\underline{1}\ \underline{10}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{00}\ \underline{1}$

Define

$\Omega'_{n,k} = \{ w \in \mathfrak{S}\{1^k, 0^{n-k}\} :$ w has no paired $\underline{01}$ $\}$.

ex. $\underline{\Psi}$

$\underline{0}\,\underline{0}\,\underline{1}\,\underline{1}$

$\underline{0}\,\underline{1}\,\underline{01}$ No.

$\underline{0}\,\underline{1}\,\underline{10}$

$\underline{1}\,\underline{00}\,\underline{1}$

$\underline{1}\,\underline{01}\,\underline{0}$ No.

$\underline{1}\,\underline{10}\,\underline{0}$

ex. (cont'd)

$\Omega(2,2)'$	$q^{inv(w)}$	$wt(w)$
0011	1	1
0110	q^2	$q(1+q)$
1001	q^2	q^2
1100	q^4	$q^3(1+q)$

$$\sum = 1 + (q+q^3)(1+q) + q^2$$
$$= q^4 + q^3 + 2q^2 + q + 1.$$

Recall

$$wt(w) = q^{a(w)} \cdot (1+q)^{p(w)}$$

$p(w) = \#$ <u>10</u> pairs in w, $a(w) =$

$a(w) = inv(w) - p(w)$.

20 : What abaut other combinatorial objects with q-analogues ?

Goal: Given a q-analogue

$$f(q) = \sum_{w \in S} q^{\sigma(w)},$$

for some statistic $\sigma(\cdot)$, find a subset $T \subseteq S$ and statistics $A(\cdot)$ and $B(\cdot)$ so that

$$f(q) = \sum_{w \in T} q^{A(w)} \cdot (1+q)^{B(w)}.$$

Goal′: Given

$$f(q) = \sum_{w \in T} q^{A(w)} \cdot (1+q)^{B(w)},$$

find ~~poset~~ theoretic and topological explanations.

The Stirling numbers of the second kind

$S(n,k) =$ # partitions of $\{1, \ldots, n\}$ into k blocks.

ex. $S(4,2)$:

1/234	12/34	
134/2	13/24	(written in standard form).
124/3	14/23.	
123/4		

The <u>q-Stirling numbers</u>

$$S_q[n,k] = S_q[n-1, k-1] + [k]\, S_q[n-1, k]$$

with $S_q[n,n] = 1 = S_q[n,1]$.

RG-words [Milne].

Encode a partition π using a restricted growth word w.

$w = w_1 \cdots w_n$ where $w_i = j$ if the elt. i is in the jth block of π.

ex. $\pi = 125/36/47 \longleftrightarrow 1123123$.

Let $\mathcal{R}(n,k)$ = set of all RG-words which encode a set partition of $\{1, \ldots, n\}$ into k parts.

For $w \in \mathcal{B}(n,k)$ let

$$wt(w) = \prod_{i=1}^{n} wt_i(w),$$

where $m_i = \max\{w_1, \ldots, w_i\}$,

$wt_1(w) = 1$ and for $2 \le i \le n$

$$wt_i(w) = \begin{cases} q^{w_i - 1} & \text{if } w_i \le m_{i-1} \\ 1 & \text{if } w_i > m_{i-1} \end{cases}$$

Theorem: [Cai–Readdy]
The q-Stirling number of the second kind is given by

$$S_q[n,k] = \sum_{w \in \mathcal{B}(n,k)} wt(w).$$

ex.

π	w	$wt(w)$
1/234	1222	$q^1 \cdot q^1 = q^2$
134/2	1211	1
124/3	1121	1
123/4	1112	1
12/34	1122	q^1
13/24	1212	q^1
14/23	1221	q^1

$$\sum = q^2 + 3q + 3 .$$

$$= S_q[4,2]$$

Remark: See Garsia-Remmel, Milne, and especially Wachs-White for a multitude of statistics that generate $S_q[n,k]$.

The wt(·) statistic is related to Wachs-White's ls(·) statistic.

Let $wt'(w) = \prod_{i=1}^{n} wt_i'(w)$, $m_i = \max\{w_1, \ldots, w_i\}$,

and

$$wt_i'(w) = \begin{cases} q^{w_i - 1}(1+q) & \text{if } w_i < m_{i-1} \\ q^{w_i - 1} & \text{if } w_i = m_{i-1} \\ 1 & \text{if } w_i > m_{i-1} \text{ or } i = 1, \end{cases}$$

Write $A(w) = \sum_{i=1}^{n} A_i(w)$ and $B(w) = \sum_{i=1}^{n} B_i(w)$

where

$$A_i(w) = \begin{cases} w_i - 1 & \text{if } w_i \le m_{i-1} \\ 0 & \text{if } w_i > m_{i-1} \text{ or } i = 1 \end{cases} \qquad B_i(w) = \begin{cases} 1 & \text{if } w_i < m_{i-1} \\ 0 & \text{otherwise.} \end{cases}$$

Allowable RG-words

def. An RG-word $w \in \mathcal{R}(n,k)$ is <u>allowable</u> if it is of the form

$$\underbrace{1 \cdots 1}_{1\text{'s}} \; 2 \; \underbrace{\cdots}_{1\text{'s},\, 3\text{'s}} \; 4 \; \underbrace{\qquad}_{1\text{'s},\, 3\text{'s},\, 5\text{'s}} \; 6 \; \cdots$$

ex. $w = 1\,1\,2\,1\,3\,3\,1\,4\,3\,5 \in \mathcal{A}(10,5)$.

$$wt(w) = 1 \cdot 1 \cdot 1 \cdot (1+q) \cdot 1 \cdot q^2 \cdot \quad \cdot (1+q) \cdot 1 \cdot q^2(1+q) \cdot 1$$

Allowable words are denoted by $\mathcal{A}(n,k)$

ex.

w	$wt'(w)$
1222	–
1211	$(1+q)^2$
1121	$(1+q)$
1112	1
1122	–
1212	–
1221.	–

$$\sum = (1+q)^2 + (1+q) + 1$$

$$= q^2 + 3q + 3$$

$$\parallel$$

$$S_q[4,2]$$

ex. $S_q[5,3]$.

w.	$wt'(w)$
12311	$(1+q)^2$
12131	$(1+q)^2$
12113	$(1+q)^2$.
12133	$(1+q)\cdot q^2$
12313	$(1+q)\cdot q^2$
12331	$q^2\cdot(1+q)$
12333	$q^2\cdot q^2$.
11213	$(1+q)$
11231	$(1+q)$
11233	q^2
11123	1

$$\sum = q^4 + 3q^2(1+q) + q^2 + 3\cdot(1+q)^2 + 2(1+q) + 1.$$

$$S_q[5,3] = q^4 + 3q^3 + 7q^2 + 8q + 6$$

Theorem: [Cai - Readdy]

$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} wt'(w)$$

$$= \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot (1+q)^{B(w)},$$

Stembridge's $q=-1$ phenomenon

B finite set

$$X(q) = \sum_{b \in B} q^{wt(b)}.$$

Set $q=-1$ to count fixed pts in an involution.

Corollary: [Cai-Readdy]
$S_q[n,k]$ when $q=-1$
counts the # of weakly increasing allowable words in $\mathcal{A}(n,k)$.

Form: $1 \cdots 1\ 2\ 3 \cdots 3\ 4\ 5 \cdots 5\ 6\ \cdots$
(No $(1+q)$ terms).

