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Fractions and the Set of Rational Numbers

We introduced the whole numbers as a system for keeping track of the number of objects in a set. Later on, we discovered that negative numbers could help us to, for example, track debits and credits in our bank account, and so we introduced the *integer* number system. Our task in this section is to describe *fractions* which help us to count parts of whole objects. We then present the *rational* number system which is the set of all fractions.

Definition. A **fraction** is an ordered pair of integers a and b with $b \neq 0$ which we write in the form $\frac{a}{b}$ or a/b. The a is called the **numerator** and the b is called the **denominator**.

Models for Fractions. A good model for fractions must do the following:

1. Specify the unit or "whole".
2. Describe how many equal parts the unit has been subdivided into. This gives the denominator.
3. Describe how many parts of the whole are present. This gives the numerator.

Some possible models are:

1. Colored regions. Choose a shape. Divide the shape into *equal* parts. The number of parts is the denominator of your fraction. Color the number of parts for the numerator. See page 346 for some example pictures.

2. Set model. Draw your universe set U so that the number of objects in the universe is the number in your denominator. Collect all the objects that you are trying to count in a set A. Then the corresponding fraction is

$$\frac{n(A)}{n(U)}$$

see the diagram on page 347.

3. Fraction strips. Give each student a strip of paper. To represent the fraction $\frac{a}{b}$, have the students use a pencil to subdivide their strips into b equal rectangles. Then have them color a of the rectangles.

4. Number lines. To represent the fraction $\frac{a}{b}$, take the standard integer number line and divide each of the intervals between the integers into b equal subintervals. Then, beginning at zero, count out a subintervals on the number line. See page 348.

Basic Properties of Fractions.

Definition. Two fractions are called **equivalent** if they represent the same quantity.

Example. Make fractions strips for the following fractions:

$$\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \dots$$

The n^{th} fraction of this sequence is $\frac{2\cdot n}{3\cdot n}$. Moreover by our fraction strips each of these fractions are really **equivalent**. That is,

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \dots = \frac{2\cdot n}{3\cdot n}.$$

In general, this is the **fundamental property of fractions**:

Theorem. for $\frac{a}{b}$ a fraction and $n \neq 0$ an integer we have that

$$\frac{a}{b} = \frac{an}{bn}.$$

In words this says that if I multiply the numerator and the denominator of a fraction by any non-zero integer I get an equivalent fraction back.

Now notice that

$$\frac{35}{21} = \frac{5 \cdot 7}{3 \cdot 7} = \frac{5}{3}$$

by the fundamental property of fractions. Thus, to reduce a fraction we can factor the numerator and the denominator and then "cancel out" all the common factors.

Suppose I give you two fractions. How can you tell if they are equivalent? The rule is given by the following theorem.

Theorem. The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. That is,

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

So are $\frac{3}{12}$ and $\frac{2}{8}$ equivalent?

A good model for these ideas are the pizza diagrams. See page 350.

Definition. A fraction is in **simplest form** if a and b have no common divisor larger than 1 and b is positive.

Example. Write $\frac{360}{600}$ is simplest form.

An easy way to do this is to successively divide out the common factors of the numerator and denominator:

$$\begin{aligned}\frac{360}{600} &= \frac{36 \cdot 10}{60 \cdot 10} = \frac{36}{60} \\ &= \frac{6 \cdot 6}{10 \cdot 6} = \frac{6}{10} \\ &= \frac{3 \cdot 2}{5 \cdot 2} = \frac{3}{5}.\end{aligned}$$

Another procedure is to divide the numerator and denominator by their GCD:

$$360 = 2^3 \cdot 3^2 \cdot 5$$

$$600 = 2^3 \cdot 3 \cdot 5^2$$

$$GCD(560, 600) = 2^3 \cdot 3 \cdot 5 = 120$$

$$\frac{360}{600} = \frac{\frac{360}{120}}{\frac{600}{120}} = \frac{3}{5}.$$

Often the easiest trick is to just write the prime factorizations of the numerator and denominator and cancel out the common factors:

$$\frac{360}{600} = \frac{2^3 \cdot 3^2 \cdot 5}{2^3 \cdot 3 \cdot 5^2} = \frac{3}{5}.$$

Common Denominators.

When working with two fractions it is often useful to rewrite both fractions in an equivalent form with a common denominator.

Example. Write $\frac{5}{8}$ and $\frac{7}{10}$ as equivalent fractions with a common denominator.

One choice is to use $8 \cdot 10$ for the common denominator. Then we would write:

$$\frac{5}{8} = \frac{5 \cdot 10}{8 \cdot 10} = \frac{50}{80} \text{ and } \frac{7}{10} = \frac{7 \cdot 8}{10 \cdot 8} = \frac{56}{10}.$$

Really any common multiple of 8 and 10 could serve as our common denominator, however, it is often useful to choice the least common multiple for our common denominator. Since LCM$(8, 10) = 40$ another answer is:

$$\frac{5}{8} = \frac{5 \cdot 5}{8 \cdot 5} \text{ and } \frac{7}{10} = \frac{7 \cdot 4}{10 \cdot 4} = \frac{28}{40}$$

For more examples, see page 353-354.

The Rational Numbers.

Definition. The set of **rational numbers** is the set of numbers that can be expressed in the form of a fraction $\frac{a}{b}$ where the a and b are integers and $b \neq 0$. Two rational numbers are **equal** if and only if they can be represented by equivalent fractions.

Notice that since the integer n can be written as $\frac{n}{1}$ the set of rational numbers includes all the integers. If we let Q denote the rational numbers, I the integers, W the whole numbers, and Z the integers we have the following subset relationship for our number systems:

$$Z \subset W \subset I \subset Q.$$

Remember that with the relations $<, \leq, >$, and $\geq$ we were able to put the integers in a useful order. Of course we can do the same thing with fractions.

Given two fractions, is is sometimes tricky to tell at first glance which is the larger number. For example, which is bigger $\frac{3}{4}$ or $\frac{5}{6}$? You can see the answer by drawing the number strips. Alternatively, rewrite $\frac{3}{4}$ and $\frac{5}{6}$ over a common denominator. Thus:

$$\frac{3}{4} = \frac{9}{12} < \frac{10}{12} = \frac{5}{6}.$$

This leads us to the following test:

Definition. Let two rational numbers be represented by the fractions $\frac{a}{b}$ and $\frac{c}{d}$ where b and d are positive. Then $\frac{a}{b}$ is less than $\frac{c}{d}$, written $\frac{a}{b} < \frac{c}{d}$, if and only if $ad < bc$.

This makes sense since if we get two fractions $\frac{a}{b}$ and $\frac{c}{d}$ we can write

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} \text{ and } \frac{b \cdot c}{b \cdot d}$$

so that $\frac{a}{b} < \frac{c}{d}$ is true if and only if $ad < bc$.