q-Combinatorics: A new view

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Let's count \[ \approx 50,000 \text{ BC}^* \]

\[
\sum_{\pi \in \mathcal{S}_n} 1 = n!
\]

\[
\sum_{S \subseteq \{1, \ldots, n\}} 1 = \binom{n}{\omega}.
\]

\[ |S| = \omega \]

* Source: Wikipedia*
Let's \( q \)-count \( [1700's \text{ Euler}^*] \).

\( q \)-analogue of \( n \in \mathbb{Z}^+ \)

\[
[n]_q = [n] = 1 + q + \ldots + q^{n-1},
\]

\( q \) an indeterminate.

\[
\lim_{q \to 1} \frac{[n]_q}{n} = 1 + \ldots + 1 = n.
\]

\[
[n]_n = [n] [n-1] \ldots [2] [1].
\]

* Theta function \( \varphi(n) = \sum_{a \in \mathbb{Z}} (\frac{n}{a}) \frac{(n)}{a} \)

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^n b^n,
\]

\(|a| |b| < 1 \).
Combinatorial interpretation

[MacMahon 1916]

\[
\sum_{\pi \in S_n} q^{\text{inv} (\pi)} = [n]!,
\]

where

\[
\text{inv} (\pi) = \# \{ (i,j) : i < j \text{ and } \pi_i > \pi_j \}
\]

for \( \pi = \pi_1 \ldots \pi_n \in S_n \).
Gaussian polynomial. (the q-binomial)

\[
[n]_q = \begin{cases} 
\frac{[n]!}{[q^n]! [n-q^n]!} & 0 \leq q^n \leq n \\
0 & q^n < 0 \quad q^n > n.
\end{cases}
\]

C母 interpretation.

\[
\sum_{\eta \in \mathcal{G}(1,0^n-q^n)}^{\text{inv } \eta} \eta = [n]_q.
\]

[MacMahon 1916]
\[ \left[ \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right]. \]

<table>
<thead>
<tr>
<th>\eta_1</th>
<th>\eta_2</th>
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<tbody>
<tr>
<td>0011</td>
<td>0</td>
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<tr>
<td>0101</td>
<td>1</td>
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<td>0110</td>
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<td>1010</td>
<td>3</td>
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<tr>
<td>1100</td>
<td>4</td>
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\[
\sum_{\eta \in \{0,1 \}^2} q^{\text{inv } \eta} = q^4 + q^3 + 2q^2 + q + 1.
\]

Check \( \left[ \begin{array}{c} 4 \\ a \end{array} \right] = \frac{\left[ 4 \right] \left[ 3 \right]}{\left[ a \right]} = \frac{(1+q)(1+q^a)(1+q+q^2)}{(1+q)} \).
The negative $q$-binomial \cite{Fw-Reiner-Stanton-Thiem, 2012}

def. \[
\begin{aligned}
\left[\begin{array}{c}
\underline{\nu} \\
\underline{\nu}
\end{array}\right]_q^\prime & \equiv (-1)^{\underline{\nu}(n-\underline{\nu})} \left[\begin{array}{c}
\nu \\
\nu
\end{array}\right]_{-q} \\
\end{aligned}
\]

\text{ex.} \quad \left[\begin{array}{c}
4 \\
2
\end{array}\right]_q^\prime = q^4 - q^3 + 2q^2 - q + 1.
Theorem: \[ \text{Fu - Reiner - Stanton - Thiem}. \]

\[
\left[ \begin{array}{c} n \\ \psi \end{array} \right] q = \sum_{w \in \Omega(n, \psi)} w + (w) \\
= \sum_{w \in \Omega(n, \psi)} q^{\alpha(w)} (q-1)^{p(w)}
\]

where \( \Omega(n, \psi) \) is a certain subset of \( \mathfrak{S}_n \times \mathfrak{S}_{n-\psi} \),

\( p(w) \) = number of 10 pairs in \( w \),

\( \alpha(w) = \text{inv}(w) - p(w) \).
Corollary: \([F-R-S-T]\)

The \(q\)-binomial can be expressed as

\[\left[\begin{array}{c}
\binom{n}{k}
\end{array}\right]_q = \sum_{w \in \Omega(n,k)} q^{a(w)} (1 + q)^{p(w)}.\]
def. Given $w = w_1 \cdots w_n \in \{1, 2, \ldots, n - 1\}$, pair

t. $n = 1$. Leave letter unpaired.