The Stirling poset of the second kind $\Pi(n,k)$

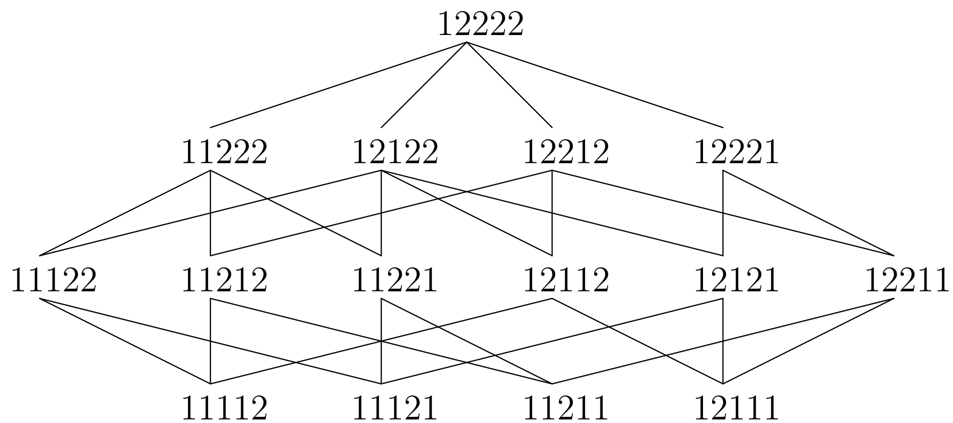
For $\pi, \sigma \in \mathcal{R}(n,k)$ let $\pi \lessdot \sigma$ if

$$\sigma = \pi_1 \pi_2 \cdots (\pi_i + 1) \cdots \pi_n .$$

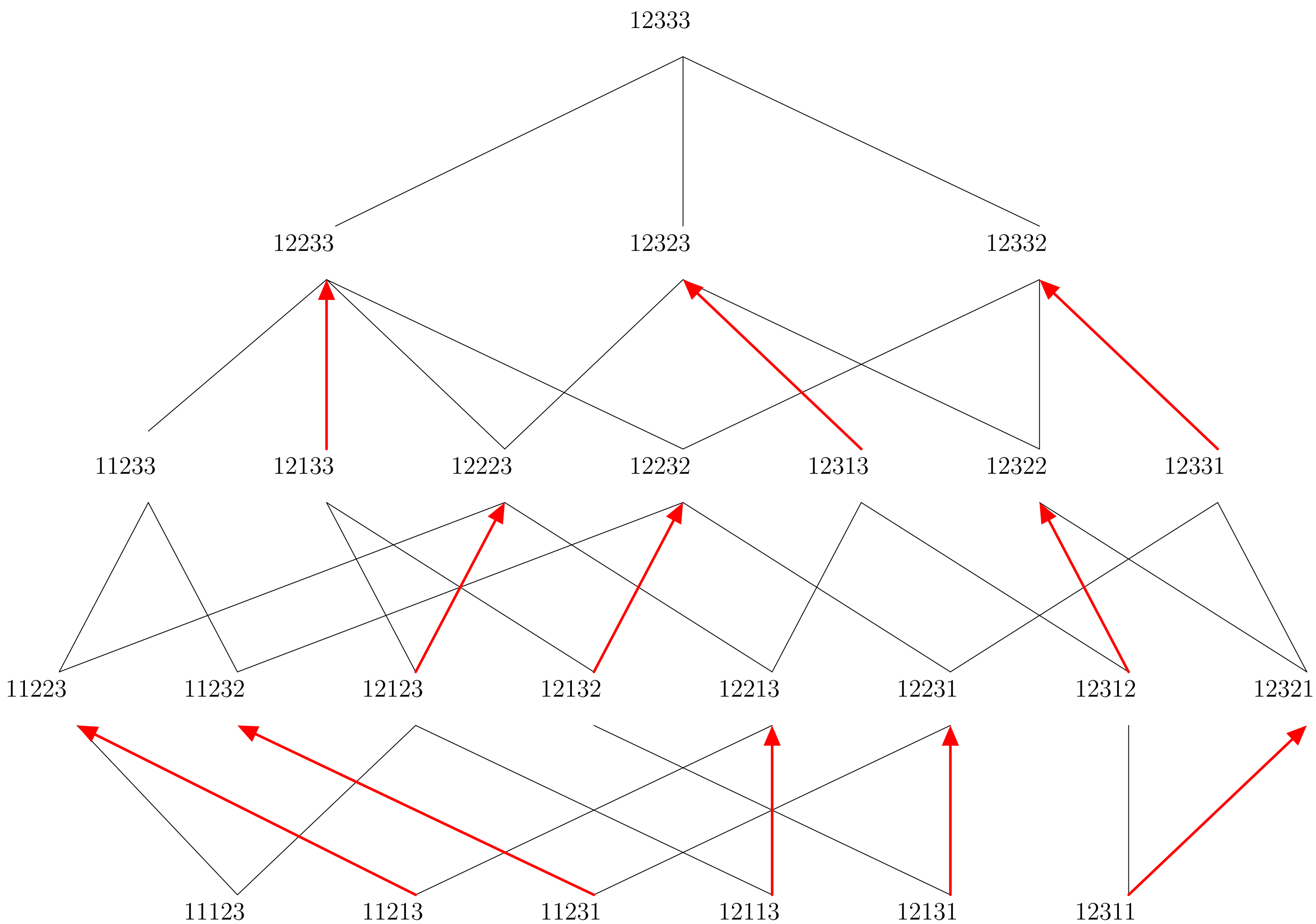
for some index i.

Clearly, $\pi \lessdot \sigma \Rightarrow wt(\sigma) = q \cdot wt(\pi)$.

Thus $\Pi(n,k)$ is a graded poset



The Stirling poset of the second kind $\Pi(5, 2)$.



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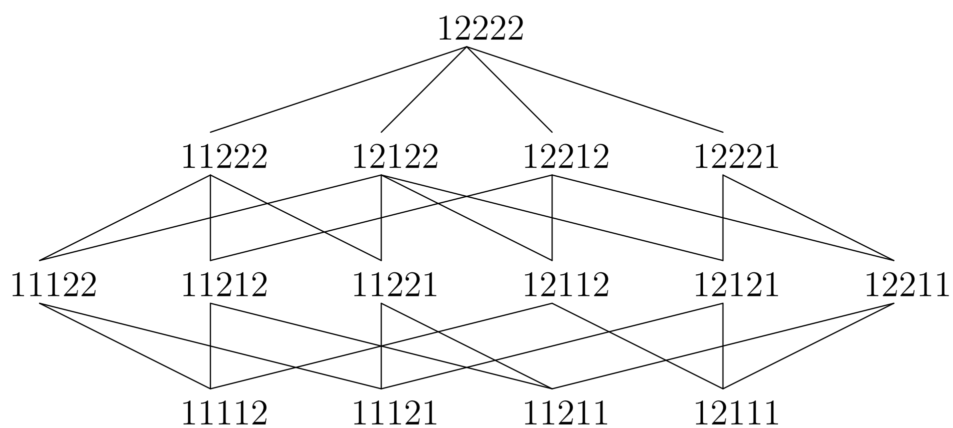
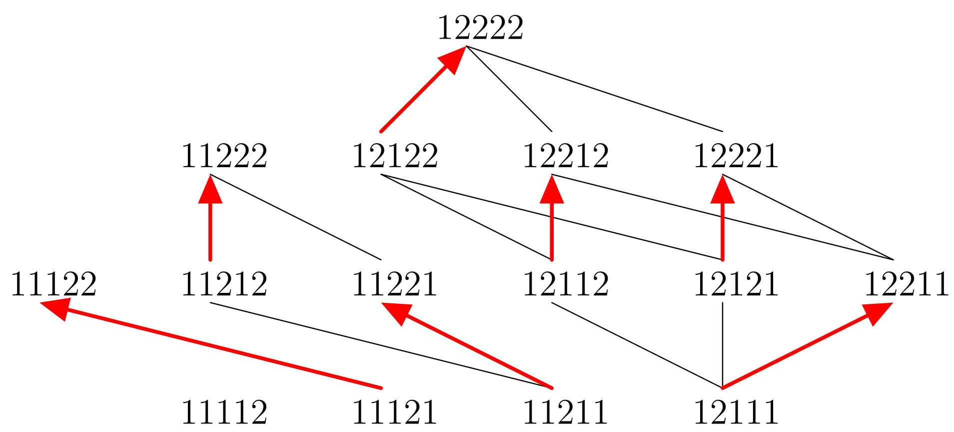
Theorem: [Cai - Readdy].
The Stirling poset of the second kind has the decomposition.

