

1 Introduction to Financial Mathematics

1.1 Introduction

The mathematical sophistication of finance has increased dramatically over last 50 years.

While software can help perform routine calculations, financiers still need to develop a deep understanding of the mathematical principles of finance. Computers and calculators are great at doing mindless computation, but they cannot replace a thinking human being!

Technology Recommendations: Familiarize yourself with at least one of these.

- (a.) Finance calculator, like TI's BA II or BA II plus
- (b.) Finance menu on the TI-83 or TI-84, or similar calculators
- (c.) Spreadsheet, like MS Excel, OpenOffice Calc, etc.

On the exam, you will be allowed to leave most answers in terms of formulas and unevaluated expressions, but for homework and studying, you will find it best to simplify the formulas and expressions as completely as possible.

1.2 Time Value of Money

A dollar today is not the same a dollar tomorrow, and certainly a dollar in 1920 is not the same as a dollar today. The fact that the value of money changes over time, and the mathematical techniques used to cope with this changing value is referred to as *the time value of money*, sometimes abbreviated to TVM.

Suppose that you have \$500 and your friend needs \$500. You agree to lend \$500 to your friend for one year, but you require your friend to pay back \$520. The fact that the amount your friend must pay back is larger than the amount he borrowed is an example of the time value of money. The discrepancy between the two amounts, $\$520 - \$500 = \$20$ is the *interest*.

There are several plausible reasons for explaining *why* money has a time value.

- (a) Inflation - general price levels tend to increase over time, so the interest is required to keep up with the increase in price levels.
- (b) Risk Premium - there is a risk that your friend won't pay you back, so the interest is used as compensation for the risk that a loan is not repaid.
- (c) Utility/Opportunity Cost - by lending your friend money, you are forfeiting the right to spend that money over the next year, and interest is used to compensate you for any missed spending opportunities.

While each of these three contribute to the time value of money, the *utility/opportunity cost* explanation is the most accurate: money continues to have a time value even when inflation and risk are non-existent. The fact that money has a time value means we need to be careful when comparing dollar amounts.

Money can be surprisingly abstract to work with. It may help to look at a more familiar example before going deeper into financial mathematics.

Example 1. *Paris, KY, to Lexington, KY, is 29.5 kilometers. Harrodsburg, KY, to Lexington, KY, is 31.8 miles.*

- (a.) *Are these distances roughly the same, since 29.5 and 31.8 are close?*
- (b.) *Is the total distance $29.5 + 31.8 = 61.3$?*

Solution:

- (a.) NO! The distances are measured in different units, so we cannot make a direct comparison.
- (b.) 61.3 what? We can't directly add miles and kilometers.

We can only directly compare two numerical values if they are measured in the same units of measure. We can only directly add two numerical values if they are measured in the same units of measure.

In order to compare two numerical values measured with different units of measure, it is necessary to convert the numerical values to a common unit of measure.

Thinking back to previous courses (or more likely, resorting to a web search!) we remind ourselves that 1 mile \approx 1.61 kilometers. We can now convert the Paris to Lexington distance from kilometers to miles by dividing by 1.61: $29.5 \text{ km} \approx 29.5/1.61 \approx 18.3$ miles. Now that the two distances are measured with a common unit, we can

- (a.) compare the distances: Paris is closer than Harrodsburg since 18.3 miles is less than 31.8 miles
- (b.) add the distances: distance from Paris to Harrodsburg (by way of Lexington) is $18.3 + 31.8 = 50.1$ miles

We arbitrarily chose to convert kilometers to miles, but we could have converted the miles to kilometers instead. Using 1 mile \approx 1.61 kilometers, we convert the Harrodsburg to Lexington distance from miles to kilometers by multiplying by 1.61: $31.8 \text{ miles} \approx 31.8 \cdot 1.61 \approx 51.2$ km. Now that the two distances are measured with a common unit, we can

- (a.) compare the distances: Paris is closer than Harrodsburg since 29.5 km is less than 51.2 km
- (b.) add the distances: distance from Paris to Harrodsburg (by way of Lexington) is $29.5 + 51.2 = 80.7$ km

In fact, we could have converted both miles and kilometers to a third set of units (feet, centimeters, rods, angstroms, light years, etc.) had it been convenient. The choice of units isn't so important. What is important is that ALL of the distances be expressed in terms of a common unit.

