MA 162: Finite Mathematics - Section 2.3

Fall 2014

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September 15, 2014

Announcements:

- Alternate Exam Request Form due Today.
- Homework 2.2 due Tuesday at 6pm.
- Homework 2.3 due Friday at 6pm.

2.2 Review - The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is non-zero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is non-zero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

2.3 - System of Equations with Infinitely Many Solutions

We saw last week that the system:

$$3x - 2y = 12$$
$$6x - 4y = 24$$

has infinitely many solutions of the form $(t, \frac{3}{2}t - 6)$ where t can be any real number. What does this look like if we use an augmented matrix and the Gauss-Jordan elimination method?

pivot

$$\left[\begin{array}{cc|c} 3 & -2 & 12 \\ 6 & -4 & 24 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cc|c} 3 & -2 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

Only having one pivot with two columns to the left of the vertical bar means we have infinitely many solutions as long as there are solutions

2.3 - System of Equations with Infinitely Many Solutions

Solve the system of equations:

$$\begin{array}{rcrcrcr} x & & & - & z & = & 1 \\ -x & + & y & + & 3z & = & 3 \\ -2x & - & 2y & - & 2z & = & -10 \end{array}$$

pivot

2nd pivot

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ -1 & 1 & 3 & 3 \\ -2 & -2 & -2 & -10 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 + 2R_1]{R_2 \mapsto R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & -2 & -4 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 + 2R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

two pivots with three columns to the left of the vertical bar means there are infinitely many solutions as long as there are solutions.

In equation form this is

$$\begin{aligned} x \quad - z &= 1 \\ y + 2z &= 4. \end{aligned}$$

Let $z = t$ (because there is no pivot in the z-column)

Then

$$\begin{aligned} x - t &= 1 \\ y + 2t &= 4 \\ z &= t \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} x &= 1 + t \\ y &= 4 - 2t \\ z &= t. \end{aligned}$$

So the solutions to this system are parameterized as $(1+t,\ 4-2t,\ t)$ where t is any real number.

2.3 - System of Equations with No Solutions

Solve the system of equations:

$$\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 0 \\ 2x & - & 3y & + & 3z & = & 2 \\ x & + & y & - & 2z & = & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 3 & 2 \\ 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -5 & 7 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

In equation form, the last row of the augmented matrix says $0=1$. This means this system has no solutions.

2.3 - Overdetermined Systems

An overdetermined system is a system of equations that has more equations than unknowns.

Each augmented matrix below is the final form after the Gauss-Jordan method for an overdetermined system with two variables (x and y in order). Find the solutions to each overdetermined system.

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

No Solutions

b/c $0 \neq 1$

One Solution

$$\boxed{\begin{array}{l} x = 5 \\ y = -2 \end{array}}$$

Infinite Solutions

$x - y = 5$

Let $y = t$. Then all solutions are of the form $(5+t, t)$ for any real number t.

2.3 - Underdetermined System

An underdetermined system is a system of equations that has more unknowns than equations.

Each augmented matrix below is the final form after the Gauss-Jordan method for an underdetermined system with four variables (x, y, z, and w in order). Find the solutions to each underdetermined system.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

No Solutions

b/c $0 \neq 11$.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 1 & 5 \\ 0 & \boxed{1} & -3 & -6 & -2 \end{array}\right]$$

Infinite Solutions

The two pivot positions are circled. Then we let the remaining colums be parameters.

Let $z = s$

Let $w = t$

Then $$\begin{aligned} x + 2z + w &= 5 \\ y - 3z - 6w &= -2 \end{aligned}$$

means

$$\begin{aligned} x &= 5 - 2z - w \\ y &= -2 + 3z + 6w \end{aligned}$$

and furthermore

$$\begin{aligned} x &= 5 - 2s - t \\ y &= -2 + 3s + 6t \end{aligned}$$

So solutions to this system are of the form

$(5-2s-t,\ 3s+6t-2,\ s,\ t)$ for any real numbers s and t.

Tan, Section 2.3, Problem 37

The management of Hartman Rent-A-Car has allocated \$1,512,000 to purchase 60 new automobiles to add to the existing fleet of rental cars. The company will choose from compact, mid-sized, and full-sized cars costing \$18,000, \$28,800, and \$39,600 each, respectively. Find formulas giving the options available to the company. Give two specific options.

Let x = # compact cars bought

Let y = # mid-sized cars bought

Let z = # full-sized cars bought

$$\begin{cases} x + y + z = 60 & \text{(total cars bought)} \\ 18000x + 28800y + 39600z = 1512000 & \text{(cost per car / total allotted)} \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 18000 & 28800 & 39600 & 1512000 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 18000 R_1}$$

pivot

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 10800 & 21600 & 432000 \end{array}\right] \xrightarrow{R_2 \mapsto \frac{1}{10800} R_2}$$

pivot

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 1 & 2 & 40 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 20 \\ 0 & 1 & 2 & 40 \end{array}\right]$$

Now let $z = t$ and parameterize the solutions

$$x - z = 20$$
$$y + 2z = 40$$

So

$$x = 20 + t$$
$$y = 40 - 2t$$
$$z = t$$

Since we are talking about cars, all of x, y, z must be non-negative integers.

In particular, $20 + t \geq 0$ $t \geq -20$

$40 - 2t \geq 0$ $t \leq 20$

$t \geq 0$ $t \geq 0$

So $0 \leq t \leq 20$ and t is an integer.

The companies options are to buy $(20+t)$ compact cars, $(40-2t)$ mid-sized cars, and (t) full sized cars where t is any integer between zero and twenty inclusive.

Two specific options are:

<u>$t = 0$</u>

20 compact cars
40 mid-sized cars
0 full-sized cars

<u>$t = 10$</u>

30 compact cars
20 mid-sized cars
10 full-sized cars.