$$\Pi(n,k) \cong \dot{\bigcup_{w \in \mathcal{A}(n,k)}} B_{|Inv(w)|}$$

where B_j is the Boolean algebra on j elts,
$Inv(w) = \{ w_i : w_j > w_i \text{ for some } j < i \}$
is the set of all entries in w that contribute to an inversion,
and $\mathcal{A}(n,k)$ are allowable RG-words in $\Pi(n,k)$.

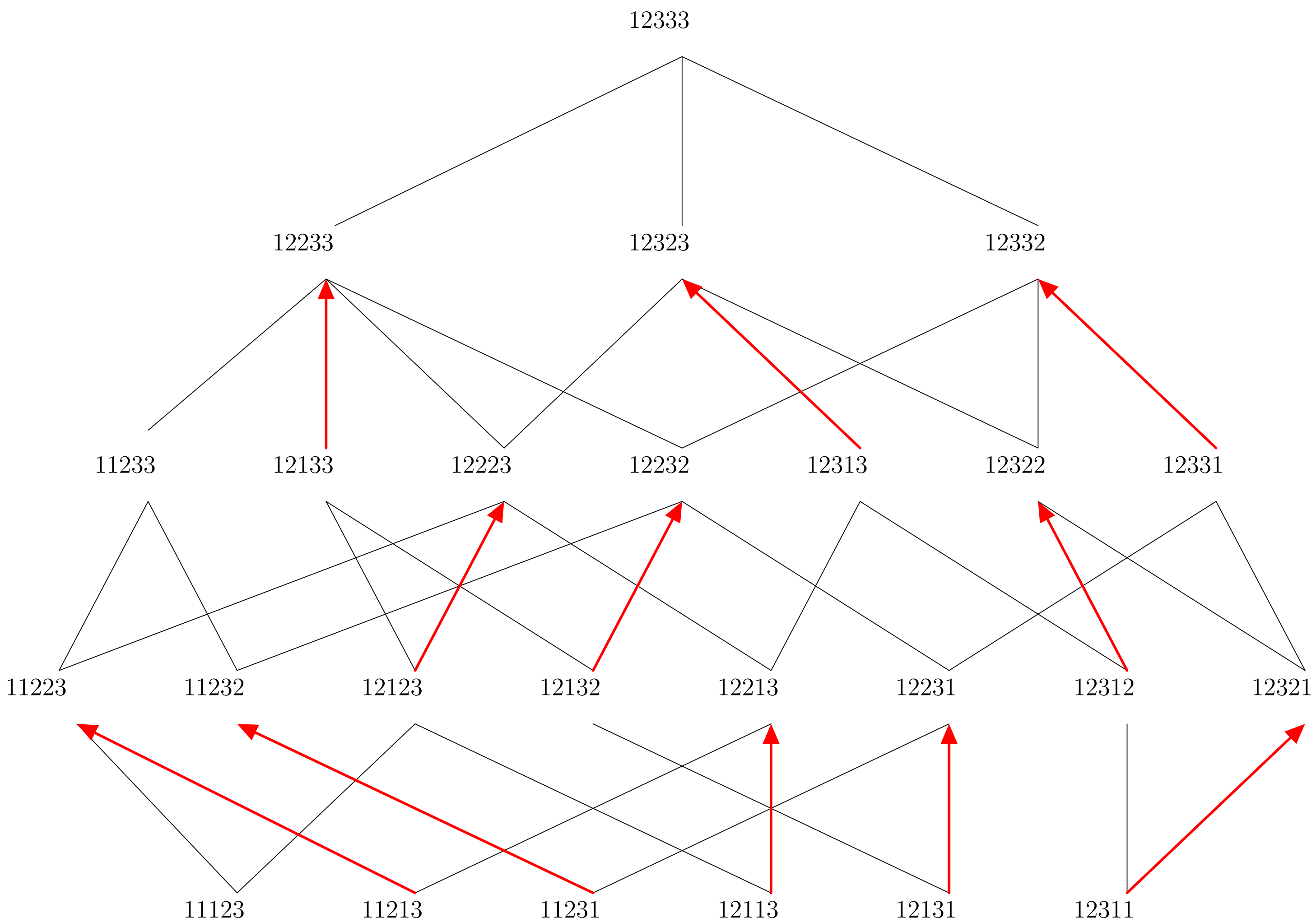


The Stirling poset of the second kind $\Pi(5,2)$.



The decomposition.

The weight of the poset is $1 + (1+q) + (1+q)^2 + (1+q)^3$.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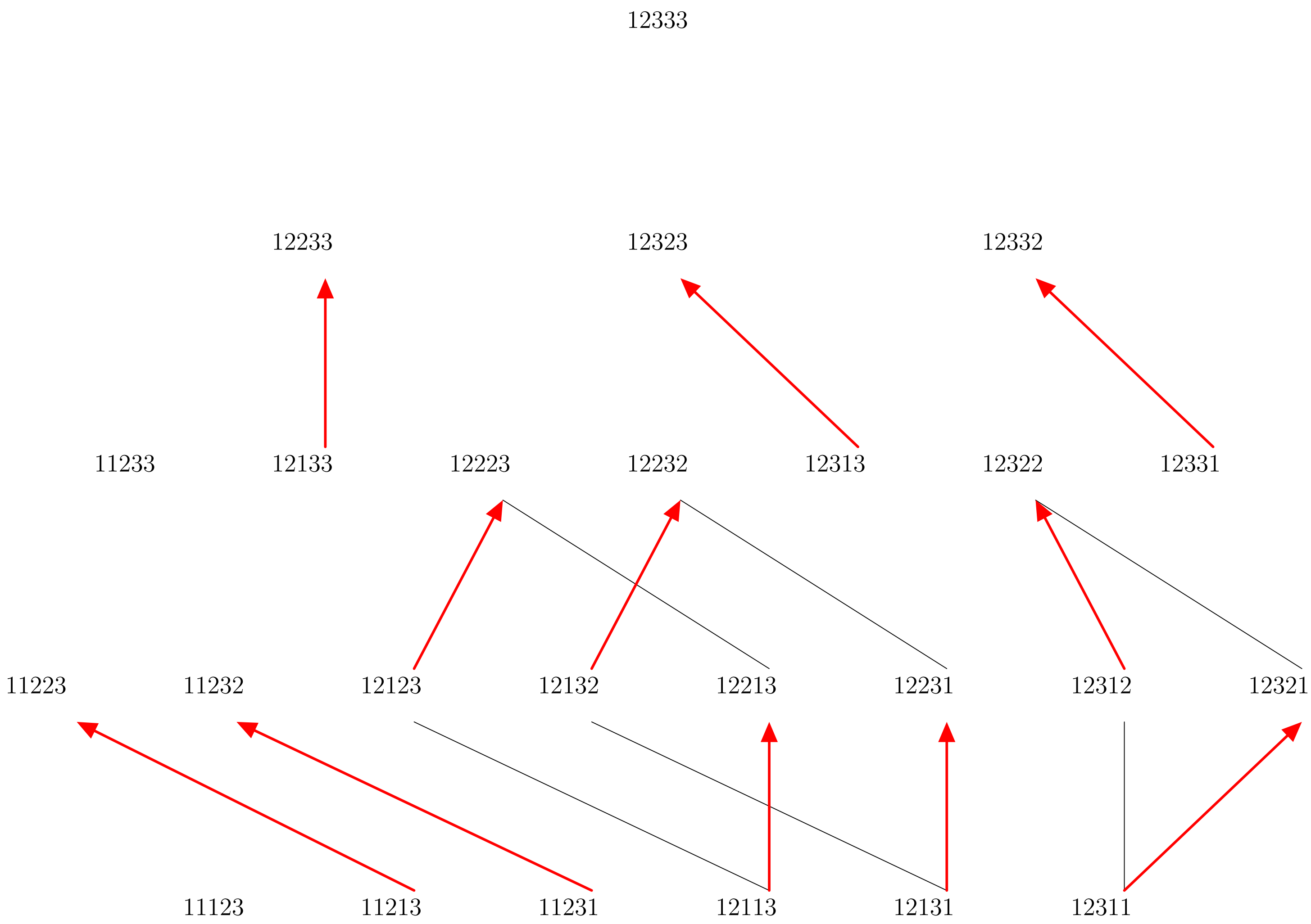
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Homological $q=-1$ phenomenon [Hersh - Shareshian - Stanton]

