5 More on Annuities and Loans

5.1 Introduction

This section introduces *Annuities.* Much of the mathematics of annuities is similar to that of loans. Indeed, we will see that a loan and an annuity are just two ways of looking at the same financial transaction.

5.2 What is an annuity?

An annuity consists of a stream of cash flows that are paid out at regular time periods – like payments per month or payments per year, etc. Rent payments, car payments, cell phone bill payments, etc. can all be thought of constituting the regular cash flows of an annuity. Monthly paychecks you receive from a steady job can also be thought of as constituting the cash flows of an annuity.

There are many different types of annuities. The following list only covers some of the many different flavors of annuities.

- *annuity certain* versus *contingent annuities:* An annuity certain has a fixed number of cash flows. For a contingent annuity, the number of cash flows made depends on other factors. For example, a pension account makes regular payments to a retiree until he or she dies. This course will be mostly concerned with annuity certains, and as such, if you see the word annuity without either modifier certain or contingent, you should assume it is an annuity certain.

- A *level payment annuity* is an annuity in which the cash flows are all of equal size. In practice, one may run across non-level annuities. For non-level annuities, the cash flows may grow over time, they may shrink over time, they may vary smoothly over time, they may change abruptly. They may change in a nice regular manner (e.g., payment increases by \$20 each year) or they may vary in less predictable ways. This course will focus primarily on level payment annuities.

- An *ordinary annuity* is an annuity in which payments are made at the end of each period. An *annuity due* is an annuity in which payments are made at the beginning of each period. The value of an annuity due can be easily computed from the value of an ordinary annuity, so we will focus primarily on ordinary annuities in this course.

- A *simple annuity* is an annuity in which the frequency of payments is identical to the frequency of interest compounding. A *complex annuity* is an annuity in which the frequency of payments and frequency of interest compounding are different. A complex

annuity can be translated into a simple annuity by replacing the interest rate with an appropriate effective interest rate[13].

- In this course, we are mostly concerned with *ordinary simple level-payment annuity certains.* We will usually suppress all of those modifiers and just say *annuity.*

Annuities and Loans both involve cash flows that are of equal size and regularly spaced over time. How are they different?

- For a loan, we generally make the payments.

- With an annuity, we generally receive the payments.

Now suppose Jacqueline takes out a 4 year loan from Third National Kentucky Auto-Financing. She borrowed $8,000 and is to make payments of $192.31 at the end of each of the next 48 months.

Self Check 4. *Check that this corresponds to 7.2% APR compounded monthly.*

As far as Jacqueline is concerned, this is a loan: she has to make payments, spread out over time. As far as Third National Kentucky Auto-Financing is concerned, this is an annuity. They receive a stream of regular cash payments over time.

Thus, loans and annuities are really the same thing. Switching between loans and annuities just involves swapping the payer and the payee. In particular, the loan formula can also be used to value annuities,

$$P = R \cdot \frac{1 - (1+i)^{-n}}{i}$$

Example 16 (Interpreting the Present Value of an annuity)**.** *Your dad just retired. His company pension plan will pay him $2000 at the end of each month for the next 15 years. He asks if he can receive the entire balance of his pension account today, so that he can merge this with other investments. How much money should his company give him today?*

Did you say $2000 \cdot 15 \cdot 12 = \$360,000$? Why is this (probably) wrong?

His pension payments form an annuity, with $R = \$2000$ and $n = 15 \cdot 12 = 180$. He should receive the *present value* of this annuity. To determine this value, we need to know the discount/interest rate. Let assume this rate is 3.0% APR compounded monthly. The present value is then

$$P = \$2000\frac{1 - (1.0025)^{-180}}{0.0025} = \$289,610.94$$

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[13]The truth of this statement depends on how interest is credited over partial interest conversion periods.

5.3 Future value of an annuity

There are times that you wish to know the future value of an annuity, i.e. the value of an annuity at the end of the cash flow. To compute the future value of an annuity, you can compute the present value and then apply an accumulation factor of $(1 + i)^n$. This results in the following formula.

Formula for Accumulated or Future Value of an Annuity

P denotes the principal of a loan (how much was borrowed)

R denotes the payment size

t the number of years (the term of the loan)

r is the nominal interest rate per year

m is the number of conversion periods per year

i is the interest rate per period, so $i = r/m$

n is the number of conversion periods in the term, so $n = t \cdot m$

Then

$$F = R \cdot \frac{(1 + i)^n - 1}{i}$$

The future value is concerned with the value of the cash flows at the end of the term of the annuity.

