An Introduction to the Black-Scholes PDE

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Financial Derivatives

Definition

A **derivative** *is a financial contract whose value is based on the value of an underlying asset.*

- Typically, a derivative gives the holder the right to buy an asset at a pre-determined price over some time horizon.
- Buyers and sellers use derivatives to offset risk in their portfolios (hedgeing).
- One of the sophisticated instruments that rose to prominence during the financial revolutions of the 70's and 80's.

Example

A dairy farmer might agree to a "forward contract" with milk processors which guarantees a fixed price for future quantities of milk produced.

- Shifts the risk of price drops from farmers to producers.
- Limits farmers ability to gain from price increases.
- In general, the *underlying* can be any asset or commodity.

The Pricing Problem

- Investors want to trade derivatives.
- The value of the derivative is based on the value of the underlying, market conditions, and terms of the contract.
- The value of a derivative itself is unclear.

Plan for Solving the Pricing Problem

- Make some assumptions about the underlying asset and the derivatives market.
- Build a model.
- Formulate the problem in terms of a partial differential equation (Black-Scholes-Merton PDE)
- Find a way to solve the PDE

Remarkable Insight

A basic transformation will turn the Black-Scholes equation into a classical PDE!

Basic Assumptions:

1. Frictionless and efficient market for derivatives.
2. Trading in assets is a continuous process.
3. Every underlying instrument has a unique, known price.
4. The price of the underlying follows a stochastic process.

Brownian Motion

Definition

A process $z(t)$ follows **standard Brownian motion** *if*

1. $z(0) = 0$
2. *z is continuous at time t with probability 1 for each t.*
3. *For all t_1, t_2 such that $0 \leq t_1 \leq t_2$, $z(t_2) - z(t_1)$ is a normally distributed random variable with mean 0 and variance $t_2 - t_1$.*
4. *The increments are independent: for all times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $z(t_2) - z(t_1), z(t_3) - z(t_2), \ldots z(t_n) - z(t_{n-1})$ are independent random variables.*

We can intuitively regard Brownian motion as a random walk with step sizes tending to zero.

Price Dynamics for the Underlying Asset

Let $S(t)$ be the value of the underlying. Our model assumes the instantaneous rate of return on S is given by:

$$\frac{dS}{S} = \mu dt + \sigma dz(t)dt$$

where

- μ is the expected return on the asset.
- σ is the variance of the return on the asset.
- $dz(t)$ represents a stochastic process, in particular assume it is Brownian motion.

Illustration of Brownian Modeling

- The log of the value of the underlying obeys Brownian motion. Let $X = \ln S$
- $dX = \mu dt + \sigma dz(t)\sqrt{dt}$
- Discrete form: $X(t_{i+1}) - X(t_i) = \mu\Delta t + \sigma dz(t_i)\sqrt{\Delta t}$
- $S(t_{i+1}) = S(t_i)e^{\mu\Delta t + \sigma dz(t_i)\sqrt{\Delta t}}$

Simulation

- Model for stock price over a single trading day: $S(t_{i+1}) = S(t_i)e^{\mu\Delta t + \sigma dz(t_i)\sqrt{\Delta t}}$
- Parameter values: $\mu = .01, \sigma = .04, \Delta t = .004, P(0) = 50$.
- $dz(t)$ is a random normal variable with mean 0, variance 1.