Claim: Stembridge's $q=-1$ phenomenon is same Euler characteristic computation.

Idea: Define a chain complex $(\mathcal{C}, \partial)$.
Ranks of chain groups are coeffs in the polynomial $X(q)$.
Euler characteristic is $X(-1)$
Also, Euler characteristic = alternating sum of ranks of homology groups.

Best scenario: $(\mathcal{C}, \partial)$ has homology concentrated in ranks of same parity & has basis indexed by fixed points of involution $= X(-1)$.

def. P graded poset

W_i = rank i elts of P

The poset P supports a chain complex $(\mathcal{C}, \partial)$

of $\mathbb{F}$-vector spaces C_i if:

C_i has basis indexed by the elts W_i

$C_i \neq 0 \iff W_i \neq \emptyset$

∂ boundary map.

For $x \in W_{i-1}$, $y \in W_i$ the coeff of

$\partial_{y,x}$ of x in $\partial_i(y)$ is zero unless $x < y$.

ex. The algebraic complex $(\mathscr{C}, \partial)$ supported by the poset $\Pi(n,k)$

For $w \in \Pi(n,k)$, let

$$E(w) = \{ i : w_i \text{ even and } w_j = w_i \text{ for some } j < i \}$$

be the set of indices of repeated even entries in w.

ex $w = 122344 \Rightarrow E(w) = \{3, 6\}$.

The boundary map ∂ on elts of $\Pi(n,k)$

$$\partial(w) = \begin{cases} \sum_{j=1}^{r} (-1)^{j-1} w_1 \cdots w_{i_j - 1} (w_{i_j} - 1) w_{i_j+1} \cdots w_n, \\ \qquad \text{where } E(w) = \{ i_1 < \cdots < i_r \} \text{ and } w \notin \mathcal{A}(n,k) \\ 0, \qquad \text{if } w \in \mathcal{A}(n,k). \end{cases}$$

ex. (cont'd)

$w = 122344$

$E(w) = \{3, 6\}$

$\partial(w) = 121344 - 122343$

Lemma: $\partial^2 = 0$.

Algebraic Morse Theory.

See [Kozlov 2005, Sköldberg 2006,
Jöllenbeck - Welker 2009].

P poset
Orient edges in Hasse diagram downwards.

A partial matching is a subset $M \subseteq P \times P$ s.t.

i. $(a,b) \in M \Rightarrow a \lessdot b$

ii. Each elt $a \in P$ belongs to at most one elt in M.

For $(a,b) \in M$ write $b = u(a)$, $a = d(b)$ "up" "down".

A partial matching is acyclic if there are no cycles in the directed Hasse diagram.

Matching M on $\Pi(n,k)$:

Let π_i be first entry in $\pi = \pi_1 \cdots \pi_n \in \mathcal{R}(n,k)$ s.t. π is weakly decreasing:

$$\pi_1 \leq \pi_2 \leq \ \cdots \leq \pi_{i-1} \geq \pi_i \ \cdots$$

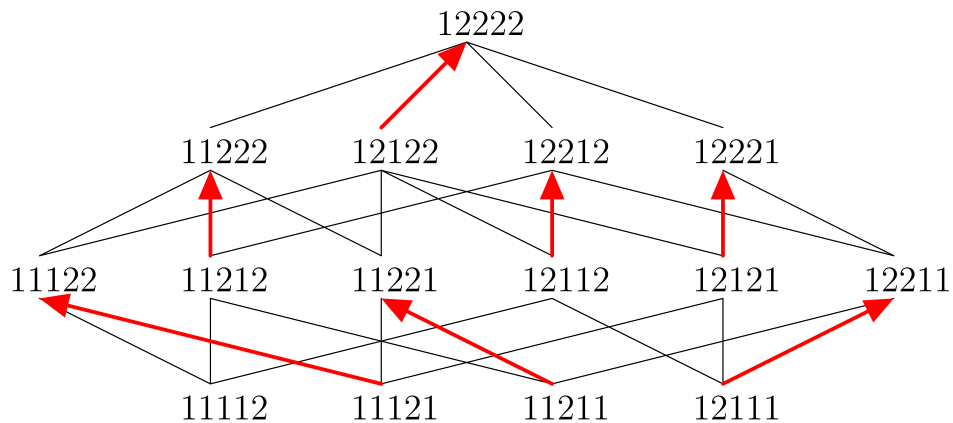
and $\pi_{i-1} \geq \pi_i$ is strict unless both π_{i-1} & π_i even.