In solving annuity problems, pay careful attention to the wording to decide if you should use the present value form or the future value form.

Example 17 (Computing and understanding future value of annuity). *Nick recently graduated and got his first real job. He decides to set up a retirement fund. He will deposit $1500 into this account at the end of each quarter from now until retirement. He will retire in 40 years. His retirement fund will earn 12% interest compounded quarterly. How much will his retirement fund be worth when he retires?*

Nick's retirement savings plan can be thought of as an annuity with $40 \cdot 4 = 160$ level cash flows of $1500. His quarterly interest rate will be $i = \frac{0.12}{4} = 0.03$. The value of his retirement fund when he retires will be the future value of this annuity.

$$\$1500 \cdot \frac{(1 + .03)^{160} - 1}{.03} = \$5,611,427.59$$

In the case of regular deposits into a savings account, the future value represents the overall value of the savings/investment at the end of the term.

What's the relation between the present value and future value of an annuity?

To help answer that, lets look at the present value of Nick's account.

$$\$1500 \cdot \frac{1 - (1 + 0.03)^{-160}}{.03} = \$49,558.41527$$

How much would Nick need to invest today, as a lump sum, in order to get the $\$5,611,427.59$ in 40 years?

$$\$5,611,427.59 = P(1 + 0.12/4)^{160}$$

so he would need to invest $P = \$5,611,427.59/(1.03)^{160} = \$49,558.41527$ today in order to have $\$5,611,427.59$ at the end of 40 years.

This is true in generally. The present value of an annuity can be interpreted as the amount we would need to invest now, as a single lump sum in order to get the same accumulated value as that annuity. If we know the future value of an annuity and we wish to find the present value of the annuity, we need only apply the appropriate discount factor. Likewise, if we know the present value and we wish to find the future value, we need only apply the correct accumulation factor.

The idea of the calculation allows us to solve problems that can't be solved by just plugging directly into formulas. Suppose that Nick will retire in 40 years, and he will stop making deposits into his retirement fund upon retiring but he determines that he won't need to start drawing off the retirement account until 48 years from now. Determine the FV of this retirement account 48 years from now.

His retirement account is acting like an annuity for the first 40 years, but then behaves like a lump sum investment for the remaining 8 years.

$$\$1500 \cdot \frac{(1 + 0.12/4)^{160} - 1}{.03} = \$5,611,427.59 = \text{FV after 40 years.}$$

This lump sum will need to grow for another 8 years.

$$\$5,611,427.59 \cdot (1 + 0.12/4)^{8 \cdot 4} = \$14,449,890.42.$$

These FV figures are huge. Are they realistic? The figures are huge partly because we are dealing with a very long time period, but also because the interest rate is high. The real question is whether this interest rate is unreasonably large. Financial history shows that a diversified portfolio of large stocks over a long period of time will have an average annual return of around 12%. Of course, with stocks, this means that one year could see a loss of 12% and the next year could then see a gain of 36%, etc.

5.4   Additional Annuity Examples

Example 18. *Tina wants to evaluate her retirement plan. She plans on investing $\$3000$ at the end of each year for the next 10 years. She then plans on investing $\$5500$ at the end*

of each year for another 35 years. She estimates that her investments will earn 9% APR compounded annually. Determine the future value of her retirement account.

We can't directly use an annuity formula, as the payment sizes change. The trick is to view this as two separate annuities, and the most obvious way to do this is to consider a ten year annuity followed by a 35 year annuity.

$$\$3000 \cdot \frac{(1.09)^{10} - 1}{0.09} = \$45,578.78915$$

$$\$5500 \cdot \frac{(1.09)^{35} - 1}{0.09} = \$1,186,409.20$$

Now we just add them, right? WRONG!!!

These two values do not occur at the same point in time! Indeed, the $\$45,578.79$ is the value of the $\$3000$ cash flow annuity at the BEGINNING of the term of the $\$5500$ cash flow annuity. We want the future value of the retirement account, so we want the value at the end of the term of the $\$5500$ cash flow annuity, and so the $\$45,478.79$ must accumulate interest for another 35 years:

$$\$45,478.78915 \cdot (1.09)^{35} = \$930,443.94.$$

The two values $\$930,443.94$ and $\$1,186,409.20$ now both occur at the correct point in time so we may now add them: $\$2,116,853.14$.

