MathExcel Worsksheet 9

So far, you have learned how to integrate using rules such as the power rule, parts, $u$ and trigonometric substitution, and partial fractions. Knowing which method to apply can be the real test. Some things to do when trying to evaluate an integral are as follows:

- Memory. You should probably know easy integrals such as $\int x^n \, dx$, $\int \cos(x) \, dx$, and $\int \frac{1}{x^2 + 1} \, dx$.

- Simplifying or rewriting the integrand. Writing $\int \frac{x^3 - 1}{x^2} \, dx = \int \left( x - \frac{1}{x^2} \right) \, dx$, and using the power rule.

- Look for a $u$-substitution. When looking at $\int \frac{x}{\sqrt{x^2 - 1}} \, dx$, it may appear like a trig. sub., but noticing that the derivative of what is under the root is in the numerator, we can let $u = x^2 - 1$.

- If we have an integrand with $\sqrt{x^2 \pm a^2}$ or $\sqrt{a^2 - x^2}$ somewhere in it, look for a trig. substitution.

- If the integrand is the product of two functions such as $e^x \sin(x)$ or $\ln(x)x$, try using parts.

- If you have an integral of a rational function such as $\int \frac{x^n}{x^2 - 4x + 4} \, dx$ that cannot be simplified, use partial fractions, keeping in mind that if the highest power in the numerator is at least as big as the highest power in the denominator, you must first divide.

- Use intuition and do lots of practice to know the type of integral.

Evaluate the following integrals.

\[
\int t \sin(t) \, dt = \int \frac{t \sin(t)}{dt} \, dt
\]

Set:

\[
\begin{align*}
u &= t \\
u' &= 1 \\
du &= dt
\end{align*}
\]

\[
\begin{align*}
&= -t \cos(t) + \int \cos(t) \, dt \\
&= -t \cos(t) + \sin(t) + C
\end{align*}
\]
2. \[
\int \frac{2s}{(s-3)^2} ds
\]
\[U = (s-3) \Rightarrow s = u + 3 \]
\[du = ds \]

\[
= \int \frac{2u + 6}{u^2} du = \int \frac{2}{u} du + \int \frac{6}{u^2} du
\]

\[
= 2 \ln |u| - \frac{6}{u} + C
\]

3. \[
\int \sin(t) \cot(t) dt
\]

\[
= \int \cos(t) dt = \sin(t) + C
\]

4. \[
\int_{-1}^{1} \frac{t}{\sqrt{4-t^2}} dt
\]
\[u = 4 - t^2 \]
\[du = -2t dt \]

\[
= \int_{3}^{1} \frac{-du}{2\sqrt{u}} = 0
\]

5. \[
\int \frac{1}{\sqrt{4-t^2}} dt
\]
\[t = 2 \sin \theta \]
\[dt = 2 \cos \theta d\theta \]

\[
= \int \frac{2 \cos \theta d\theta}{\sqrt{4 - 4 \sin^2 \theta}} = \int \frac{d\theta}{\cos \theta}
\]
\[= \theta + C
\]
\[\theta = \sin^{-1} \left( \frac{t}{2} \right) + C
\]

6. \[
\int_{0}^{1} \sqrt{4-t^2} dt
\]

Intuition: Since \(\sqrt{4-t^2}\) is the top half of a circle with radius 2, we have that

\[
\int_{0}^{2} \sqrt{4-t^2} dt = \text{area of } C = \frac{\pi 2^2}{4} = \pi
\]

(Could also use trig. sub.)
7. \[ \int e^x \, dx = e^x + C \] \text{Constant rule}

8. \[ \int x^4 \, dx = \frac{x^5}{5} + C \] \text{Power rule}

9. \[ \int e^x \ln 7 \, dx = \frac{e^x (\ln 7)}{\ln 7} + C \] \text{by a \( u \)-sub.}

10. \[ \int \csc^2(t) \, dt = -\cot(t) + C \] \text{Memory}

11. \[ \int \frac{t^2 - 2t + 1}{t-1} \, dt = \int \frac{(t-1)^2}{t-1} \, dt = \int (t-1) \, dt = \frac{t^2}{2} - t + C \] \text{Could also use partial fractions}

12. \[ \int \tan^2(t) \, dt = \int (\sec^2(t) - 1) \, dt = \tan(t) - t + C \] \text{Rewrite}
13. \[ \int \frac{1}{t^2-1} \, dt = \int \frac{d}{t(t+1)} = \int \left( \frac{A}{t-1} + \frac{B}{t+1} \right) \, dt \]

\[ = A \ln |t-1| + B \ln |t+1| + C \]

Since \[ A(t+1) + B(t-1) = 1 \]
\[ 2A = 1 \quad \text{and} \quad 2B = 1 \]

So, \[ \int \frac{dt}{t^2-1} = \ln \left( \sqrt{\frac{t-1}{t+1}} \right) + C \]

14. \[ \int \frac{1}{x^2 \sqrt{4 + x^2}} \, dx = \int \frac{2 \sec^2 \theta \, d\theta}{4 \tan \theta \cdot 2 \sec \theta} = \frac{1}{4} \int \frac{\sec \theta \, d\theta}{\tan^2 \theta} \]

\[ = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta \]

\[ = \frac{1}{4} \int \frac{du}{u^2} \]

\[ = -\frac{1}{4u} + C = \frac{-1}{4 \sin \theta} + C = \frac{-\sqrt{4 + x^2}}{4x} + C \]

15. \[ \int (t-2)e^t \, dt = (t-2)e^t - \int e^t \, dt \]

\[ = (t-2)e^t - e^t + C \]

16. Prove that

\[ \int \frac{1}{\sqrt{a^2 - \theta^2}} \, d\theta = \frac{1}{|a|} \sec^{-1} \left( \frac{\theta}{a} \right) + C \]

for any constant \( a \neq 0 \).

Set \( \theta = \text{asec}(x) \) for \( 0 < x < \frac{\pi}{2} \) or \( \pi < x < \frac{3\pi}{2} \)

\[ d\theta = \text{asec}(x) \tan(x) \, dx \]

Then, \[ \int \frac{d\theta}{\sqrt{a^2 - \theta^2}} = \int \frac{\text{asec}(x) \tan(x) \, dx}{\text{asec}(x) \sqrt{a^2 \sec^2(x) - a^2}} = \int \frac{dx}{|a|} \]

\[ = \frac{x}{|a|} + C = \frac{\text{sec}^{-1} \left( \frac{x}{a} \right)}{|a|} + C \]