tt. $n > 2 + u$ odd: Pair $\underline{w_1 w_2}$
Repeat on $w_3 \cdots w_n$

ttt. $n > 2 + u$ even: Pair $w_1$.
Repeat on $w_2 \cdots w_n$.

ex. \[0110010101\]
\[110001001\]
Define

\[ q_{n,k} = \begin{cases} \text{true} & \text{if } w \in \mathcal{L}(A_k) \land w \text{ has no paired } 01 \gamma. \\
\text{false} & \text{otherwise} \end{cases} \]

ex. \[ \gamma \]

\begin{align*}
0011 & \quad \text{No.} \\
0101 & \quad \text{No.} \\
0110 & \quad \text{No.} \\
1001 & \\
1010 & \quad \text{No.} \\
1100 & \\
\end{align*}
\[ \sum = 1 + (q + q^3)(1 + q) + q^2 \]

\[ = q^4 + q^3 + 2q^2 + q + 1. \]

Recall

\[ w^+(w) = q \cdot (1 + q)^{p(w)} \]

\[ p(w) = \# \text{ of pairs in } w, \quad a(w) = \]

\[ a(w) = \text{inv}(w) - p(w). \]
20: What about other combinatorial objects with q-analogues?
Goal: Given a \( q \)-analogue
\[
f(q) = \sum_{w \in S} \sigma(w) q^w,
\]
for some statistic \( \sigma(\cdot) \), find a subset \( T \subseteq S \) and statistics \( A(\cdot) \) and \( B(\cdot) \) so that
\[
f(q) = \sum_{w \in T} q^w (1 + q)^{A(w)} B(w).
\]
Goal: Given

\[ f(q) = \sum_{w \in T} q^A(w) (1 + q)^B(w) \]

find poset theoretic and topological explanations.
The Stirling numbers of the second kind

\[ S(n, k) = \text{# partitions of } \{1, \ldots, n\} \text{ into } k \text{ blocks} \]

Ex. \[ S(4, 2) : \]
\[
\begin{align*}
1/234 & \quad 12/34 \\
134/2 & \quad 13/24 \\
124/3 & \quad 14/23. \\
123/4 & \\
\end{align*}
\]
(written in standard form).

The \( q \)-Stirling numbers

\[ S_q[n, k] = S_q[n-1, k-1] + [k] S_q[n-1, k] \]

with \[ S_q[n, n] = 1 = S_q[n, 1]. \]
RG-words [Milne].

Encode a partition \( \nu \) using a restricted growth word \( w \).

\[ w = w_1 \cdots w_n \quad \text{where} \quad w_i = j \quad \text{if the elt. } \nu_i \text{ is in the } j\text{th block of } \nu. \]

ex. \( \nu = 125/36/47 \leftrightarrow 1123123. \)

Let \( \mathcal{R}(n,k) = \text{set of all RG-words which encode a set partition of } k1, \ldots, nk \text{ into } k \text{ parts.} \)
For \( \mathcal{B}(n, k) \), let
\[
wh(w) = \prod_{i=1}^{n} w_{z_i}(w),
\]
where \( m_{z_i} = \max \{ w_i, \ldots, w_{z_i} \} \),
\( w_{z_1}(w) = 1 \) and for \( 2 \leq z \leq n \)
\[
w_{z_i}(w) = \begin{cases} 
q^{w_{z_i}-1} & \text{if } w_{z_i} \leq m_{z_i-1} \\
n & \text{if } w_{z_i} > m_{z_i-1}
\end{cases}
\]

Theorem: [Cai-Rea/dy]
The \( q \)-Stirling number of the second kind is given by
\[
S_q[n, k] = \sum_{w \in \mathcal{B}(n, k)} wh(w).
\]
\[ \begin{align*}
\text{ex.} & & W & & w^+(w) \\
1/234 & & 1223 & & q' \cdot q' = q^2 \\
134/2 & & 1211 & & 1 \\
124/3 & & 1121 & & 1 \\
123/4 & & 1112 & & 1 \\
12/34 & & 1122 & & q' \\
13/24 & & 1212 & & q' \\
14/23 & & 1221 & & q' \\
\end{align*} \]

\[ \sum = q^2 + 3q + 3 \]

\[ S_q[4,2] \]
Remark: See Garin, Remmel, Milne, and especially Waichs-White for a multitude of statistics that generate $S_q [n, 6]$.