Dollar amounts should also be tagged with a time. Keep this distance example in mind when you think about money. Time changes the value of money.

First Principle of Financial Mathematics

- (a.) The values of two monetary figures can only be directly compared if they occur at the same point in time.
- (b.) The values of two monetary figures can only be added or subtracted if they occur at the same point in time.
- (c.) In order to compare the values of two monetary figures occurring at different points in time, it is necessary to apply a conversion factor.

Many errors in basic finance arise by erroneously comparing two monetary values that occur at very different times. Financial mathematics is concerned with correctly making comparisons of monetary figures at different points in time.

It is usually safe to ignore the time value of money when dealing with short periods of time, like several days or even months. Over long periods of time (10 years, say) the effect of the time value of money can be enormous. The effect of the time value of money is even observable over moderate periods of time, like 1 year.

Example 2. *Someone wins a lottery jackpot worth \$18M. They can elect to receive either*

- (a) *\$600,000 per year for the next 30 years*
- (b) *\$9M today*

You may be tempted to think that the lump sum payments is a rip-off, since you only get half the jackpot. However, we cannot meaningfully compare these two options, at least not

directly, since they involve cash flows occurring at different points in time. The first option forms an *annuity*. We'll learn how to value annuities in a later section.

Where does this \$18M figure come from? Lottery officials simply add all of the nominal payments to obtain $\$600,000 \cdot 30 = \$18M$. The \$18M figure is essentially meaningless: Not only is it wrong to directly *compare* monetary values occurring at different times, but adding monetary values at different points in time is also a violation of the First Principle of Financial Mathematics.

The “true value” of this jackpot is the \$9M. The \$18M “value” is a nonsensical figure. Lottery officials like to use this made up figure to make the jackpot seem more impressive than it really is.

Returning to the lottery problem, the \$18M jackpot is obtained by adding $\$_{2014}600,000$ in 2014 with $\$_{2015}600,000$ in 2015 with $\$_{2016}600,000$ in 2016, etc¹.

$$\$_{2014}600,000 + \$_{2015}600,000 + \cdots \$_{2043}600,000$$

The numerical value of the sum is 18,000,000, but 18,000,000 what? Dollars in 2014 are not the same as dollars in 2015, or dollars in 2043, etc, so this sum is essentially meaningless since the sum does not involve a consistent set of units.

In order to meaningfully add all of these dollar figures, we will need to choose a specific time, perhaps January 2014, then convert each dollar amount to its January 2014 equivalent. How do we convert money from one time to another? That's what the rest of the chapter is concerned with.

¹In practice, we do not use subscripts on dollar signs to indicate the time. It is your responsibility to know when each cash flow occurs!

2 Time Value of Money, Accumulation and Discounting Factors

First Principle of Financial Mathematics

- (a.) The values of two monetary figures can only be directly compared if they occur at the same point in time.
- (b.) The values of two monetary figures can only be added or subtracted if they occur at the same point in time.
- (c.) In order to compare the values of two monetary figures occurring at different points in time, it is necessary to apply a conversion factor.

The previous section focused on the first two parts, which basically tell us “what not to do.” In practice it is necessary to make comparisons at different points in time. This section will focus on the third part, which indicates how to make the comparisons when the cash flows occur at different times.

If we wish to compare two distances, but the distances are measured in different units, say miles and kilometers, then before making that comparison it is necessary to multiply one of the distances by an appropriate conversion factor. The analogue in financial mathematics is to multiply by either an accumulation factor or a discounting factor.