For π_i even:

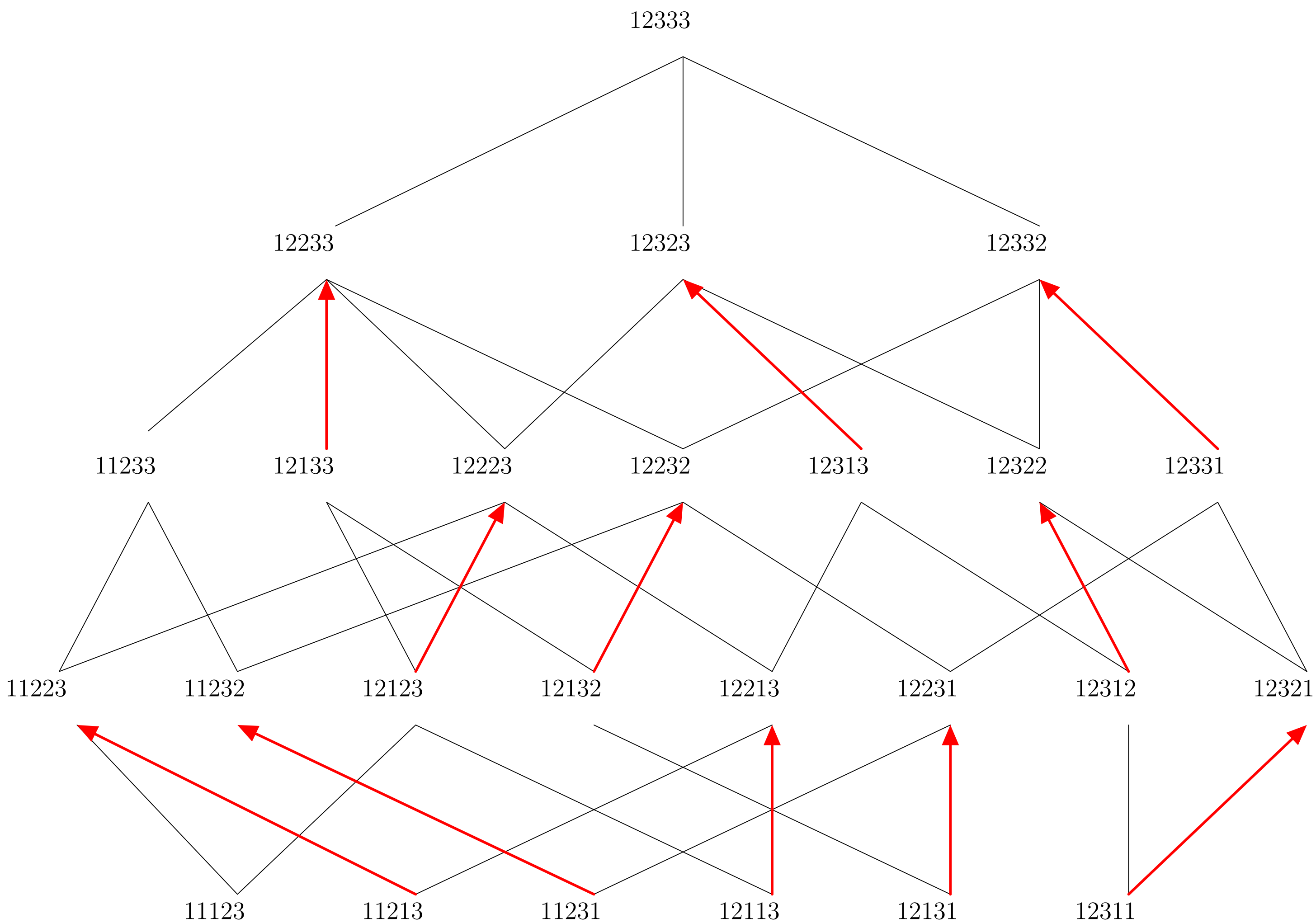
$$d(\pi) = \pi_1 \, \pi_2 \cdots \pi_{i-1} \, (\pi_i - 1) \, \pi_{i+1} \cdots \pi_n .$$

For π_i odd.

$$u(\pi) = \pi_1 \, \pi_2 \cdots \pi_{i-1} \, (\pi_i + 1) \, \pi_{i+1} \cdots \pi_n .$$



The matching.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Lemma: The unmatched words in $\Pi(n,w)$ are of the form

$$1 \cdots 1 \; 2 \; 3 \cdots 3 \; 4 \; 5 \cdots 5 \; 6 \cdots$$

Lemma: Let a and b be two distinct elements in the Stirling poset of the second kind $\Pi(n,w)$ s.t.

$$a \prec u(a) \;\&\; b \prec u(b).$$

Then $a >_{lex} b$.

Theorem: [Cai - Readdy].
The matching described for $\Pi(n,w)$ is an acyclic matching.

Lemma: [Hersh – Shareshian – Stanton].

P graded poset supporting an algebraic complex $(\mathcal{C}, \partial)$.

Assume P has a Morse matching M s.t. for all $q = M(p)$ with $q < p$ one has $\partial_{p,q} \in \mathbb{F}^{\times}$.

If all unmatched elts occur in ranks of the same parity then.

$\dim H_i(\mathcal{C}, \partial) = |P_i^{\text{un}M}|$, that is, the # of unmatched elts of rank i.

Lemma: The weighted generating function of the unmatched words in $\Pi(n,k)$ is given by the q^2-binomial coefficient

$$\sum_{w \in \mathcal{U}(n,k)} wt(w) = \begin{bmatrix} n-1-\lfloor k/2 \rfloor \\ \lfloor k-1/2 \rfloor \end{bmatrix}_{q^2}.$$

Theorem: [Cai- Readdy]
The algebraic complex $(\mathcal{E}, \partial)$ supported by $\Pi(n,k)$ has basis for homology given by the increasing allowable RG-words in $\mathcal{A}(n,k)$.
Furthermore

$$\sum_{i \geq 0} \dim(H_i)\, q^i = \begin{bmatrix} n-1-\lfloor k/2 \rfloor \\ \lfloor k-1/2 \rfloor \end{bmatrix}_{q^2}.$$

q-Stirling number of the first kind

$$c[n,k] = c[n-1,k-1] + [n-1]\,c[n-1,k]$$

with $c[n,0] = \delta_{n,0}$.

Recall Stirling number $c(n,k)$ counts # $\pi \in \mathfrak{S}_n$ with k disjoint cycles.

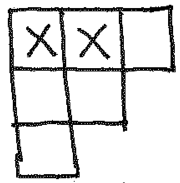
Theorem: [de Médicis - Leroux].

$$c[n,k] = \sum_{T \in \mathcal{P}(n-1,\,n-k)} q^{s(T)}$$

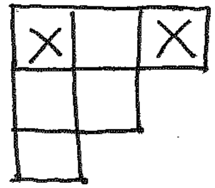
$\mathcal{P}(m,n)$ = set of ways to place n rooks on a length m staircase board with no two rocks in same column.

For $T \in \mathcal{P}(m,n)$, $s(T)$ = # of squares to the south of the rocks in T.

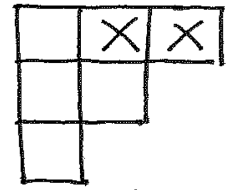
ex. $c[4,2] = q^3 + 3q^2 + 4q + 3$



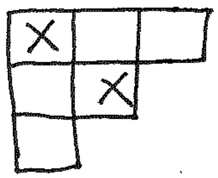
q^3



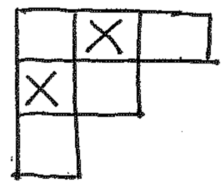
q^2



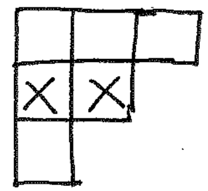
q^1



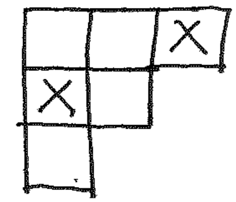
q^2



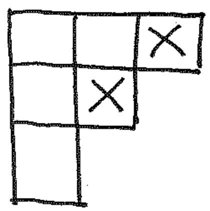
q^2



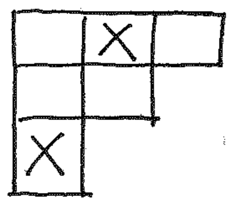
q^1



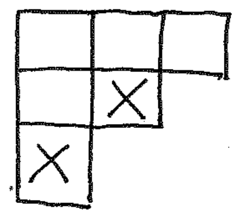
q^1



$1 = q^0$



q^1



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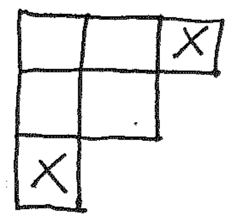

