Examples for Graphical Solutions to Linear Programming Problems

1. A farmer is going to plant apples and bananas this year. It costs $40 per acre to plant apples and $60 per acre to plant bananas and the farmer has a maximum of $7400 available for planting. To plant apples trees requires 20 labor hours per acre; to plant banana trees requires 25 labor hours. Suppose the farmer has a total of 3300 labor hours available. If he expects to make a profit of $150 per acre on apples and $200 per acre on bananas, how many acres each of apples and bananas should he cultivate?

Solution. We begin by translating this problem into linear programming problem. Our goal is to maximize profit. Let $A$ be the number of acres of apples planted and $B$ the number of acres of bananas planted. Then if $P$ is profit, our objective function is $P = 150A + 200B$. Our constraints are defined in terms of total cost and labor we have:

$$\begin{align*}
\text{Maximize:} & \quad P = 150A + 200B \\
\text{Subject to:} & \quad 40A + 60B \leq 7400 \\
& \quad 20A + 25B \leq 3300
\end{align*}$$

We draw our feasible set:

![Feasible set diagram](image)

Figure 1: The solution set is the shaded region.

Then we check the value of $P$ at the vertices of our solution set:
So we maximize $P$ at the point $(65, 80)$. Hence the farmer should plant 60 acres of apples and 80 acres of bananas.

2. Find the maximum and minimum for $P = 10x + 12y$ subject to:

$$
\begin{align*}
3x + 5y &\geq 20 \\
3x + y &\leq 16 \\
-2x + y &\leq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
$$

Solution. First we draw our feasible set:

We draw our feasible set:

![Graph of the feasible set](image)

Figure 2: The solution set is the shaded region.

Now we complete our table:

<table>
<thead>
<tr>
<th>Vertex $(x, y)$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3, 7)$</td>
<td>114</td>
</tr>
<tr>
<td>$(5, 1)$</td>
<td>62</td>
</tr>
<tr>
<td>$\left(\frac{15}{13}, \frac{45}{13}\right)$</td>
<td>$\frac{666}{13} \approx 51.23$</td>
</tr>
</tbody>
</table>
So our maximum occurs at (3, 7) and our minimum at \((\frac{15}{13}, \frac{43}{13})\) and the values at these points are given in the table.

Example 1: (From class) In college there is a trade off between time spent studying and leisure. Suppose you can measure your utility (level of satisfaction) with each of these activities in terms of “utils”. Suppose you get 3 utils per hour from studying and 8 utils per hour from leisure and that in order to stay in college you must study at least three times as much as you spend doing leisure activities each day. If you require 8 hours of sleep per night and you get no utility from activities outside of studying and leisure, how should you allocate your time (each day) to maximize your utility?

Solution. Translate this problem into a linear program. Let \(S\) be the number of hours spent studying and \(L\) be the number of hours spent doing leisure activities. Since we want to maximize our utility, our objective function will be \(U = 3S + 8L\). Then we can write this problem as:

\[
\begin{align*}
\text{Maximize:} & \quad U = 3S + 8L \\
\text{Subject to:} & \quad S - 3L \geq 0 \\
& \quad S + L \leq 16
\end{align*}
\]

Figure 3: The solution set is the shaded region.

Now we complete our table:
So we will maximize our daily utility by spending 12 hours studying and 4 hours in leisure.

Example 2: (From class) Maximize and minimize the objective function \( Z = 4x + 5y \) on the feasible set drawn below.

![Graph showing the feasible set shaded.](image)

Figure 4: The solution set is the shaded region.

Now we complete our table:

<table>
<thead>
<tr>
<th>Vertex ((x, y))</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 4))</td>
<td>24</td>
</tr>
<tr>
<td>((3, 6))</td>
<td>42</td>
</tr>
<tr>
<td>((5, 4))</td>
<td>40</td>
</tr>
<tr>
<td>((3, 1))</td>
<td>17</td>
</tr>
</tbody>
</table>

So our maximum occurs at \((3, 6)\) and our minimum at \((3, 1)\) and the values of \(Z\) at these points are given in the table.