The Coarea formula for $W^{1,p}$ functions

A Masters Exam Presentation

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Notation.

- Let $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection

$$\pi(x, x_{n+1}, ..., x_{n+m}) = x.$$

- For a map $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, let $\bar{f} : \Omega \to \mathbb{R}^{n+m}$ be the graph mapping:

$$\bar{f}(x) = (x, f(x)).$$

- Let $\Gamma_f = \bar{f}(\Omega)$ be the graph of f.

Notation.

- The Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$, denoted $Jf(x)$, is obtained via the polar decomposition theorem for linear maps and is given by:

$$Jf(x) = \sqrt{\det{(Df)^T Df}}.$$

- For $f : \Omega \to \mathbb{R}^m$, we say $f \in W^{1,p}(\Omega, \mathbb{R}^m)$ provided $\int_\Omega (|u|^p + |\nabla u|^p)dx < \infty$.

The Lorentz spaces: a "refined scale" of L^p spaces.

Definition

If f is a measurable function on Ω, given the distribution function

$$\mu_f(s) = \mathcal{L}^n(\{x \in \Omega : |f(x)| > s\})$$

set

$$\|f\|_{L^{m,1}(\Omega)} = \int_0^\infty \mu_f(s)^{1/m} ds.$$

Then f belongs to the Lorentz space $L^{m,1}(\Omega)$ if $\|f\|_{L^{m,1}(\Omega)} < \infty$.

The Lorentz spaces: a "refined scale" of L^p spaces.

- $L^{1,1}(\Omega) = L^1(\Omega)$.
- If $p > m > 1$, $L^p(\Omega) \subsetneq L^{m,1}(\Omega) \subsetneq L^m(\Omega)$

Hausdorff measure

Definition

Let $0 \leq \delta \leq \infty$ and $0 \leq q < \infty$. The q dimensional Hausdorff-δ measure of the set E, $\mathcal{H}^q_\delta(E)$, is the infimum of the sums:

$$\sum_{j=1}^{\infty} \alpha_q \left(\frac{\mathrm{diam} E_j}{2} \right)^q$$

over all countable coverings of E by sets $\{E_j\}_{j=1}^{\infty}$ with $\mathrm{diam}(E_j) \leq \delta$, and α_q the ball constant.

Hausdorff content and measure

Definition

We call $\mathcal{H}^q_\infty(E)$ the q dimensional Hausdorff content of E.

Definition

The q dimensional Hausdorff measure of E, denoted $\mathcal{H}^q(E)$, is defined as

$$\lim_{\delta \to 0} \mathcal{H}^q_\delta(E).$$

Hausdorff countable recitifiability

Definition

For integer q a set $E \subset \mathbb{R}^n$ is said to be countably $\mathcal{H}^q$ rectifiable if there exist subsets $E_k \subset \mathbb{R}^q$ and Lipschitz mappings $g_k : E_k \to \mathbb{R}^n$ with the property that

$$\mathcal{H}^q \left(E \backslash \cup_{k=1}^{\infty} g_k(E_k) \right) = 0.$$

The classical coarea formula

Theorem

For Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$ with $n \geq m$ and $A \subset \mathbb{R}^m$ measurable:

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))dy.$$

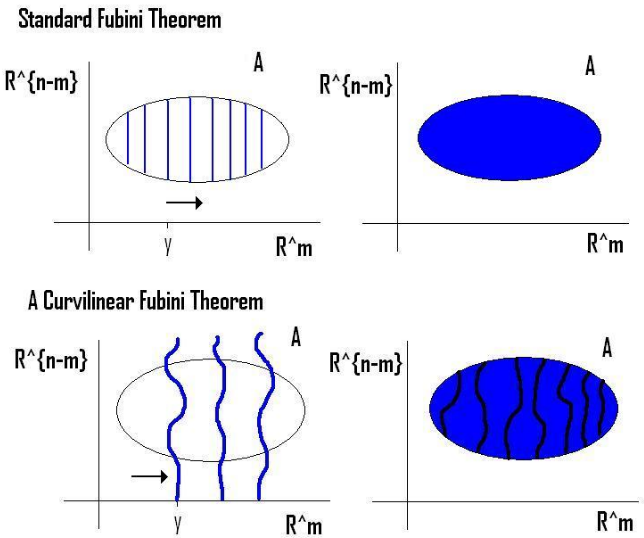
Curvilinear Fubini

The coarea formula is a generalization of Fubini's theorem.
To see this, let $\Pi : \mathbb{R}^m \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ be the projection map onto the first m-components. Then using the polar decomposition theorem, $Jf(x) = 1$ and for a fixed $y \in \mathbb{R}^m$,

$$\mathcal{H}^{n-m}(A \cap f^{-1}(y)) = \int_{\mathbb{R}^{n-m}} \chi_{A_y}(x)dx$$

where $A_y = \{x \in A | x = (y, \hat{x}) \text{ where } \hat{x} \in \mathbb{R}^{n-m}\}$ and so

$$\mathcal{L}^n(A) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-m}} \chi_{A_y(x)} dxdy.$$



Spherical coordinates integration formula.