Example 3. Which is better, receiving \$1,000 in 1970 or \$1,500 in 1985?

In this case units of measure are “dollars in 1970” versus “dollars in 1985.” To convert dollars from one time to another time, we need to apply either a discounting factor or an accumulation factor. If we assume money increases by 3% per year (the financial world calls this a *discount rate*² of 3% per year even though the nominal value increases), the 1970 dollars can be converted to 1985 dollars by multiplying by $(1.03)^{15}$. Thus, 1000 dollars in 1970 is equivalent to $\$1000 \cdot 1.03^{15} = \1557.97 dollars in 1985. Under the 3% per year discount rate, \$1000 in 1970 is actually more valuable than the \$1500 in 1985.

If we prefer, we could convert the 1985 dollars to 1970 dollars by multiplying by the reciprocal, $(1.03)^{-15}$. Thus, \$1500 dollars in 1985 is equivalent to $\$1500 \cdot (1.03)^{-15} = \frac{\$1500}{1.03^{15}} = \$962.79$ dollars in 1970. Again, we see that \$1000 in 1970 is more valuable than \$1500 in 1985, at least if we assume 3% per year discount rate.

²Later sections will talk about different discount rates

In fact, there is nothing special about 1970 or 1985. We could convert both figures to 2014 terms, then compare:

(a) \$1000 in 1970 is equivalent to $\$1000 \cdot 1.03^{44} = \3671.45 dollars in 2014.

(b) \$1500 in 1985 is equivalent to $\$1500 \cdot 1.03^{29} = \3534.85 dollars in 2014.

Comparing the values in 2014, we again see that the \$1000 in 1970 is more valuable than the \$1500 in 1985.

Self Check 1. *Using the same factor of 1.03 per year,*

(a.) *Determine the equivalents of \$1000 in 1970 and \$1500 in 1985 in terms of dollars in 2005.³*

(b.) *Determine the equivalents of \$1000 in 1970 and \$1500 in 1985 in terms of dollars in 1980.⁴*

Second Principle of Financial Mathematics

Let A and B represent two financial transactions occurring at possibly different times.

- (a) If, at some point in time t , the value of A valued time t is less than the value of B valued at time t , then the value of A valued at any other time s will also be less than the value of B valued at time s .
- (b) If, at some point in time t , the value of A valued time t is equal to the value of B valued at time t , then A and B will have the same value when valued at any other point in time.

This principle assures us that the time chosen to make the comparison does not matter.

The previous example illustrated the first part of this principle. As for the second, lets again consider a 3% per year discount rate and compare \$1000 in 2000 with \$1060.90 in 2002. Converting the 2000 figure to dollars in 2002:

$$\$1000 \cdot (1.03)^2 = \$1000 \cdot (1.0609) = \$1060.90.$$

Thus, the two amount are equivalent, at least when converted to dollars in 2002.

Now lets convert both amounts to dollars in 2014:

$$\$1000 \cdot (1.03)^{14} = \$1512.59$$

³In 2005, $\$_{1970}1000 = \$1000 \cdot (1.03)^{35} = \$_{2005}2813.86$ and $\$_{1985}1500 = \$1500 \cdot (1.03)^{20} = \$_{2005}2709.17$.

⁴In 1980, $\$_{1970}1000 = \$1000 \cdot (1.03)^{10} = \$_{1980}1343.92$ and $\$_{1985}1500 = \$1500 \cdot (1.03)^{-5} = \$_{1980}1293.91$.

and

$$\$1060.90 \cdot (1.03)^{12} = \$1512.59$$

Thus, both cash flows have the same value when valued in 2014. Indeed, you may want to check for yourself that they have the same value when valued at any time you choose.

