MA/CS 321 : Intro to Numerical Methods

Problem Set 7

Due: Monday November 25, 2013

Note: Turn in all code that you have written and turn in all output generated by your functions/scripts. Your functions/scripts must be commented and output must be formatted nicely.

1. Newton’s Method (E3.2.36) Apply Newton’s method on these test problems, print a table showing the rate of convergence:

(a) \( f(x) = x^2 \), Hint: The first derivative is zero at the root, and convergence may not be quadratic.

(b) \( f(x) = x + x^{4/3} \), Hint: There is no second derivative at the root and convergence may fail to be quadratic.

(c) \( f(x) = x + x^2 \sin(2/x) \) for \( x \neq 0 \) and \( f(0) = 0 \) and \( f'(x) = 1 + 2x \sin(2/x) - 2 \cos(2/x) \) for \( x \neq 0 \) and \( f'(0) = 1 \). Hint: The derivative of this function is not continuous at the root, and convergence may fail.

2. Newton’s Method for a system (E3.2.37) Left \( F(x) = [x^2 - x_2 + c; x_2^2 - x_1 + c] \). Each component of \( F(x) \) is the equation for a parabola. Any point \((x^*, y^*)\) such that \( F((x^*, y^*)) = [0; 0] \) corresponds to points where the two parabolas intersect. Use Newton’s method to find such points for the following values of \( c \), \( c = 1/2, 1/4, -1/2, -1, -1 \). Give the Jacobian matrix for each. Also for each of these values plot the resulting curves showing the points of intersection.

3. Secant Method for aiming a cannon The following system of differential equations describes the motion of a cannonball subject to linear drag, that is the drag due to air is proportional to the speed of the cannonball,

\[
x'' = -c \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad y'' = -g - c \frac{y'}{\sqrt{x'^2 + y'^2}},
\]

where \( y(0) = y_0, x(0) = x_0, y'(0) = v_0 \sin(\theta), \) and \( x'(0) = v_0 \cos(\theta) \). The constant \( c \) determines how much drag there is. The angle \( \theta \) is the angle that the cannon makes with horizontal axis.

(a) In the absence of drag, \( c = 0 \), the equations of motion can be solved exactly,

\[
y(t) = y_0 + v_0 \sin(\theta) t - \frac{1}{2} gt^2, \quad x = v_0 \cos(\theta) t.
\]

Let \( y_0 = 0 \), if we want the cannonball to hit a target at \((d, 0)\), determine an equation for the \( \theta \) that will ensure the cannonball will hit its target.

(b) Linearize the equations of motion for the cannonball for \( c \neq 0 \), so that you have a first order system of equations of the form,

\[
z' = f(z).
\]

Your linearized system should have four variables and four equations. What are the initial conditions?
(c) What is the Jacobian of $f(z)$ in your linearized system.

(d) Write a function that uses Backward Euler to solve the equations of motion. Use Newton’s method to solve the system of equations for each step of Backward Euler. Let $g = 9.8$, $c = 1$, and $v_0 = 60$, $\theta = \pi/4$. For a given time step $\Delta t$, your code should determine a time interval in which the cannonball hits the ground. Use a linear interpolant for $y(t)$ in this interval to determine an approximate time and place of impact, $T_I$ and $x(T_I)$. Pick your step size so that $T_I$ is accurate to the nearest hundredth of a second.

(e) Use the secant method to determine a $\theta$ such that the cannonball will hit a target at $(300, 0)$. Now $T_I(\theta)$ is a function of $\theta$ and you are finding a root of the function $f(\theta) = d - x(T_I(\theta))$. You must use the secant method here because it would be difficult to determine a formula for the derivative of $x(T_I(\theta))$ as a function of $\theta$, since $x(T_I)$ is determined by numerically solving the differential equation. Choose your step size so that you determine $\theta$ to the nearest thousandth of a radian.