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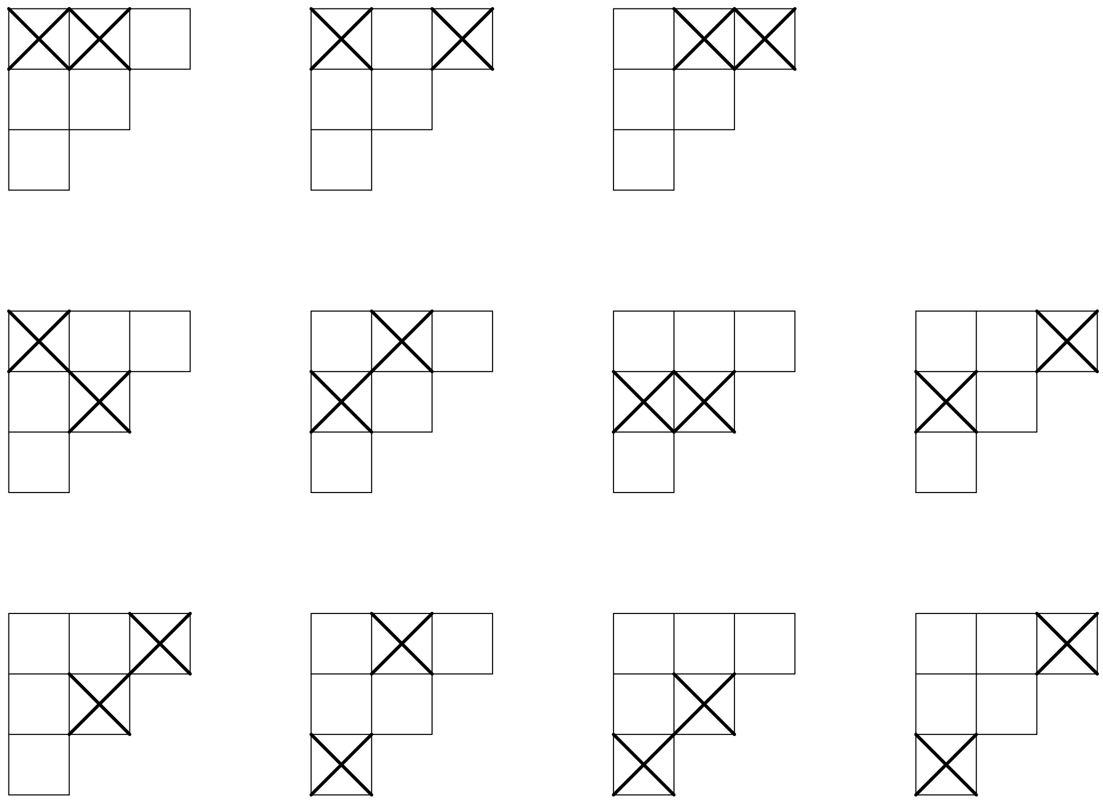
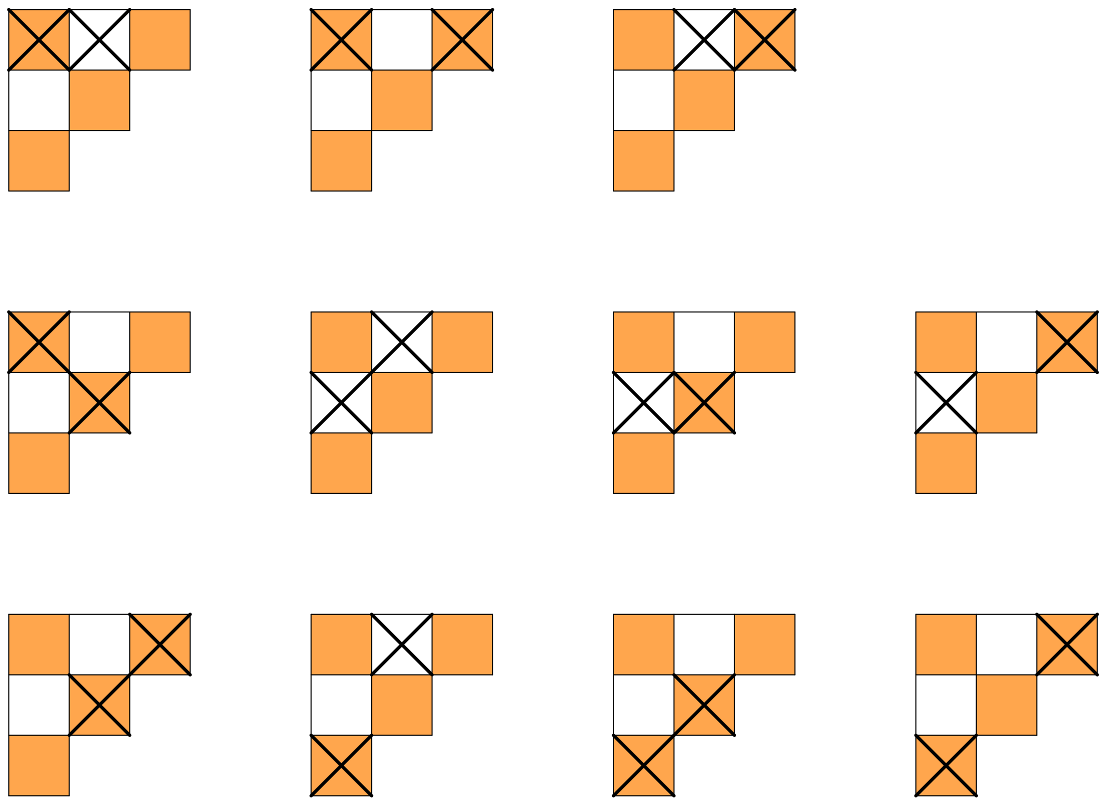
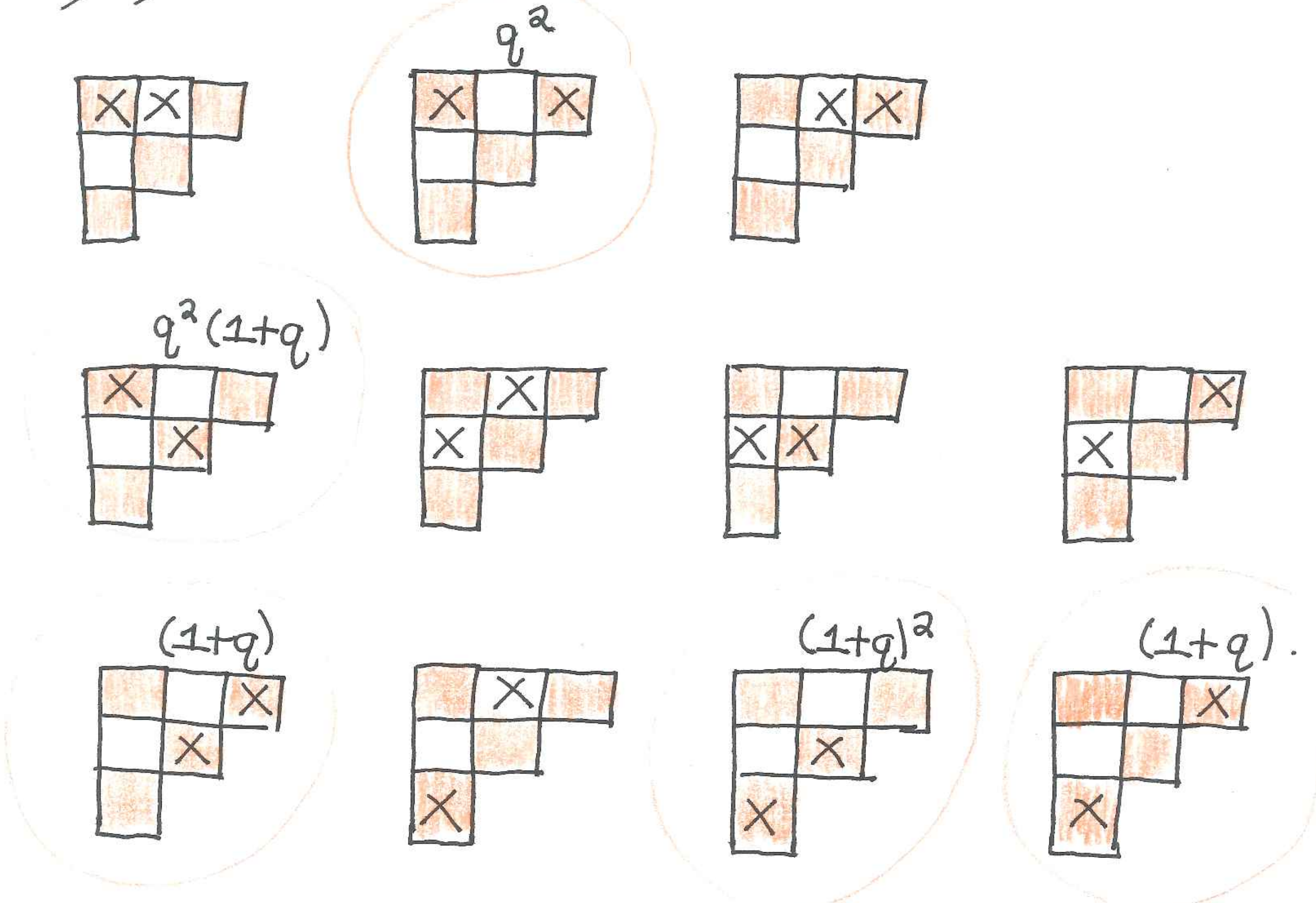
Find a subset $Q(n-1, n-k)$ of $P(n-1, n-k)$:



