

3 More on Accumulation and Discount Functions

3.1 Introduction

In previous section, we used $(1.03)^{\# \text{ of years}}$ as the accumulation factor. This section looks at other accumulation factors, including various forms of *compound interest*.

3.2 Interest earned versus interest rate

\$20,000 is invested for 1 year. Suppose that at the end of that 1 year, the investment is worth \$20,783. The *interest earned* is the difference between the accumulated amount and the amount invested (often called the principal). In this case, interest earned is equal to \$783. In particular, interest, or interest earned, is a *dollar* amount.

When discussing interest, it is common to speak of the percentage change, which leads to the *interest rate*:

$$\frac{783}{20,000} = 0.03915 = 3.915\%$$

Be careful to distinguish between interest earned and interest rate.

3.3 Compound interest

Compound interest involves using an exponential accumulation function. If $\$P$ is invested for t years at an APR of r year compounded annually, the accumulated value (i.e. *future value*) is

$$A = P \cdot (1 + r)^t$$

From a financial point of view, this means that compound interest involves earning interest on any interest that has already accrued, in addition to earning interest on the principal. The next worked example should help clarify this last statement.

Suppose that \$1000 is invested at 3% APR compounded annually. After one year, the accumulated value is $\$1000 \cdot (1.03)^1 = \1030 and the interest earned in the first year is $\$1030 - \$1000 = \$30$.

At the end of two years, the accumulated value is $\$1000(1.03)^2 = \1060.90 and the interest earned over the first two years is then $\$1060.90 - \$1000 = \$60.90$.

Thus, the interest earned during the second year $\$60.90 - \$30 = \$30.90$, obtained by subtracting the interest in the first year from the interest over the entire two years.

Notice the account earned more interest in the second than it had in the first year, even though the interest rate had not changed. Why did it earn more in the second year? Because the balance has changed. During the first year, the 3% interest rate was applied to the original \$1000 investment. During the second year, the 3% interest rate was applied to both the original \$1000 investment AND to the \$30 of earned interest from the first year.

Doing some algebra, we see that

$$\begin{aligned} \$1060.90 &= \$1000 \cdot (1.03)^2 \\ &= \$1000 \cdot (1.03) \cdot (1.03) \\ &= \$1030 \cdot (1.03) \\ &= \$1000 \cdot (1.03) + \$30 \cdot (1.03). \end{aligned}$$

Thus, we can think of the \$1060.90 as arising from investing \$1030 for one year at 3% APR, earning 3% on both the original principal \$1000 and the interest after one year \$30.

3.4 Compound interest with different compounding frequencies

The *compounding frequency* refers to how often interest is credited. Annual compounding means that interest is paid at the end of the year. Quarterly compounding means that interest is paid at the end of every three months. Monthly compounding means interest is paid at the end of every month.

How quickly an amount of money grows under compound interest depends on both the stated interest rate and the frequency of compounding. To see how the compounding frequency affects the accumulated value, we compare two options:

- (a.) Invest \$1000 for one year at 4% APR compounded annually.
- (b.) Invest \$1000 for one year at 4% APR compounded semiannually.

APR stands for *Annual Percentage Rate*. Under annual compounding, the accumulated value is $\$1000 \cdot (1.04)^1 = \1040 . Semiannual compounding means that interest will be paid every half of a year. In this case, interest will be paid out twice per year but each interest payment involves half of the annual percentage rate. More specifically, after the first half of the year, the account will earn 2% interest, and after the second half of the year the account will earn the other 2% interest⁶.

The second option has a larger return than the first. Why? At the end of the first six months, the \$1000 grows to $\$1000(1.02) = \1020 . During the second six months, the 2% interest rate is applied to the \$1020 so the accumulated value is $\$1020 \cdot (1.02) = \1040.40 .

⁶Notice that this is another example of adding quantities where the units are similar but different, like the lottery. You are finding 2% of two different quantities, so it does not make sense to 2%+2% to obtain 4% APR. Alas, this is the financial world's convention, so we are stuck with it.

