Financial Derivatives

Definition

A derivative is a financial contract whose value is based on the value of an underlying asset.

- Typically, a derivative gives the holder the right to buy an asset at a pre-determined price over some time horizon.
- Buyers and sellers use derivatives to offset risk in their portfolios (hedging).
- One of the sophisticated instruments that rose to prominence during the financial revolutions of the 70’s and 80’s.

The Pricing Problem

- Investors want to trade derivatives.
- The value of the derivative is based on the value of the underlying, market conditions, and terms of the contract.
- The value of a derivative itself is unclear.

Example

A dairy farmer might agree to a “forward contract” with milk processors which guarantees a fixed price for future quantities of milk produced.

- Shifts the risk of price drops from farmers to producers.
- Limits farmers ability to gain from price increases.
- In general, the underlying can be any asset or commodity.
Plan for Solving the Pricing Problem

- Make some assumptions about the underlying asset and the derivatives market.
- Build a model.
- Formulate the problem in terms of a partial differential equation (Black-Scholes-Merton PDE)
- Find a way to solve the PDE

**Remarkable Insight**

A basic transformation will turn the Black-Scholes equation into a classical PDE!

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Basic Assumptions:

- Frictionless and efficient market for derivatives.
- Trading in assets is a continuous process.
- Every underlying instrument has a unique, known price.
- The price of the underlying follows a stochastic process.

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Brownian Motion

**Definition**

A process \( z(t) \) follows **standard Brownian motion** if

- \( z(0) = 0 \)
- \( z \) is continuous at time \( t \) with probability 1 for each \( t \).
- For all \( t_1, t_2 \) such that \( 0 \leq t_1 \leq t_2 \), \( z(t_2) - z(t_1) \) is a normally distributed random variable with mean 0 and variance \( t_2 - t_1 \).
- The increments are independent: for all times \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \), \( z(t_2) - z(t_1), z(t_3) - z(t_2), \ldots, z(t_n) - z(t_{n-1}) \) are independent random variables.

We can intuitively regard Brownian motion as a random walk with step sizes tending to zero.

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Price Dynamics for the Underlying Asset

Let \( S(t) \) be the value of the underlying. Our model assumes the instantaneous rate of return on \( S \) is given by:

\[
\frac{dS}{S} = \mu dt + \sigma dz(t)dt
\]

where

- \( \mu \) is the expected return on the asset.
- \( \sigma \) is the variance of the return on the asset.
- \( dz(t) \) represents a stochastic process, in particular assume it is Brownian motion.
Illustration of Brownian Modeling

- The log of the value of the underlying obeys Brownian motion. Let $X = \ln S$
- $dX = \mu dt + \sigma dz(t)\sqrt{dt}$
- Discrete form: $X(t_{i+1}) - X(t_i) = \mu \Delta t + \sigma dz(t_i)\sqrt{\Delta t}$
- $S(t_{i+1}) = S(t_i)e^{\mu \Delta t + \sigma dz(t_i)\sqrt{\Delta t}}$

Simulation

- Model for stock price over a single trading day: $S(t_{i+1}) = S(t_i)e^{\mu \Delta t + \sigma dz(t_i)\sqrt{\Delta t}}$
- Parameter values: $\mu = .01, \sigma = .04, \Delta t = .004, P(0) = 50.$
- $dz(t)$ is a random normal variable with mean 0, variance 1.

Example 1

Figure: Example 1

Example 2

Figure: Example 2
Example 3

Figure: Example 3

Deriving the PDE

To derive the PDE:

- $S$ be the price of the underlying.
- $V(S, t)$ be the value of the derivative.
- Form a portfolio $\Pi$ by selling the derivative and buying $\Delta$ units of the underlying.
- The value of your portfolio is $\Pi(t) = V(t) - \Delta S(t)$.
- By linearity: $d\Pi = d(V - \Delta S) = dV - \Delta dS$
- Need to find a way to compute $dV$.

Ito’s Lemma

Lemma (Ito’s Lemma)

Let $V = V(S(t), t)$ where $S$ satisfies

$$dS = \mu S dt + \sigma S dz(t)dt$$

Then:

$$dV = \left( \mu V_S + V_t + \frac{\sigma^2}{2} V_{SS} \right) dt + \sigma V_S dz(t)dt.$$
Deriving the PDE

We have:

\[ d\Pi = \left( \mu S [V_s - \Delta] + V_t + \frac{\sigma^2}{2} S^2 V_{SS} \right) dt + \sigma S (V_s - \Delta) dW. \]

We would like to eliminate the random term \( dW \). Since \( \Delta \) is arbitrary, we set \( \Delta = V_s \) and obtain:

\[ d\Pi = \left( V_t + \frac{\sigma^2}{2} S^2 V_{SS} \right) dt \]

Substituting:

\[ r\Pi dt = \left( V_t + \frac{\sigma^2}{2} S^2 V_{SS} \right) dt \]

\[ r(V_s - \Delta S) = V_t + \frac{\sigma^2}{2} S^2 V_{SS} \]

\[ rV = V_t + \frac{\sigma^2}{2} S^2 V_{SS} + rSV_s \]

The last equation is the Black-Scholes-Merton PDE.

Deriving the PDE

Fundamental Economic Assumption: No arbitrage. Investing in the portfolio should be no different than the risk-free alternative.

Let \( r \) be the prevailing interest rate on risk free bonds (say government bonds).