The $w(\cdot)$ statistic is related to Waichs-White's $I_s (\cdot)$ statistic.
Let \( w^t'(w) = \prod_{\varepsilon=1}^{n} w^t_{\varepsilon}'(w) \), \( m_{\varepsilon} = \max R w_{1, \ldots, w_{\varepsilon} y} \), and

\[
w^t_{\varepsilon}'(w) = \begin{cases} 
q \left(1 + q\right) & \text{if } w_{\varepsilon} < m_{\varepsilon-1} \\
q^{w_{\varepsilon}-1} & \text{if } w_{\varepsilon} = m_{\varepsilon-1} \\
1 & \text{if } w_{\varepsilon} > m_{\varepsilon-1} \text{ or } \varepsilon = 1.
\end{cases}
\]

Write \( A(w) = \sum_{\varepsilon=1}^{n} A_{\varepsilon}'(w) \) and \( B(w) = \sum_{\varepsilon=1}^{n} B_{\varepsilon}'(w) \), where

\[
A_{\varepsilon}'(w) = \begin{cases} 
w_{\varepsilon} - 1 & \text{if } w_{\varepsilon} \leq m_{\varepsilon-1} \\
0 & \text{if } w_{\varepsilon} > m_{\varepsilon-1} \text{ or } \varepsilon = 1
\end{cases}
\]

\[
B_{\varepsilon}'(w) = \begin{cases} 
1 & \text{if } w_{\varepsilon} < m_{\varepsilon-1} \\
0 & \text{otherwise}
\end{cases}
\]
Allowable RG-words

Def. An RG-word we \( w \in \alpha(n, w) \) is allowable if it is of the form

\[
\begin{array}{ccccccc}
1 & \cdots & 1 & 2 & \cdots & 4 & \cdots & 6 & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& 1, 2 & 3, 4 & 5, 6 & 7, 8 & 9, 10 & 11, 12 & 13, 14 & 15, 16 \\
\end{array}
\]

ex. \( w = 1 1 2 1 3 3 1 4 3 5 \in \alpha(10, 5) \).

\( wt(w) = 1 \cdot 1 \cdot 1 \cdot (1+q) \cdot 1 \cdot q^2 \cdot \cdots \cdot (1+q) \cdot 1 \cdot q^2 (1+q) \cdot 1 \)

Allowable words are denoted by \( \alpha(n, w) \)
\[
\begin{array}{ccc}
\text{ex.} & w & w^+(w) \\
1222 & - & \\
1211 & (1+q)^2 & \\
1121 & (1+q) & \\
1112 & 1 & \\
1122 & - & \\
1212 & - & \\
1221 & - & \\
\end{array}
\]

\[
\sum = (1+q)^2 + (1+q) + 1 \\
= q^2 + 3q + 3 \\
\subseteq [4,2]_q
\]
\[ \sum_{\mathfrak{w}} w \cdot \begin{array}{c|c} w^t(w) \\ \hline 12311 & (1+q)^2 \\ 12131 & (1+q)^2 \\ 12113 & (1+q)^2 \\ 12133 & (1+q) \cdot q^2 \\ 12313 & (1+q) \cdot q^2 \\ 12331 & q^2 \cdot (1+q) \\ 12333 & q^2 \cdot q^2 \\ 11213 & (1+q) \\ 11231 & (1+q) \\ 11233 & q^2 \cdot q^2 \\ 11123 & q^2 \\ 1 & 1 \end{array} \]

\[ \sum = q^4 + 3q^3(1+q) + q^2 + 3 \cdot (1+q)^2 + 2(1+q) + 1. \]

\[ S_q[5,3] = q^4 + 3q^3 + 7q^2 + 8q + 6 \]
Theorem: \([\text{Carleman's Additivity}]\)

\[
S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} \lambda^w \cdot (1 + q)
\]

\[
= \sum_{w \in \mathcal{A}(n,k)} A(w) \cdot B(w),
\]
Stembridge's \( q = -1 \) phenomenon

Let \( B \) be a finite set:

\[
X(q) = \sum_{b \in B} q^{\text{wt}(b)}.
\]

Set \( q = -1 \) to count fixed points in an involution.

Corollary: [Cox- Readdy]

\( S_q^q [n,w] \) when \( q = -1 \)

counts the number of weakly increasing allowable words in \( A_n(n, w) \).

Form: \( 1 \ldots 1 2 3 \ldots 3 4 5 \ldots 5 6 \ldots \)

(No \((1 + q)\) terms).
The Stirling poset of the second kind $T(n, \psi)$

For $\psi, \sigma \in \mathcal{P}_0(n, \psi)$ let $\psi \leq \sigma$ if

$$\sigma = \psi_1, \psi_2, \ldots, (\psi_{\tau}+1), \ldots, \psi_n.$$  

for some index $\tau$.