Consider a checkerboard coloring

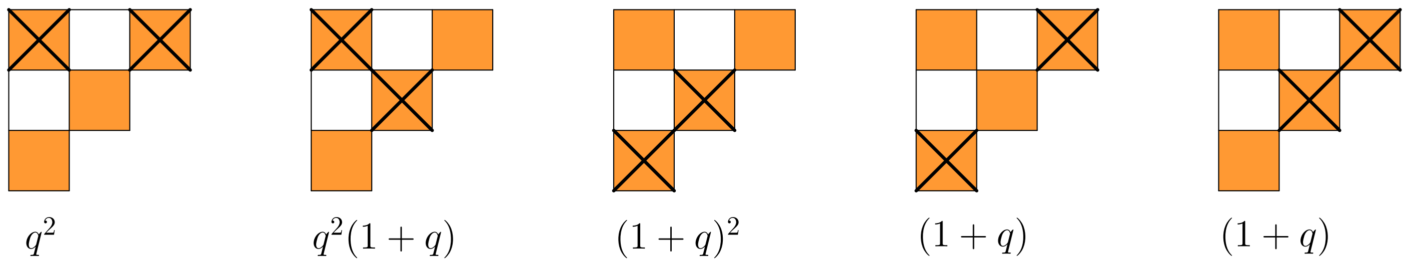


To find a ~~subset~~ $Q(n-1, n-k)$ of $P(n-1, n-k)$:



$$C[4,2] = q^2(1+q) + (1+q)^2 + q^2 + 2\cdot(1+q)$$

$$\overset{\checkmark}{=} q^3 + 3q^2 + 4q + 3.$$



Computing the q-Stirling number of the first kind $c_q[4,2]$ using $\mathcal{Q}(4,2)$.

Theorem: [Cai- Readdy].

$$c[n,k] = \sum_{T \in Q(n-1,n-k)} q^{s(T)} (1+q)^{r(T)}$$

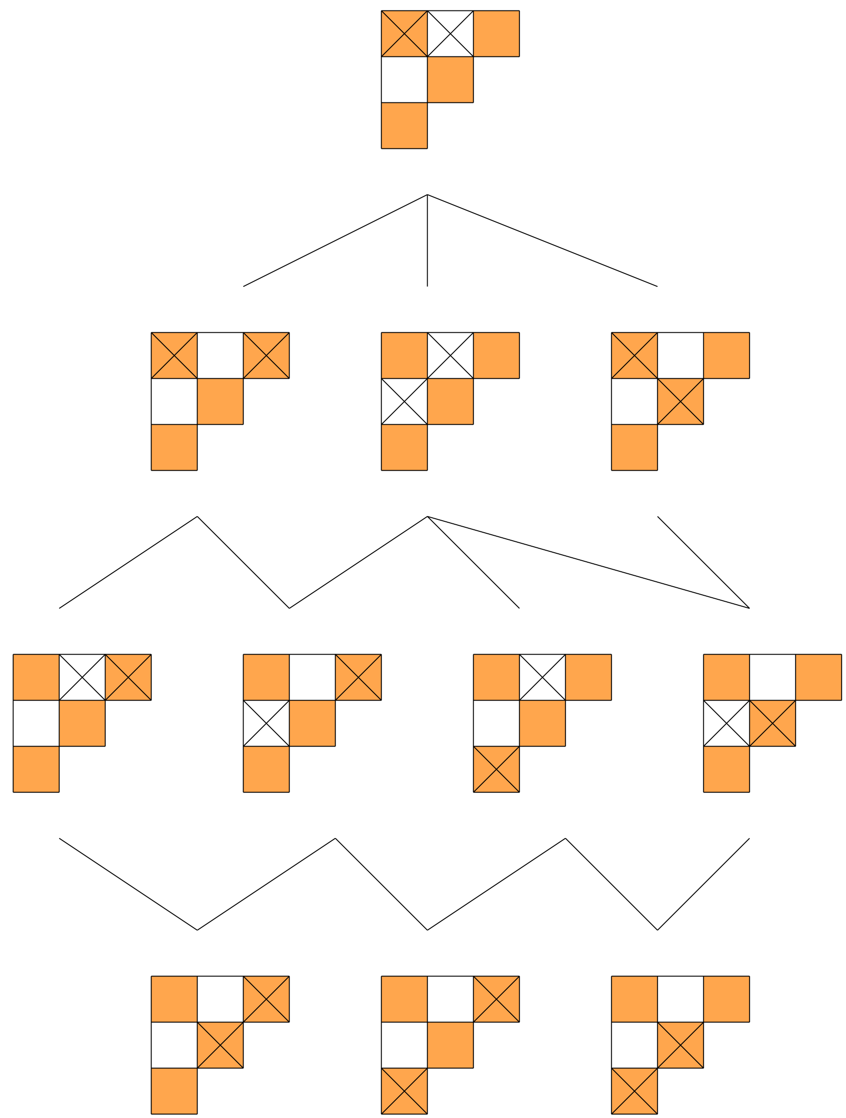
where $Q(n-1, n-k) \subseteq \mathcal{P}(n-1, n-k)$ are rook placements on the alternating shaded staircase board (shaded alternatingly starting from lowest diagonal),

$s(T)$ = # squares to the south of the rooks in T

$r(T)$ = # rooks not in first row.