Example 1

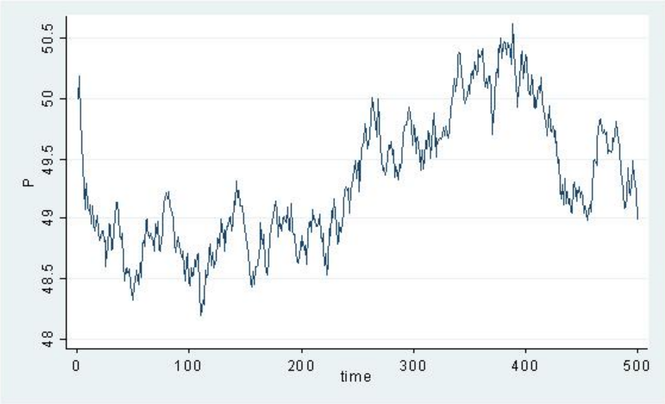

Figure: Example 1

Example 2

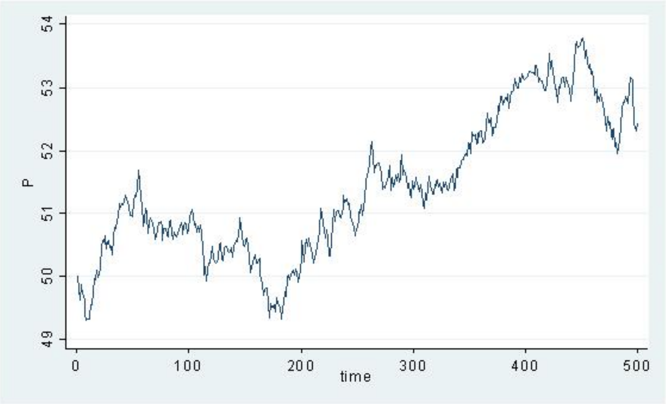

Figure: Example 2

Example 3

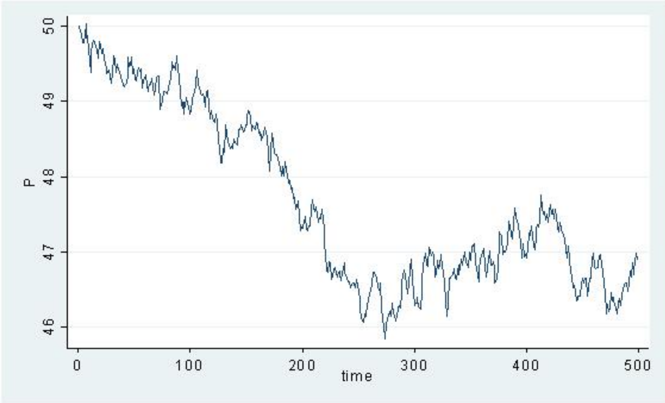

Figure: Example 3

Deriving the PDE

To derive the PDE:

- S be the price of the underlying.
- $V(S,t)$ be the value of the derivative.
- Form a portfolio Π by **selling** the derivative and **buying** Δ units of the underlying.
- The value of your portfolio is $\Pi(t) = V(t) - \Delta S(t)$.
- By linearity: $d\Pi = d(V - \Delta S) = dV - \Delta dS$
- Need to find a way to compute dV.

Ito's Lemma

Lemma (Ito's Lemma)

Let $V = V(S(t),t)$ where S satisfies

$$dS = \mu S dt + \sigma S dz(t) dt$$

. Then:

$$dV = \left(\mu V_S + V_t + \frac{\sigma^2}{2} V_{SS}\right) dt + \sigma V_S dz(t) dt.$$

Deriving the PDE

Substituting:

$$dV = \left(\mu S V_S + V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt + \sigma S V_S dW.$$

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \left[\left(\mu S V_S + V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt + \sigma S V_S dW\right] \\ &\quad - \Delta\left[\mu S_t dt + \sigma S V_S dW\right] \\ &= \left(\mu S[V_s - \Delta] + V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt + \sigma S(V_S - \Delta) dW \end{aligned}$$

Deriving the PDE

- We have:

$$d\Pi = \left(\mu S[V_s - \Delta] + V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt + \sigma S(V_S - \Delta) dW.$$

- We would like to eliminate the random term dW. Since Δ is arbitrary, we set $\Delta = V_S$ and obtain:

$$d\Pi = \left(V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt$$

Deriving the PDE

- Fundamental Economic Assumption: No arbitrage. Investing in the portfolio should be no different than the risk-free alternative.
- Let r be the prevailing interest rate on risk free bonds (say government bonds).
- Difference in return should be zero:

$$0 = r\Pi dt - d\Pi$$

So

$$r\Pi dt = d\Pi$$

Deriving the PDE

Substituting:

$$r\Pi dt = \left(V_t + \frac{\sigma^2}{2} S^2 V_{SS}\right) dt$$

$$r(V - \Delta S) = V_t + \frac{\sigma^2}{2} S^2 V_{SS}$$

$$rV = V_t + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_s$$

The last equation is the Black-Scholes-Merton PDE.

The PDE

In summary:

- $S(t)$ be the value of the underlying at time t.
- $V(S(t), t)$ be the value of the derivative at time t.
- r be the zero risk interest rate.
- σ be the volatility of the underlying.