Clearly, $\psi \leq \sigma \Rightarrow wt(\sigma) = \psi \cdot wt(\psi)$.

Thus $T(n, \psi)$ is a graded poset.
The Stirling poset of the second kind $\Pi(5, 2)$. 
Theorem: [Carleheddy].

The Stirling poset of the second kind has the decomposition:

\[ \mathcal{T}(n, k) \cong \bigcup_{w \in A(n, k)} |\text{Inv}(w)| \bigcup_{w \in B} \]

where \( B \) is the Bodean algebra on \( j \) elements,
\( \text{Inv}(w) = \{ w_i : w_j > w_k \text{ for some } j < k \} \)

is the set of all entries in \( w \) that contribute to an inversion,
and \( A(n, k) \) are allowable RG-words in \( \mathcal{T}(n, k) \).
The Stirling poset of the second kind \( \Pi(5, 2) \).
The decomposition.

The weight of the poset is $1 + (1 + q) + (1 + q)^2 + (1 + q)^3$. 
Homological $q = -1$

phenomenon [Hersh - Shareghi/M - Stanton]

Claim: Stembridge's $q = -1$ phenomenon is the same Euler characteristic computation.

Idea: Define a chain complex $(\delta, \partial)$.

Ranks of chain groups are coeff in the polynomial $X(q)$.

Euler characteristic is $X(-1)$.

Also, Euler characteristic = alternating sum of ranks of homology groups.

Best scenario: $(\delta, \partial)$ has homology concentrated in ranks of same parity & have basis indexed by fixed parts of involution = $X(-1)$. 
The poset $P$ supports a chain complex $(C_i, d_i)$ of $F$-vector spaces $C_i$ if:

- $C_i$ has a basis indexed by the elements $W_i$.
- $C_i \neq 0$ if and only if $W_i \neq \emptyset$.

There is a boundary map $d_i$ for $i \in W_{i-1}$ and $y \in W_i$. The coefficient of $y, x$ of $x$ in $d_i(y)$ is zero unless $x \leq y$.
The algebraic complex \((\mathcal{C}, \partial)\) supported by the poset \(\Pi(n, \mathcal{C})\)

For \(w \in \Pi(n, \mathcal{C})\), let

\[ E(w) = \{ i : \text{ \(w_i\) even and } w_j = w_i \text{ for some } j < i \} \]

be the set of indices of repeated even entries in \(w\).

\[ w = 122344 \Rightarrow E(w) = \{ 3, 6 \} \]

The boundary map \(\partial\) on \(w\) of \(\Pi(n, \mathcal{C})\)

\[ \partial(w) = \sum_{j=1}^{n} (-1)^{j-1} w_1 \ldots w_{j-1} (w_j - 1) w_{j+1} \ldots w_n, \]

where \(E(w) = \{ \tau_1, \ldots, \tau_r \}\) and \(w \notin A(n, \mathcal{C})\)

\[ 0, \quad \text{if } w \in A(n, \mathcal{C}). \]
ex. (cont’d)

\( w = 122,344 \)
\( E(w) = 93,639 \)
\( \overline{w} = 121,344 - 122,343 \)

Lemma: \( \overline{e}^2 = 0 \).
Algebraic Module Theory.


Port

Orient edges in Hasse diagram downwards.

A partial matching is a subset $M \subseteq P \times P$ s.t.

1. $(a, b) \in M \implies a \nleq b$

2. Each elt $a \in P$ belongs to at most one elt in $M$.

For $(a, b) \in M$ write $b = w(a), \ a = d(b)$ “up” “down”.

A partial matching is acyclic if there are no cycles in the directed Hasse diagram.
Matching in $T(n, w)$:

Let $\phi_i$ be first entry in $\phi = \phi_1 \ldots \phi_n \in T(n, w)$

s.t. $\phi_i$ is weakly decreasing,

\[
\phi_1 \leq \phi_2 \leq \ldots \leq \phi_{i-1} \geq \phi_i \ldots
\]

and $\phi_{i-1} \geq \phi_i$ is strict unless both $\phi_{i-1}$ and $\phi_i$ are even.