Compound interest, compounded m times per year

Suppose $\$P$ is invested for t years at an annual interest rate of r per year compounded m times per year. We let

$$i = \frac{r}{m} \quad i \text{ is the interest rate per period}$$
$$n = m \cdot t \quad n \text{ in the number of periods}$$

Then the accumulated value, A , is given by

$$A = P \cdot (1 + i)^n$$

Equivalently,

$$A = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}$$

Example 6. $\$1200$ is invested for 5 years at 2.6% APR compounded weekly. Determine the accumulated value.

In order to use the formula, we need to identify all of the variables. $r = 0.026$, $t = 5$ and $P = 1200$. There are 52 weeks in a year⁷, so $m = 52$. Then $n = 5 \cdot 52 = 260$ and $i = \frac{0.026}{52} = 0.0005$. The accumulated value is then

$$\$1200 \cdot (1 + 0.0005)^{260} = \$1366.55$$

Third Principle of Financial Mathematics

Assume the APR of an account earning compound interest is positive. Given a fixed amount of principal invested in the account, the future value of the investment increases if at least one of the following occurs (and nothing else changes):

- (a.) the time invested increases.
- (b.) the compounding frequency increases.
- (c.) the nominal interest rate increases.

Given a fixed future value of an investment in the account, the present value of the investment decreases if at least one of the following occurs (and nothing else changes):

- (a.) the time invested increases.

⁷Pay attention to assumptions. Some banks assume there are only 50 weeks in a year. There may also be discrepancies about the number of days in one year. Banks will often assume there are 360 days in a year and that each month has exactly 30 days.

(b.) the compounding frequency increases.

(c.) the nominal interest rate increases.

3.5 Comparing interest rates and the Annual Effective Rate of Interest

In the previous example we saw that \$1000 invested for one year at 4% APR compounded semiannually results in an accumulated value of \$1040.40. Notice that \$1000 invested for one year at 4.04% APR compounded annually will also result in an accumulated value of \$1040.40, and in fact, any amount of money invested for any amount of time will have the same accumulated value whether the interest rate is 4% APR compounded semiannually or 4.04% APR compounded annually. We therefore say that 4% APR compounded semiannually and 4.04% compounded annually are *equivalent*.

The fact that two different nominal rates of interest with different compounding frequencies can result in the same returns makes it complicated to compare different interest rates.

In the same way that we can only directly compare monetary values if they occur at the same period in time, it is only meaningful to compare interest rates if they have the same compounding frequency. In order to compare interest rates with different frequencies, you must do a conversion with at least one of the interest rates. (It's the miles versus kilometers thing all over again!)

Example 7. Compare an investment at 9.8% annual rate compounded weekly to an investment at 9.9% annual rate compounded quarterly. Which is a better investment if you invest your money for exactly one year?

Investing \$1 for one year at the 9.8% annual rate compounded weekly will yield

$$\begin{aligned} \$1 \cdot \left(1 + \frac{0.098}{52}\right)^{52} &= \$1.102861 \\ \frac{\$0.102861}{\$1} &= 10.2861\% \end{aligned}$$

This means the 9.8% annual interest compounded weekly behaves just like 10.2861% annual interest compounded annually, known as the *effective annual interest rate*, r_{eff} .

Investing \$1 for one year at the 9.9% annual rate compounded quarterly will yield

$$\begin{aligned} \$1 \cdot \left(1 + \frac{0.099}{4}\right)^4 &= \$1.102736 \\ \frac{\$0.102736}{\$1} &= 10.2736\% \end{aligned}$$

This means the 9.9% annual interest compounded quarterly behaves just like 10.2736% annual interest compounded annually, yielding $r_{\text{eff}} = 10.2736\%$.

Now that both interest rates are expressed in terms of a common compounding frequency, we can now safely compare the rates. The 9.8% APR compounded weekly is a higher interest rate than the 9.9% APR compounded quarterly, despite the fact that it had a lower nominal rate.

When comparing interest rates, we usually convert all of the interest rates to their annual compounding equivalent. This can be done as above. If you prefer formulas: to find r_{eff} , the effective annual compound interest rate which corresponds to interest of r APR compounded m times per year, use

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

So what's the deal with all these different compounding frequencies anyway? And why don't all banks and lenders just express the interest in terms of the effective annual rate?

Annual effective rates are useful for making comparisons between different investments, whereas a nominal APR compounded with a frequency other than annual may be more useful for describing how the interest is being credited.