2.1 Accumulation and Discounting Factors terminology

Going back to the previous example, we saw that to convert dollars in 1985 to dollars in 2014, we multiplied by $(1.03)^{29}$. This numerical value is an *accumulation factor*. To convert dollars in 1985 to dollars in 1970, we multiplied by $(1.03)^{-15}$. This numerical value is a *discount factor*.

- Accumulation factors and discounting factors are both used to convert dollar figures from one point in time to another.
- To convert an earlier time to a later time, you multiply by an accumulation factor.
- To convert a later time to an earlier time, you multiply by a discount factor.
- Accumulation factors and discount factors are reciprocals of each other.
- Thus, you can convert an earlier time to a later time by dividing by the discount factor (which is the same as multiplying by an accumulation factor).
- Likewise, you can convert later time to an earlier time by dividing by the accumulation factor (which is the same as multiplying by a discount factor).

2.2 Is the 3% figure realistic?

The focus on this chapter is on correctly applying techniques of financial mathematics. Whether a given discount factor is realistic, and how to determine a realistic discount factor are best addressed in courses on finance.

- In reality, discount factors may vary over different time periods.
- Discount factors may vary from country to country, currency to currency, industry to industry.
- Even different departments or divisions of a single company may use different discount factors.

2.3 Some more calculations with accumulation and discount factors

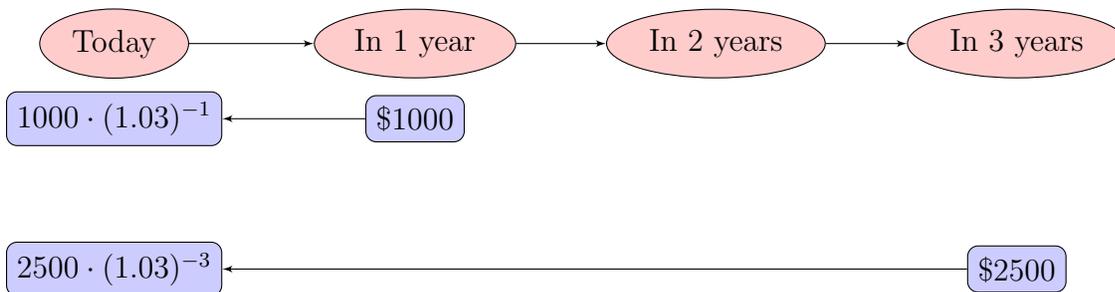
Example 4. You are to receive \$1000 one year from now, and another \$2500 three years from now. How much is this stream of cash flows worth today? How much is the cash flow worth in three years (after you receive the second cash flow)? (Assume the discount rate is still 3% per year.)

For each of the cash flows, we would like to know its value today and also in four years. The table below illustrates the time value of each cash flow. In bold are the origins of the cash flows. Each cell in a row is the accumulation factor (1.03) times the value to its left (i.e. the value one year previous).

Year	Today	In one year	In two years	In three years	In four years
First Cash Flow	\$970.87	\$1,000.00	\$1,030.00	\$1,060.90	\$1,092.73
Second Cash Flow	\$2,287.85	\$2,356.49	\$2,427.18	\$2,500.00	\$2,575.00
Total Value	\$3,258.73	\$3,356.49	\$3,457.18	\$3,560.90	\$3,667.73

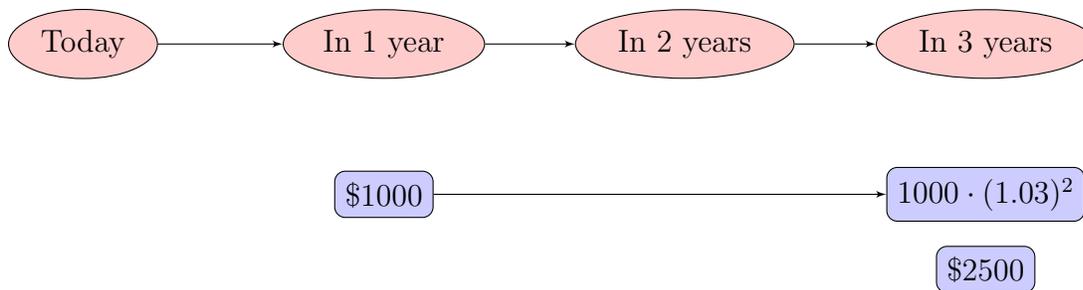
Thus, the total value of the cash flows today is