For $\phi_i$ even:

\[
d(\phi) = \phi_1 \phi_2 \ldots \phi_{i-1} \left( \phi_i - 1 \right) \phi_{i+1} \ldots \phi_n.
\]

For $\phi_i$ odd,

\[
u(\phi) = \phi_1 \phi_2 \ldots \phi_{i-1} \left( \phi_i + 1 \right) \phi_{i+1} \ldots \phi_n.
\]
The matching.
Lemma: The unmatched words in $T(n,k)$ are of the form

$$1 \ldots 1 \ 2 \ldots 3 \ 4 \ldots 5 \ 6 \ldots$$

Lemma: Let $a$ and $b$ be two distinct elements in the Stirling poset of the second kind $T(n,k)$ s.t.

$$a < w(a) \leq b \leq w(b).$$

Then $a >_{\text{lex}} b$.

Theorem: [Can–Readdy]
The matching described for $T(n,k)$ is an acyclic matching.
Lemma: [Hersh–Shareshian–Stanton].

A graded poset supporting an algebraic complex $(E, d)$,

Assume $P$ has a Morse matching $M$ such that for all $q = M(p)$ with $q < p$, one has $d_{p, q} \in \mathbb{F}^+$.

If all unmatched elements occur in ranks of the same parity, then

$$\dim H_\ast(E, d) = |\{ p \in M \} |$$

that is, the # of unmatched elements of rank $\ast$. 
Lemma: The weighted generating function of the unmatched words in $T(n, \omega)$ is given by the $q^2$-binomial coefficient

$$\sum_{\text{well}(n, \omega)} w^+(\omega) = \left[ \begin{array}{c} n-1 - \lfloor \omega/2 \rfloor \\ \lfloor \omega-1/2 \rfloor \end{array} \right]_q.$$

Theorem: [Cai-Reasddy]

The algebraic complex $(\mathcal{C}, \partial)$ supported by $T(n, \omega)$ have bases for homology given by the increasing allowable RG-words in $A(n, \omega)$.

Furthermore

$$\sum_{\varepsilon \geq 0} \dim (H_\varepsilon) q^\varepsilon = \left[ \begin{array}{c} n-1 - \lfloor \omega/2 \rfloor \\ \lfloor \omega-1/2 \rfloor \end{array} \right]_q.$$
$q$-Stirling number of the first kind

\[ S[n, v] = S[n-1, v-1] + [v] \cdot S[n-1, v] \]

with \( S[n, 0] = S_n, 0 \).

Recall Stirling number \( S(n, v) \) counts \( \# \) of \( \pi \in \mathfrak{S}_n \) with \( v \) disjoint cycles.

**Theorem:** [de Médicis - Leroux].

\[ S[n, v] = \sum_{T \in \mathfrak{P}(n-1, n-v)} q^{|T|} \]

\( \mathfrak{P}(m, n) \) set of ways to place \( n \) rocks on a length \( m \) stairs, no two rocks in same column.

For \( T \in \mathfrak{P}(m, n) \), \( s(T) = \# \) of squares to the south of the rocks in \( T \).
\text{ex. } c[4, a] = q^3 + 3q^2 + 4q + 3
Find a subset $Q(n - 1, n - k)$ of $P(n - 1, n - k)$:
Consider a checkerboard coloring
To find a subset $Q(n-1, n-k)$ of $P(n-1, n-k)$:

\[\begin{align*}
q^2 \\
q^2(1+q) \\
(1+q) \\
\end{align*}\]

\[\begin{align*}
(1+q)^2 \\
(1+q)
\end{align*}\]

\[c[4,3] = q^2 (1+q) + (1+q)^2 + q^2 + 2 \cdot (1+q)\]

\[= q^3 + 3q^2 + 4q + 3.\]
Computing the \( q \)-Stirling number of the first kind \( c_q[4, 2] \) using \( Q(4, 2) \).
Theorem: [Cai-Readdly]

\[ c[n,k] = \sum_{T \in Q(n-1,n-k)} q^{s(T)} (1+q)^{r(T)} \]

where \( Q(n-1,n-k) \subseteq R(n-1,n-k) \) are rook placements on the alternating shaded staircase board (shaded alternatingly starting from lowest diagonal),

\[ s(T) = \# \text{ squares to the south of the rooks in } T \]

\[ r(T) = \# \text{ rooks not in first row} \]
The Stirling poset of 
the first kind \( \% \(m,n\) \)

For \( T, T' \in \% \(m,n\) \) let \( T \leq T' \) if 
\( T' \) can be obtained from \( T \) by moving one rook to the left (west) or up (north).
Define a matching \( m \):

For \( T \in P(m,n) \), let \( r \) be the first rock (reading left to right) that is not in a shaded square in first row.