Then the Black-Scholes PDE is:

$$rV = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S$$

Boundary Conditions

Assume that the derivative contract gives the owner the right to buy the underlying at fixed price K (strike price) at anytime upto and including time T. Then we have the following boundary conditions:

$$\begin{cases} V(0,t) = 0, \text{ for all } t \\ V(S,t) \to S \text{ as } S \to \infty \\ V(S,T) = max(S-K,0) \end{cases}$$

Black-Scholes IBVP

Goal: Solve the following initial boundary value problem:

$$rV = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S$$

$$\begin{cases} V(0,t) = 0, \text{ for all } t \\ V(S,t) \sim S \text{ as } S \to \infty \\ V(S,T) = max(S-K,0) \end{cases}.$$

We will do this by transforming the Black-Scholes PDE into the heat equation.

The Heat Equation

The heat equation in one space dimensions with Dirchlet boundary conditions is:

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = u_0(x) \end{cases}$$

and its solution has long been known to be:

$$u(x,t) = u_0 * \Phi(x,t)$$

where

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4kt}}$$

is the fundamental solution and $*$ is the convolution operator.

Transforming the PDE

Let

$$\begin{cases} \tau = \sigma^2/2(T-t) \\ x = \ln(S/K) \\ V(S,t) = Ku(x,\tau) \end{cases}$$

Then by the multivariate chain rule:

$$V_S = Ku = K(u_x x_s + u_\tau \tau_S) = \frac{Ku_x}{S} = e^{\ln(S/K)}u_x = e^{-x}u_x$$

$$V_{SS} = \frac{-K\sigma^2}{2}u_\tau$$

$$V_t = \frac{-K\sigma^2 u_\tau}{2}$$

Transforming the Black-Scholes PDE

The original PDE is:

$$rV = V_t + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_s.$$

Substituting and simplifying obtain:

$$u_\tau = u_{xx} + (k-1)u_x - kv$$

where $k = \frac{2r}{\sigma^2}$. Not quite the heat equation...but closer.

Another substitution

Let $w(x,\tau) = e^{ax+b\tau} w(x,\tau)$. Then

$$\begin{cases} w_x = e^{ax+b\tau}(au(x,t) + u_x) \\ w_{xx} = e^{ax+b\tau}(a^2 u(x,\tau) + 2\alpha u_x + u_{xx}) \\ w_\tau = e^{ax+b\tau}(bu(x,\tau) + u_\tau) \end{cases}$$

Another substitution

With these substitutions:

$$w_\tau = (\alpha^2 + (k-1)a - k - b)w + (2a + k - 1)u_x + u_{xx}$$

- If $2a + k - 1 = 0$ and $\alpha^2 + (k-1)a - k - b = 0$ the this is the heat equation.
- But a, b are arbitrary and basic algebra gives the solution $a = (1 - k/2)$ and $b = -(k+1)^2/4$.
- So we can make $w_\tau = w_{xx}$

The transformed PDE

Performing the substitutions on the boundary conditions obtain:

$$w_\tau = w_{xx}, \qquad x \in \mathbb{R}, \tau \in (0, T\sigma^2/2)$$

$$\begin{cases} w(x,0) = \max\{e^{(k+1)x/2} - e^{(k-1)x/2}, 0\}, & x \in \mathbb{R} \\ w(x,\tau) \to 0 \text{ as } x \to \pm\infty, & \tau \in (0, T\sigma^2/2) \end{cases}$$

This is the heat equation with Dirchlet boundary conditions!

The solution

Solving the heat equation with the boundary data and transforming back to the variables S, t:

Theorem (The Black-Scholes European Call Pricing Formula)

Let $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2} dz$, $d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$. *Then:*

$$V(S,t) = SN(d_1) - Ke^{-rT}N(d_2)$$

Interpretation

- $V(S,t) = SN(d_1) - Ke^{-rT}N(d_2)$
- Given the current price of the underlying asset S, the conditions of the option (T, K), and the interest rate on a suitable government bond, the value of the derivative can be calculated by this formula.
- Implicitly derived this equation for a European Call Option. Easy extensions to a variety of other derivatives.

Sample Computation:

Example

- A European call style option is made for a security currently trading at \$ 55 per share with volatility .45. The term is 6 months and the strike price is \$ 50. The prevailing no-risk interest rate is 3 %. What should the price per share be for the option?
- $S = 55,\ K = 50,\ T = .5,\ \sigma = .45,\ r = .03.$
- $d_1 = 0.50577047542718$; $d_2 = 0.18757242389323$
- $V(S,t) = SN(d_1) - Ke^{-rT}N(d_2)$
- Price of the option should be about \$ 9.99.

References

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