$$\begin{aligned}
 \text{Total Value Today} &= \text{First Cash Flow Value Today} + \text{Second Cash Flow Value Today} \\
 &= 1000 \cdot (1.03)^{-1} + 2500 \cdot (1.03)^{-3} \\
 &= 970.87 + 287.85 \\
 &= \$3258.73.
 \end{aligned}$$



In addition, the total value of the cash flows in three years is

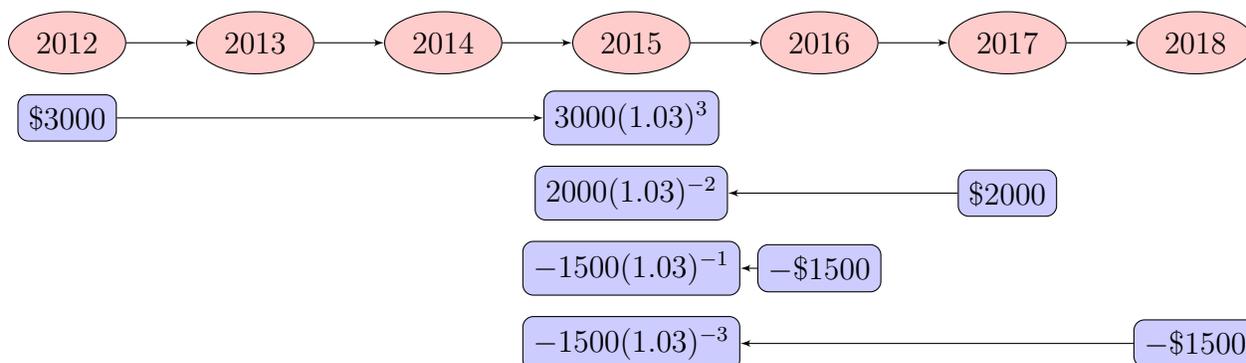
$$\begin{aligned}
 \text{Total in 3 years} &= \text{First Value in 3 years} + \text{Second Value in 3 years} \\
 &= 1000 \cdot (1.03)^2 + 2500 \cdot (1.03)^0 \\
 &= 1060.90 + 2500 \\
 &= \$3560.90.
 \end{aligned}$$



Recapping the above calculation,

- The dollar figures occur at different times, so we had to apply accumulation or discount factors before we could compare or combine.
- The solution involved a *time diagram*: Get in the habit of drawing these. They can greatly simplify very complex finance problems.
- One of the values we computed is called the *Present Value* of the cash flows. The present value represents the equivalent amount you should receive today in exchange for the staggered cash flow.
- Present value is easier to understand in terms of borrowing money: Suppose you have to pay \$1000 in one year, and another \$2500 three years from now. The present value is how much you would have to pay today in order to pay off the debt in full⁵.

Example 5. *It is January 22, 2014. You received \$3000 on January 22, 2012 and you will receive \$2000 on January 22, 2017. You will have to pay \$1500 on January 22, 2016 and another \$1500 on January 22, 2018. How much will this stream of cash flows be worth on January 22, 2015? (Assume the accumulation factor is still (1.03) per year.) (HINT: Use positive numbers for money you receive and negative numbers for money you pay out.)*



Thus the total value of this set of cash flows is

$$\$3000(1.03)^3 + \$2000(1.03)^{-2} - \$1500(1.03)^{-1} - \$1500(1.03)^{-3} = \$2334.35.$$

⁵In practice, early payoff fees may apply. Such fees will be ignored in this course.