Match \( T \) to \( T' \) where \( T' \) is obtained from \( T \) by moving \( r \) one square down if \( r \) is not in a shaded square, or one square up if \( r \) is in a shaded square but not in first row.
Lemma: The unmatched rock placement in $P(m,n)$ have all of the rocks occur in shaded squares in the first row.

Theorem: [Cai- Readdy].
1. The matching described for $P(m,n)$ is acyclic.
2. $\sum_{T \in P(m,n) \text{ unmatched}} \text{wt}(T) = q^{n(n-1)} \left[ \begin{array}{c} \text{LMH} \end{array} \right]_{n} q^{2}.$
For $T \in \mathcal{P}(m,n)$, let

\[ N(T) = \{ r_i : \text{the rook } r_i \text{ in } T \text{ is not in a shaded square } j \}. \]

\[ I(T) = \{ r_j : r_j \in N(T) \text{ and } r_j < r_2 < \ldots < r_i \in N(T) \}. \]

The boundary map $\partial$ on $\mathcal{P}(m,n)$:

\[ \partial(T) = \sum_{r_j \in N(T)} (-1)^{j-1} T_{r_j}. \]

where $T_{r_j}$ is obtained by moving the rook $r_j$ in $T$ down by one square.
Theorem: [Cai- Readdy]

The algebraic complex $(\mathcal{C}, d)$ supported by $P(m,n)$ has basis for homology given by the rock placements in $Q(m,n)$ having all of the rocks occur in shaded squares in the first row.

Furthermore,

$$
\sum_{i \geq 0} \text{dim} \left( H_i \right) q^i = q^{n(n-1)} \left[ \begin{array}{c}
\frac{m+n-1}{2} \\
\end{array} \right] q^2.
$$
Orthogonality

Recall the signed $q$-Stirling numbers of the first kind.

\[ s_q[n, \ell] = (-1)^{n-\ell} \, c[n, \ell], \]

Known generating polynomials.

\[ (x)_n, q = \sum_{\ell=0}^{n} s_q[n, \ell] \, x^\ell \]

\[ x^n = \sum_{\ell=0}^{n} s_q[n, \ell] \, (x)_\ell, q \]

where

\[ (x)_n, q = \prod_{m=0}^{n-1} \left( x - [m]_q \right). \]
Define the \((q,t)\) Stirling numbers of the first and second kind by

\[
\begin{align*}
S_{q,t}^{(n,\nu)} &= (-1)^{n-\nu} \sum_{T \in \mathcal{Q}(n-1, n-\nu)} \sigma(T) r(T) \\
S_{q,t}^{(n,\nu)} &= \sum_{\text{\text{wed}}(n,\nu)} A(w) B(w)
\end{align*}
\]

respectively, where \(t = q + 1\).
Let
\[
[k]_{q,t} = \begin{cases} 
(q^{k-2} + q^{k-4} + \ldots + 1) \cdot t & \text{for } k \text{ even} \\
q^{k-1} + (q^{k-3} + q^{k-5} + \ldots + 1) \cdot t & \text{for } k \text{ odd.}
\end{cases}
\]

Theorem: [Cai–Readdy].
The generating polynomials for the \((q, t)\)-Stirling numbers are

\[
(ax)_{n,q,t} = \sum_{k=0}^{n} S_{q,t}[n,k] \cdot ax^{k}
\]

\[
x^n = \sum_{k=0}^{n} S_{q,t}[n,k] \cdot (ax)^{k,q,t}
\]

where

\[
(ax)_{n,q,t} = \prod_{m=0}^{n-1} (ax - [m]_{q,t}).
\]
Theorem: [de Médicis - Leroux].
The signed $q$-Stirling numbers $s_q[n, k]$ and the $q$-Stirling numbers $S_q[n, k]$ are orthogonal, that is,
\[ \sum_{k=m}^{n} s_q[n, k] S_q[k, m] = S_{m, n} \]
and
\[ \sum_{k=m}^{n} S_q[n, k] s_q[k, m] = S_{m, n} \]
Furthermore, this orthogonality holds bijectively.
Theorem: [Car- Readdy].

The \((q,t)\)-Stirling numbers are orthogonal, that is,

\[
\sum_{n=1}^{\infty} S_{q,t}^{[n,v]} \cdot S_{q,t}^{[v,m]} = S_{m,n}
\]

and

\[
\sum_{n=1}^{\infty} S_{q,t}^{[n,v]} \cdot S_{q,t}^{[v,m]} = S_{m,n}.
\]

Furthermore, this orthogonality holds bijectively.
Thank you!