Example 8. *You are going to invest \$200 and you have two options:*

- (a) *Invest at 5.2% APR compounded quarterly.*
- (b) *Invest at 5.3023% APR compounded annually.*

What advantages, if any, does the first option have over the second?

A quick calculation shows these two interest rates are equivalent. (More specifically, the second option is the annual effective rate corresponding to the first option. Check this, and seek help immediately if you cannot confirm this calculation!)

What happens if you need to withdraw your investment after the first quarter? Technically, quarterly compounding means that interest is credited to your account at the end of each quarter and annual compound interest means interest is credited to your account at the end of each year. Under the first option, your investment will be worth $\$200 \cdot \left(1 + \frac{0.052}{4}\right) = \202.60 .

You earned interest for one full quarter. As for the second option, the result will depend on the fine print. A strict interpretation of annual compounding means that your account is worth \$200 at the end of the first quarter. It is worth \$200 at the end of the first three quarters. It is worth \$200 at the end of the first 364 days. You don't see any interest until that year is complete.

A looser interpretation of annual compounding might allow early withdrawals and fractional periods. In this case, you earned 5.3023% APR compounded annually, but you only earned at that rate for 1/4 of a year, so your accumulated value would be

$$200 \cdot (1.053023)^{1/4} = \$202.60$$

Unless explicitly stated otherwise, throughout this course we will assume the looser interpretation of compound interest for fractional periods. Thus, if \$ P is invested for n periods, earning interest of i each period, the accumulated value will be assumed to be

$$A = P \cdot (1 + i)^n$$

regardless of whether n is a whole number or not.

3.6 Continuous Compounding

Taking the previous discussion to the extreme, we can imagine evaluating the performance of an investment over extremely short periods of time. For example, suppose an account earns 4.7% APR compounded annually, and the account credits interest over fractional periods. Determine the interest earned on \$1000 over the course of a single day, and determine the equivalent interest rate expressed as an annual rate compounded daily.

$$A = 1000(1.047)^{1/365} = \$1000.125841$$

so interest earned in one day is \$0.125841. To find the equivalent annual rate compounded daily, we first find the daily rate:

$$\frac{\text{interest earned}}{\text{principal}} = \frac{0.125841}{1000} = 0.000125841 = 0.0125841\%$$

To get the nominal annual rate compounded daily, we just multiply the daily interest rate by the number of days in a year,

$$0.0125841\% \cdot 365 = 4.593197\%$$

Pushing this even further, let's consider what happens if we invest for a single hour. One hour is 1/24 of a day, so one hour is $\frac{1}{24} \cdot \frac{1}{365} = \frac{1}{8760}$ of a year.

$$A = 1000(1.047)^{1/8760} = \$1000.005243$$

so interest earned in one hour is \$0.005243. Now find the hourly interest rate:

$$\frac{\text{interest earned}}{\text{principal}} = \frac{0.005243}{1000} = 0.000005243 = 0.0005243\%$$

To get the nominal annual rate compounded hourly, we just multiply the hourly interest rate by the number of hours in a year,

$$0.0005243 \cdot 8760 = 4.5929\%$$

Notice these last two numerical interest rates are very close. Increasing the frequency of the compounding does increase the effective annual rate, but as the frequency increase infinitesimally, the amount of the increase becomes infinitesimally small (approaching a limiting interest rate, for those of you familiar with calculus). This leads to the notion of a continuously compounded interest rate. The continuously compounded rate of interest is a measure of interest earned over an infinitesimally small length of time. Basically, it is a measure of how fast the value of the investment is growing now, at this precise moment in time. For those acquainted with calculus, you may recognize that we are discussing an instantaneous rate of change, or derivative.

Continuous Compounding

If the annual effective rate of interest is r , then the continuously compounded rate of interest is given by

$$\delta = \ln(1 + r)$$

where \ln denotes the *natural logarithm*.

If $\$P$ is invested for t years at a continuously compounded interest rate of δ , then the accumulated value is

$$A = P \cdot e^{\delta t}$$

Note that some people use the term *force of interest* instead of *continuously compounded interest rate*.

Example 9.

