MA 162: Finite Mathematics - Sections 2.6

Fall 2014

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September 24, 2014

Announcements:

- Homework 2.6 due next Tuesday at 6pm.

Multiplicative Inverses

- If a is a non-zero real number, then there exists a unique real number a^{-1} (also written $\frac{1}{a}$) such that

$$a^{-1}a = aa^{-1} = 1$$

- Essentially this means you can divide by a to solve equations.
- For example,

$$\begin{aligned} 3x &= 7 \\ 3^{-1}3x &= 3^{-1}7 \\ x &= \frac{7}{3} \end{aligned}$$

- We want something similar for matrices.

Inverse of a Matrix

Defn: Let A be a square matrix of size n (meaning A is $n \times n$). A square matrix A^{-1} of size n such that

$$A^{-1}A = AA^{-1} = I_n$$

is called the inverse of A.

- Remember that I_n is the $n \times n$ identity matrix.
- Any matrix X for which the multiplication is defined satisfies

$$I_n X = X$$

and

$$X I_n = X$$

Inverse of a Matrix

- Are

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} \text{-}5/2 & 3/2 \\ 2 & \text{-}1 \end{bmatrix}$$

  inverse matrices?

More About Inverse Matrices

- Not every square matrix has an inverse. A square matrix without an inverse is called **singular**. A square matrix that has an inverse is called **non-singular**.

- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not have an inverse.

- $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ does not have an inverse.

A Method for Finding a Matrix Inverse

- If you want to find the inverse of a square matrix A, then augment A with an identity matrix of the same size to form $[A|I]$ and use Gauss-Jordan to reduce this matrix.

- If you can reduce the augmented matrix $[A|I]$ to the form $[I|B]$, then B is the inverse of matrix A.

- If you can not reduce the augmented matrix $[A|I]$ to the form $[I|B]$, then A does not have an inverse.

- If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse.

A Method for Finding a Matrix Inverse

- Consider the matrix

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

from earlier. We already know that the inverse of this matrix is

$$\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

but let's try to use our new method.

- Augment the given matrix with an identity matrix of the same size and reduce this matrix.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$$

A Method for Finding a Matrix Inverse

- Reduce

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$$

A Formula for the Inverse of a 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

- A^{-1} exists if and only if $ad - bc \neq 0$
- $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
- In the previous example, $ad - bc = -2 \neq 0$ so the matrix has an inverse and the inverse matrix is

$$-\frac{1}{2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$$

- This method does not translate well to larger matrices!

An Example of Finding an Inverse (Tan, Section 2.6, #12)

Find the inverse (if it exists) of the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & -2 \\ -5 & 0 & -2 \end{bmatrix}$$

Solving Systems with Inverses

- Recall that you can write a system of equations as a matrix equation in the following way. Consider the system of equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & - & y & + & z & = & -3 \\ x & - & 2y & - & z & = & -1 \end{array}$$

- Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}$$

- Then the system of equations can be represented by the matrix equation

$$AX = B$$

Solving Systems with Inverses

- Then the matrix equation

$$AX = B$$

can be solved similarly to what we do for equations with numbers. That is, multiply both sides of the equation on the left by A^{-1}.

- Find A^{-1} and then

$$A^{-1}AX = A^{-1}B$$

implies

$$X = IX = A^{-1}B$$

Solving Systems with Inverses

Then to solve the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & - & y & + & z & = & -3 \\ x & - & 2y & - & z & = & -1 \end{array}$$

we can find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \text{ and multiply it by } B = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}$$

Tan, Section 2.6, #40

Bob, a nutritionist who works for the University Medical Center, has been asked to prepare special diets for two patients, Susan and Tom. Bob has decided that Susan's meals should contain at least 400 mg of calcium, 20 mg of iron, and 50 mg of vitamin C, whereas Tom's meals should contain at least 350 mg of calcium, 15 mg of iron, and 40 mg of vitamin C. Bob has also decided that the meals are to be prepared from three basic foods: Food A, Food B, and Food C. The special nutritional contents of these foods are summarized in the accompanying table. Find how many ounces of each type of food should be used in a meal so that the minimum requirements of calcium, iron, and vitamin C are met for each patient's meals.

	Calcium	Iron	Vitamin C
Food A	30	1	2
Food B	25	1	5
Food C	20	2	4