MA 162: Finite Mathematics - Section 3.3/4.1

Fall 2014

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Announcements:

- Homework 3.3 due Tuesday at 6pm.
- Homework 4.1 due Friday at 6pm.
- Exam scores were emailed on Friday.

Solving LP Problems (Tan, Section 3.3, #36)

A financier plans to invest up to $\$500,000$ in two projects. Project A yields a return of 10% on the investment, whereas Project B yields a return of 15% on the investment. Because the investment in Project B is riskier than the investment in Project A, the financier has decided that the investment in Project B should not exceed 40% of the total investment. How much should she invest in each project to maximize the return on her investment? What is the maximum return?

4.1 - Standard Linear Programming Problems

A **standard maximization problem** is one in which

- The objective function is to be maximized.
- All of the variables involved in the problem are non-negative.
- All other linear constraints may be written so that the expression involving the variables is less than or equal to a non-negative constant.

4.1 - Slack Variables

- A part of the Simplex Algorithm that we will discuss today is the introduction of slack variables.
- A slack variable is used to change an inequality into an equality.
- Suppose $x + y \leq 50$ with $x \geq 0$ and $y \geq 0$.
- We can replace this inequality with the equality $x + y + z = 50$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$.
- z is called a slack variable.
- Slack variables usually represent the amount of a resource *leftover*.

Tan, Section 4.1, #35

National Business Machines Corporation manufactures two models of portable printers: A and B. Each model A costs \$100 to make, and each model B costs \$150. The profits are \$30 for each model A and \$40 for each model B portable printer. If the total number of portable printers demanded each month does not exceed 2500 and the company has earmarked no more than $\$600,000/$month for manufacturing costs, find how many units of each model National should make each month to maximize its monthly profit. What is the largest monthly profit?

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- Let x be the number of units of model A produced.
- Let y be the number of units of model B produced.
- Printers made: $x + y \leq 2500$
- Costs: $100x + 150y \leq 600000$
- Non-negativity: $x \geq 0$, $y \geq 0$.
- Objective: Maximize $P = 30x + 40y$.

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- Let u be the number of printers less than 2500 made.
- Let v be the amount of money not spent from the budget.
- Printers made: $x + y + u = 2500$
- Costs: $100x + 150y + v = 600000$
- Non-negativity: $x \geq 0$, $y \geq 0$, $u \geq 0$, $v \geq 0$.
- Objective: Maximize $P = 30x + 40y$.

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- Put these equations into the following augmented matrix:
- $$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$
- The bottom row is the objective function rewritten as: $-30x - 40y + P = 0$.
- Right now x and y are called **non-basic variables** and u, v, and P are called **basic variables**.

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- The corner points of the feasible region correspond to letting the non-basic variables be equal to 0.

- Right now, we would let $x = 0$ and $y = 0$ which tells us that $u = 2500$, $v = 600000$ and $P = 0$.

- This means that if we make zero units of model A and zero units of model B then there are 2500 printers not made, 600000 unused money from the budget, and 0 profit.

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- Next we want to switch which variables are basic and non-basic in a way that increases profit.

- How to choose which columns to switch: Look for the largest negative number in the bottom row of your augmented matrix (if all entries are positive then you are done).

- $$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

- In this case we will use column 2 because this causes a larger increase in profit.

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- How do we choose which row to pivot about? Look at the each row with a positive entry in the chosen column.

- For each row, divide the right-hand side entry by the entry in the chosen column. The row with the smallest ratio is the row to choose.

- $$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

- In this problem, we have $2500/1 = 2500$ and $600000/150 = 4000$. We choose to pivot in the first row because this is the smaller ratio.

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- Pivot about the entry in Row 1, Column 2.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 + 40R_1]{R_2 \mapsto R_2 - 150R_1}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 2500 \\ -50 & 0 & -150 & 1 & 0 & 225000 \\ 10 & 0 & 40 & 0 & 1 & 100000 \end{array}\right]$$

- Now we have x and u as the non-basic variables so we have $x = 0$, $y = 2500$, $u = 0$, $v = 225000$, and $P = 100000$.
- Since we have no negative numbers left in the bottom row, we are done. This means the solution we got above is optimal.
- In plain English, this means the maximum profit occurs when National makes 2500 model B printers and 0 model A printers. This results in a profit of $100000 and $225000 unspent from the budget.

The Simplex Algorithm

1. Setup the initial simplex tableau.
2. Determine whether the optimal solution has been reached by examining all entries in the last row to the left of the vertical line.
   - If all the entries are non-negative, the optimal solution has been reached. Proceed to Step 4.
   - If there are one or more negative entries, the optimal solution has not been reached. Proceed to Step 3.
3. Perform the pivot operation. Locate the pivot element and convert that column to a unit column. Return to Step 2.
4. Determine the optimal solution(s). Non-basic variables get set to zero and the other variables can be read off the final table.