- (a.) Determine the future value of \$2300.00 invested at 3% interest compounded continuously after 5 years.
- (b.) Determine the continuously compounded rate of interest which is equivalent to 6% APR compounded annually.
- (c.) Determine the continuously compounded rate of interest which is equivalent to 3.6% APR compounded monthly.

Solutions:

- (a.) To compute the future value of \$2300.00 with compounded continuously, we use the continuously compounding formula, with $P = 2300$, $\delta = 0.03$ and $t = 5$.

$$A = \$2300 \cdot e^{0.03 \cdot 5} = \$2300e^{0.15} = \$2672.22$$

Thus, the \$2300.00 will be worth \$2672.22 after 5 years.

- (b.) For this question, we must realize that 1.06 is the accumulation factor after 1 year for any principal. In the continuously compounding formula, $e^{\delta \cdot 1} = e^\delta$ is the accumulation factor after 1 year for any principal. So they must be equal. $e^\delta = 1.06$ so $\delta = \ln(1.06) = 0.058269$. Thus, $e^{0.058269t} = (1.06)^t$.
- (c.) This part involves two steps. We know how to find the continuously compounded rate of interest given the APR for annual compounding, so here we need to first find the annual effective rate corresponding to 3.6% APR compounded monthly.

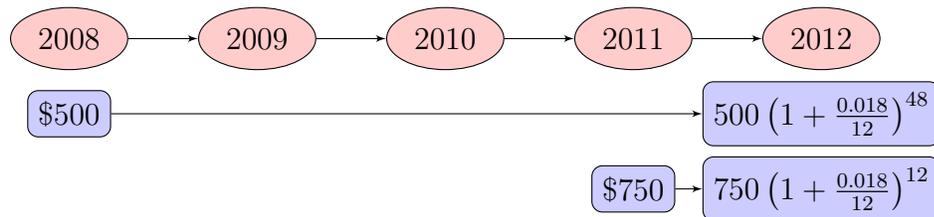
$$(1 + r_{\text{eff}}) = \left(1 + \frac{0.036}{12}\right)^{12}$$

so $r_{\text{eff}} = 3.659998\%$ is the effective annual rate. Therefore (by the same reasoning in part (b)) $e^\delta = 1.03659998$. The continuous rate is then $\delta = \ln(1.03659998) = 0.035946$, or 3.5946%.

3.7 Some more examples with compound interest

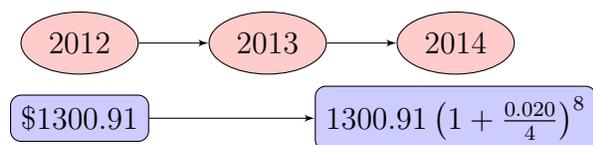
Example 10. *At the beginning of 2008, you invested \$500 in a bank account which paid 1.8% APR compounded monthly. At the beginning of 2011, you invested an additional \$750 in this same account. At the beginning of 2012, you closed this account and invested the entire balance in a different account. The new account earns 2.0% APR compounded quarterly. Determine your balance at the beginning of 2014.*

Let's organize this information with a time diagram.



The amount taken out at the beginning of 2012 is $500 \left(1 + \frac{0.018}{12}\right)^{48} + 750 \left(1 + \frac{0.018}{12}\right)^{12} \approx 1300.91$.

This amount is placed in an account for two years.



Therefore, at the beginning of 2014, your balance will be $1300.91 \left(1 + \frac{0.020}{4}\right)^8 \approx 1353.87$.

Self Check 2. *You would like to have \$1200 to pay for a trip you will take on June 1, 2014. You invested \$1142.18 on June 1, 2012 at 2.5% APR compounded annually. On June 1, 2013, the interest rate was reduced to 0.8% APR compounded annually.*

- (a) *Verify that you would have had enough money for the trip had the interest rate stayed at 2.5%*
- (b) *With the change in interest, by how much will you fall short of the \$1200 on June 1, 2014?⁸*
- (c) *In an attempt to be proactive, you decide to add money to the account on June 1, 2013 to counterbalance the decreased interest rate. How much would you need to add on June 1, 2013 in order for the accumulated value to reach \$1200 on June 1, 2014?⁹*

⁸Answer: \$19.90

⁹Answer:\$19.74