09-21-2009

Algorithms for Multiplying and Dividing Whole Numbers

In this section we will discuss several algorithms for multiplying and dividing whole numbers.

Multiplication:

- For small numbers we can physically represent multiplication as repeated addition with units, strips, and squares. Whenever we reach ten of one kind of manipulative piece we should **exchange**. 10 units $\rightarrow$ 1 strip, 10 strips $\rightarrow$ 1 mat, 10 mats $\rightarrow$ one cube,...

- For larger numbers **place value cards** are a reasonable model. To use place value cards we will take advantage of the *distributive property.*

  1. Example: $5 \cdot 147$

     Mark off a card with squares for ones, tens, hundreds,... We will record all of our multiplication on this card.

     $$5 \times 147 = 5(100 + 40 + 7) = 5 \cdot 100 + 5 \cdot 40 + 5 \cdot 7 = 5(100) + 20(10) + 35$$

     So we record 35 ones, 20 tens and 5 100s. Then begining in the ones square we make exchanges so that there are no more than 9 markers in any square.

     Here we'll trade 30 for 3 tens. This gives us 23 tens and so we'll trade 20 of them for 2 hundreds.

  2. The instructional algorithm is:

     $$\begin{array}{rr} & 147 \\ \times & 5 \\ \hline & 35 \\ & 200 \\ & 500 \\ \hline & 735 \end{array}$$

  3. The final alogrithm is:

     $$\begin{array}{rr} & 23 \\ & 147 \\ \times & 5 \\ \hline & 735 \end{array}$$

- For larger numbers the models become complicated so we present only the instructional and final algorithm.
  Example: Compute 16×321.

– Expanded notation:

$$\begin{aligned}16\cdot 321 &= (10+6)\cdot 321\\ &= 10\cdot 321+6\cdot 321\\ &= 10(300+20+1)+6(300+20+1)\\ &= 3000+200+10+1800+120+6\\ &= 3000+200+10+1000+800+100+20+6\\ &= 4000+1100+30+6\\ &= 4000+1000+100+30+6\\ &= 5000+100+30+6\\ &= 5136\end{aligned}$$

– Instructional Algortihm

$$\begin{array}{rr} & 321\\ \times & 16\\ \hline & 6\\ & 120\\ & 1800\\ & 10\\ & 200\\ & 3000\\ \hline & 5136\end{array}$$

– Final Algorithm: standard multiplication algortihm for multi-digit numbers.

$$\begin{array}{rr} & 1\\ & 321\\ \times & 16\\ \hline & 1926\\ & 321\\ \hline & 5136\end{array}$$

Division:

- Theorem: (The division algortihm) If a and b are whole numbers with b not zero, there exists exactly one pair of whole numbers q and r with $0 \leq r \leq q$ such that $a = b \cdot q + r$. The number q is the quotient and the number r is the remainder of a divided by b.

- For small numbers we know a variety of methods to compute $a \div b$. For large numbers, however, these methods become unmanagable. We need a procedure for large numbers and this procedure is called **long division**.

- Long division is really just repeated subtraction. For example compute $1000 \div 9$ by repeated subtraction. First subtract 100 $9's$ or 900 from 1000. This leaves 100. Take 10 $9's$ from 100 to get 10. Take one more nine to get 1. In total, we subtracted 9, 111 times from 1000 with a remainder of 1. So our answer is 111 R 1. To check this answer we plug our result into the division algorithm formula. $1000 = 111 \cdot 9 + 1$.

  The **scaffold algorithm** is a systematic framework for organizing computations like the above.

- Scaffold algorithm for $1000 \div 9$.

$$
\begin{array}{r}
111 \\
\hline
1 \\
10 \\
100 \\
9\overline{)1000} \\
900 \\
\hline
100 \\
90 \\
\hline
10 \\
9 \\
\hline
1
\end{array}
$$

Since $0 \leq 1 < 9$ our answer is 111 R 1.

- Scaffold algorithm for long division.

$$
\begin{array}{r}
121 \\
\hline
120 \\
100 \\
6\overline{)729} \\
600 \\
\hline
129 \\
120 \\
\hline
9 \\
6 \\
\hline
3
\end{array}
$$

Since $0 \leq 3 \leq 6$ we stop and our answer is 121 R 3.

- The scaffold alogirthm is now standard among elementary teachers. There is a notationally compact alogrithm which you may be familiar with, however. This method saves a little writting but unfortunately tends to obscure what's actually going on. The compressed algortihm:

$$
\begin{array}{r}
121 \\
\hline
6\overline{)729} \\
6 \\
\hline
129 \\
12 \\
\hline
9 \\
6 \\
\hline
3
\end{array}
$$

You can see that there's a lot left out of this algorithm and when students learn and practice the notionally compact form they are likely to lose sight of what division is really all about: repeated subtractions.

- Here's another example with the scaffold algorithm. Compute $32516 \div 312$.

$$
\begin{array}{r}
99 \\ \hline
9 \\
90 \\
312\overline{)31116} \\
28080 \\ \hline
3036 \\
2808 \\ \hline
228
\end{array}
$$

Since $0 \leq 228 < 312$ we stop and our answer is 99 R 228.

- If the *divisor is a single digit* there is a simple procedure to compute $a \div b$. The procedure is called the **short division alogrithm**. You can see an example in the text, p. 192.