Difference in return should be zero:

\[ 0 = r \Pi dt - d\Pi \]

So

\[ r\Pi dt = d\Pi \]

The PDE

In summary:

- \( S(t) \) be the value of the underlying at time \( t \).
- \( V(S(t), t) \) be the value of the derivative at time \( t \).
- \( r \) be the zero risk interest rate.
- \( \sigma \) be the volatility of the underlying.

Then the Black-Scholes PDE is:

\[ rV = V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + rSV_s \]
Boundary Conditions

Assume that the derivative contract gives the owner the right to buy the underlying at fixed price $K$ (strike price) at anytime up to and including time $T$. Then we have the following boundary conditions:

\[
\begin{align*}
V(0, t) &= 0, \text{ for all } t \\
V(S, t) &\to S \text{ as } S \to \infty \\
V(S, T) &= \max(S - K, 0)
\end{align*}
\]

Black-Scholes IBVP

Goal: Solve the following initial boundary value problem:

\[
rV = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S
\]

\[
\begin{align*}
V(0, t) &= 0, \text{ for all } t \\
V(S, t) &\sim S \text{ as } S \to \infty \\
V(S, T) &= \max(S - K, 0)
\end{align*}
\]

We will do this by transforming the Black-Scholes PDE into the heat equation.

The Heat Equation

The heat equation in one space dimensions with Dirichlet boundary conditions is:

\[
\begin{align*}
&u_t = u_{xx} \\
u(x, 0) = u_0(x)
\end{align*}
\]

and its solution has long been known to be:

\[
u(x, t) = u_0 * \Phi(x, t)
\]

where

\[
\Phi(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}
\]

is the fundamental solution and $*$ is the convolution operator.

Transforming the PDE

Let

\[
\begin{align*}
\tau &= \frac{\sigma^2/2}{T - t} \\
x &= \ln(S/K) \\
V(S, t) &= Ku(x, \tau)
\end{align*}
\]

Then by the multivariate chain rule:

\[
\begin{align*}
V_S &= Ku = K(u_x x_s + u_x \tau_S) = K\frac{u_x}{S} = e^{\ln(S/K)} u_x = e^{-x} u_x \\
V_{SS} &= \frac{-K\sigma^2}{2} u_x \\
V_t &= \frac{-K\sigma^2 u_x}{2}
\end{align*}
\]
Transforming the Black-Scholes PDE

The original PDE is:

\[ rV = V_t + \frac{\sigma^2}{2}S^2V_{SS} + rSV_s. \]

Substituting and simplifying obtain:

\[ u_{\tau} = u_{xx} + (k - 1)u_x - kv \]

where \( k = \frac{2r}{\sigma^2} \). Not quite the heat equation...but closer.

Another substitution

Let \( w(x, \tau) = e^{ax + b\tau}w(x, \tau) \). Then

\[
\begin{align*}
  w_x &= e^{ax + b\tau}(au(x, \tau) + u_x) \\
  w_{xx} &= e^{ax + b\tau}(a^2u(x, \tau) + 2\alpha u_x + u_{xx}) \\
  w_{\tau} &= e^{ax + b\tau}(bu(x, \tau) + u_{\tau})
\end{align*}
\]

Another substitution

With these substitutions:

\[ w_{\tau} = (\alpha^2 + (k - 1)a - k - b)w + (2a + k - 1)u_x + u_{xx} \]

- If \( 2a + k - 1 = 0 \) and \( \alpha^2 + (k - 1)a - k - b = 0 \) the this is the heat equation.
- But \( a, b \) are arbitrary and basic algebra gives the solution
  \( a = (1 - k/2) \) and \( b = -(k + 1)^2/4 \).
- So we can make \( w_{\tau} = w_{xx} \)

The transformed PDE

Performing the substitutions on the boundary conditions obtain:

\[
\begin{align*}
  w_{\tau} &= w_{xx}, \quad x \in \mathbb{R}, \tau \in (0, T\sigma^2/2) \\
  w(x, 0) &= \max\{e^{(k+1)x/2} - e^{(k-1)x/2}, 0\}, \quad x \in \mathbb{R} \\
  w(x, \tau) &\to 0 \text{ as } x \to \pm\infty, \quad \tau \in (0, T\sigma^2/2)
\end{align*}
\]

This is the heat equation with Dirichlet boundary conditions!
The solution

Solving the heat equation with the boundary data and transforming back to the variables $S, t$:

**Theorem (The Black-Scholes European Call Pricing Formula)**

Let $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2} dz$, $d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}$, $d_2 = d_1 - \sigma \sqrt{T}$.

Then:

$$V(S, t) = SN(d_1) - Ke^{-rT} N(d_2)$$

Interpretation

- $V(S, t) = SN(d_1) - Ke^{-rT} N(d_2)$
- Given the current price of the underlying asset $S$, the conditions of the option ($T, K$), and the interest rate on a suitable government bond, the value of the derivative can be calculated by this formula.
- Implicitly derived this equation for a European Call Option. Easy extensions to a variety of other derivatives.

Sample Computation:

**Example**

- A European call style option is made for a security currently trading at $55 per share with volatility $0.45$. The term is 6 months and the strike price is $50$. The prevailing no-risk interest rate is $3\%$. What should the price per share be for the option?
- $S = 55$, $K = 50$, $T = 0.5$, $\sigma = 0.45$, $r = 0.03$.
- $d_1 = 0.50577047542718$; $d_2 = 0.1875724389323$
- $V(S, t) = SN(d_1) - Ke^{-rT} N(d_2)$
- Price of the option should be about $9.99.

References