There is a different way we could have decomposed Tina's payments into two annuities which would have prevented us from having to worry about that annoying factor:

First, there is an annuity with cash flows of $\$3000$ at the end of each year for the full 45 years. Then there is second annuity with cash flows of $\$2500$ at the end of each year for the last 35 years. Over the last 35 years, the two annuities combine to act like an annuity with cash flow of $\$5500$. The future values are then $\$3000 \cdot \frac{(1.09)^{45} - 1}{0.09} = \$1,577,576.20$ and $\$2500 \cdot \frac{(1.09)^{35} - 1}{0.09} = \$539,276.89$ Adding them, we get $\$2,116,853.09$. The $\$0.05$ discrepancy with the previous method is a round-off error.

Example 19. *The Brown's have a new born daughter and have decided to start saving for her college education. They estimate she will attend school for 4 years and her schooling will cost $\$30,000$ per year. They will deposit a fixed amount of money into a savings account at the end of each year for 18 years. The $\$30,000$ will come due at the end of each of the 19th, 20th, 21st, and 22nd years. They estimate that they will be able to invest at a nominal interest rate of 4% compounded annually during the entire 22 years. How much money will they need to deposit at the end of each year for the first 18 years?*

Here we have two annuities: the Browns make deposits into one annuity for the first 18 years. They receive cash flows from the second annuity for 4 years.

The first annuity is used to fund the second annuity. This means the two annuities will have the same value when valued at any point in time. It is now a matter to find a convenient time to equate the values the two annuities. At the beginning of the 19th year: the value of the 18 year annuity will be given by the future value formula, the value of the 4 year annuity will be given by the present value formula. The PV of the 4 year annuity is then $\$30,000 \cdot \frac{1-1.04^{-4}}{0.04} = \$108,896.86$. The future value formula for the 18 year annuity now reads

$$\$108,896.86 = R \cdot \frac{1.04^{18} - 1}{0.04} = R \cdot 25.645413.$$

Solving for R yields $\$4,246.25$. Thus, the Browns should invest $\$4,246.25$ per year in order to pay for their daughter's education.

Example 20. *Fugu Furniture is advertising their fantastic furniture fun-stravaganza sale. They advertise that if you get their 48 month financing on a bedroom set, you will have no payments for the first 12 months. Suppose the interest is 4.2% APR, compounded monthly.*

You buy a new oak bedroom set. They tell you your monthly payments will be $\$59$ due at the end of each month. Determine the implied price of the bedroom set and determine your interest charges.

You suspect the No payments for 12 months is just a gimmick, and you opt for a regular 36 month loan. Determine the size of your monthly payments and determine your interest charges.

First let's value the bedroom set in one year.

$$59 \cdot \frac{1 - (1 + \frac{0.42}{12})^{-24}}{\frac{0.42}{12}} \approx \$1355.88.$$

In one year the bedroom set will be worth $\$1355.88$. That means now the bedroom set is worth $\$1355.88 \cdot \left(1 + \frac{0.42}{12}\right)^{-12} \approx \$1300.21$. Your total interest charges would be $\$59 \cdot 24 - 1300.21 = \$115.79$.

If you decide to go with a regular 36 month loan, we'll use the annuity formula to figure out your payments.

$$\$1300.21 = R \cdot \left(\frac{1 - 1.0035^{-36}}{0.0035}\right)$$
$$= R \cdot 33.76891$$

Solving for R yields a monthly payment of $\$38.50$. Your total interest charges would be $\$38.5 \cdot 35 - 1300.21 = \$85.79$.

We end by summarizing many of the results of calculations from the previous two sections of notes.

Fourth Principle of Financial Mathematics

Assume the APR of an account earning compound interest is positive.

1. The numerical value of the present value is not greater than the numerical value of the future value.

2. To increase the difference between the present value and the future value, you can: (a) increase the APR, (b) increase the term, (c) increase the frequency of the payments.

3. Let S be the sum of the payments; i.e. $S = n \cdot R$. Then the present value is not greater than the sum of the payments.

4. Doubling the size of the regular payment will: (a) decrease the term of the loan by more than 1/2 (b) decrease the interest paid by more than 1/2.