The Stirling poset of the first kind $\Gamma(m,n)$

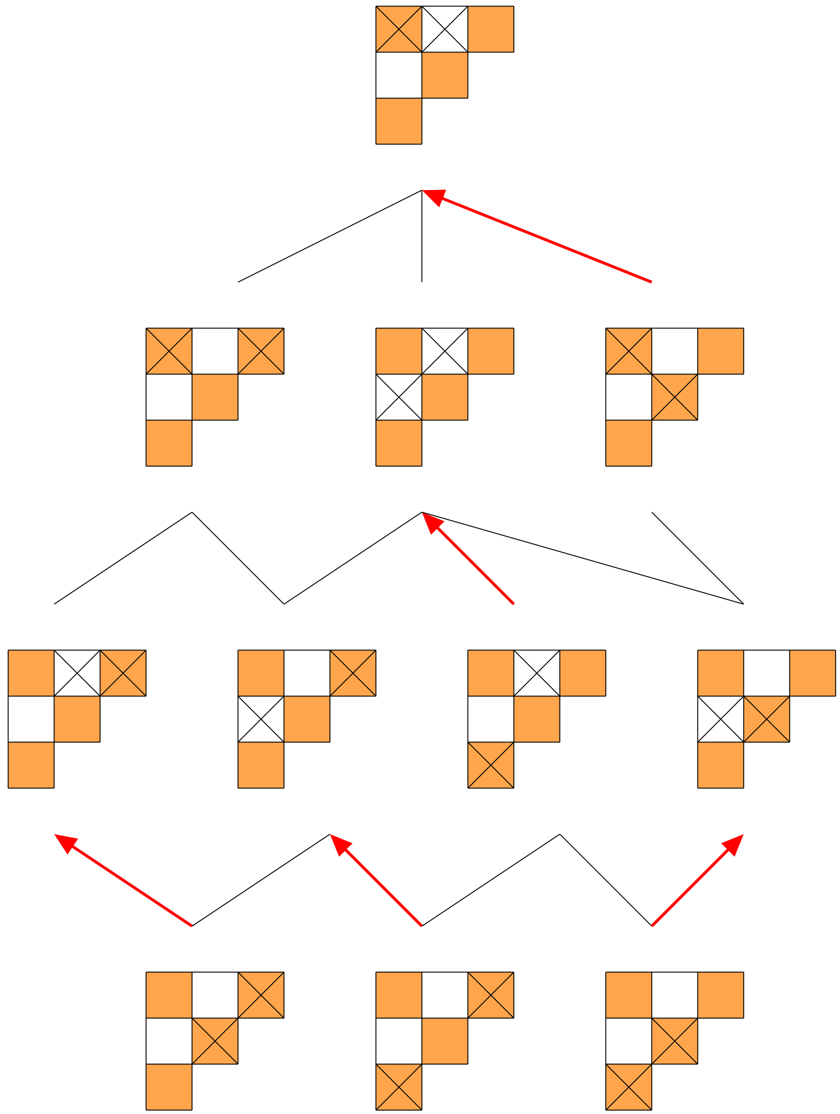
For $T, T' \in \mathcal{P}(m,n)$ let $T \lessdot T'$ if T' can be obtained from T by moving one rook to the left (west) or up (north)



Define a matching m:

For $T \in P(m,n)$, let r be the first rook
(reading left to right) that is not in a
shaded square in first row.

Match T to T' where T' is obtained
from T by moving r one square down if
r is not in a shaded square, or one square
up if r is in a shaded square but not in
first row.



Lemma: The unmatched rook placements in $\nabla(m,n)$ have all of the rooks occur in shaded squares in the first row.

Theorem: [Cai - Readdy].

i. The matching described for $\nabla(m,n)$ is acyclic.

ii. $$\sum_{\substack{T \in \nabla(m,n) \\ T \text{ unmatched}}} wt(T) = q^{n(n-1)} \begin{bmatrix} \lfloor m+1/2 \rfloor \\ n \end{bmatrix}_{q^2}$$

For $T \in \mathcal{P}(m,n)$, let

$N(T) = \{r_i :$ the rook r_i in T is not in a shaded square $\}$.

$I(T) = \{i_j : r_{i_j} \in N(T)$ and $i_1 < i_2 < \cdots < i_{|N(T)|}\}$.

The boundary map ∂ on $\mathcal{P}(m,n)$:

$$\partial(T) = \sum_{r_{i_j} \in N(T)} (-1)^{j-1} T_{r_{i_j}}.$$

where $T_{r_{i_j}}$ is obtained by moving the rook r_{i_j} in T down by one square.

Theorem: [Cai–Readdy]

The algebraic complex $(\mathcal{E}, \partial)$ supported by $\Gamma(m,n)$ has basis for homology given by the rook placements in $\mathcal{P}(m,n)$ having all of the rooks occur in shaded squares in the first row.

Furthermore,

$$\sum_{i \geq 0} \dim(H_i)\, q^i = q^{n(n-1)} \begin{bmatrix} \lfloor m+1/2 \rfloor \\ n \end{bmatrix}_{q^2}.$$

Orthogonality

Recall the signed q-Stirling numbers of the first kind.

$$s_q[n,k] = (-1)^{n-k}\, c[n,k].$$

Known generating polynomials.

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k]\, x^k$$

$$x^n = \sum_{k=0}^{n} S_q[n,k]\, (x)_{k,q}$$

where

$$(x)_{n,q} = \prod_{m=0}^{n-1} (x - [m]_q).$$

def. Define the (q,t) Stirling numbers of the first and second kind by

$$s_{q,t}[n,k] = (-1)^{n-k} \sum_{T \in Q(n-1,n-k)} q^{s(T)} t^{r(T)}$$

$$S_{q,t}[n,k] = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot t^{B(w)},$$

respectively, where $t = q+1$.

Let

$$[k]_{q,t} = \begin{cases} (q^{k-2} + q^{k-4} + \dots + 1)\cdot t & \text{for } k \text{ even} \\ q^{k-1} + (q^{k-3} + q^{k-5} + \dots + 1)\,t & \text{for } k \text{ odd.} \end{cases}$$

Theorem: [Cai–Readdy].
The generating polynomials for the (q,t)-Stirling numbers are

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k]\cdot x^{k}$$

$$x^{n} = \sum_{k=0}^{n} S_{q,t}[n,k]\,(x)_{k,q,t}$$

where

$$(x)_{n,q,t} = \prod_{m=0}^{n-1} (x - [m]_{q,t}).$$

Theorem: [de Médicis – Leroux].

The signed q-Stirling numbers $s_q[n,k]$ and the q-Stirling numbers $S_q[n,k]$ are orthogonal, that is,

$$\sum_{k=m}^{n} s_q[n,k]\, S_q[k,m] = \delta_{m,n}$$

and

$$\sum_{k=m}^{n} S_q[n,k]\, s_q[k,m] = \delta_{m,n}$$

Furthermore, this orthogonality holds bijectively.

Theorem: [Cai – Readdy].

The (q,t)-Stirling numbers are orthogonal, that is,

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \delta_{m,n}$$

and

$$\sum_{k=m}^{n} S_{q,t}[n,k] \, s_{q,t}[k,m] = \delta_{m,n}.$$

Furthermore, this orthogonality holds bijectively.

Thank you!