For continuous, integrable functions $f : \mathbb{R}^m \to \mathbb{R}$, the spherical coordinates integration formula is a special case of the coarea formula:

$$\int_{\mathbb{R}^n} f \quad dx = \int_0^\infty \left(\int_{\partial B(x_0, r)} f \quad ds \right) dr.$$

This works by setting $u(x) = |x - x_0|$ in the equivalent coarea formulation

$$\int_{\mathbb{R}^n} f(x)Ju(x)dx = \int_{\mathbb{R}^m} \left(\int_{u^{-1}(y)} f(x)d\mathcal{H}^{n-m} \right) dy.$$

Main Question.

The purpose of this talk is to establish the coarea formula for functions more general than Lipschitz functions.

In particular, we'll investigate the possibility of taking $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^m)$.

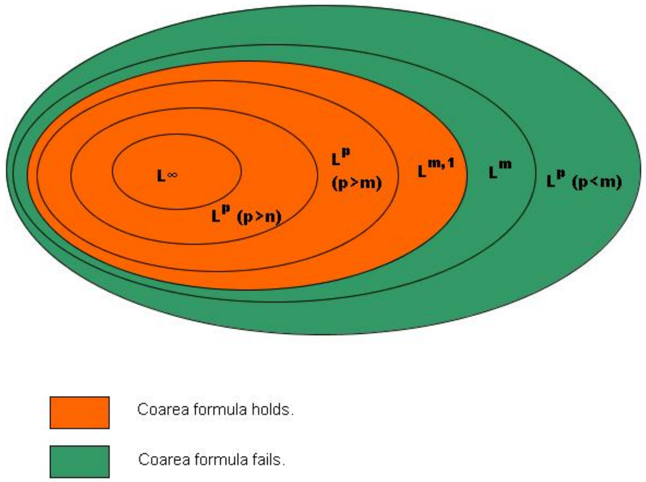
Possible extensions.

- The classical coarea formula is easily extend for $f \in W^{1,p}(\Omega \subset \mathbb{R}^n, \mathbb{R}^m)$ when $p > n$.
- In general, n may be much larger than m. Can this estimate be sharpened?

Cases.

Let $f \in L^p(\Omega, \mathbb{R}^m)$:

1. $|\nabla f| \in \mathcal{L}^p$ $p > m$ coarea works.
2. $|\nabla f| \in \mathcal{L}^m$ fails.: Possible to construct a continuous $W^{1,m}(\mathbb{R}^n)$ function with $Jf = 0$ a.e. which maps every set of the form $I \times \mathbb{R}^{n-m}$ onto an m-cube. This breaks the coarea formula.
3. But if $|\nabla f| \in L^{m,1}$ (Lorentz space) the coarea formula holds.



The goal.

Theorem (Main Result)

Suppose that $1 \leq m \leq n$ and that $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, and that $|\nabla f| \in L^{m,1}(\Omega)$. Then $f^{-1}(y)$ is countably $\mathcal{H}^{n-m}$ rectifiable for a.e. $y \in \mathbb{R}^m$, Γ_f is countably $\mathcal{H}^n$ rectifiable and for every measurable $E \subset \Omega$::

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} H^{n-m}(A \cap f^{-1}(y))dy.$$

The proof and development of this theorem is primarily from the paper: Maly, J., Swanson, D., and Ziemer, W. *The coarea formula for Sobolev mappings*, Trans. Amer. Math. Soc. **355** (2003), 477-492.

Outline for the rest of the talk.

We will prove the coarea formula for $f \in W^{1,p}(\Omega, \mathbb{R}^m)$ with $p > m$ and then show how the same proof can be extended for this more general case.

1. Sketch the classical proof.
2. Generalize the classical proof.
3. Prove the case for $p > m$.
4. Indicate how to modify our proof for the $f \in W^{1,1}, \nabla f \in L^{m,1}$ case.

How do we prove the classical coarea formula?

Use differentiability to decompose the set $A = Z \cup A_+ \cup A_0$ where

1. Z is the $\mathcal{L}^n$ measure zero set where Jf does not exist.
2. A_+ is the set where $Jf > 0$.
3. A_0 is the set where $Jf = 0$.

Then:

1. Use a Lusin type property to show that the coarea formula holds on the zero measure set Z.
2. Use an affine approximation to show that the coarea formula holds on the positive level set A_+.
3. Use an enlargement procedure to reduce the A_0 case to the A_+ case.

Proof sketch.

Proof. (Sketch.) By Rademacher, since f is Lipschitz, it is differentiable a.e.. Hence the set Z where $Jf(x)$ is undefined has $\mathcal{L}^n$ measure zero. We note that the coarea formula will hold on Z if f satisfies the following Lusin property:

Definition (Lusin Property (N))

A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$ will be said to satisfy the Lusin (N) property if

$$(N) \qquad \mathcal{L}^n(B) = 0 \Rightarrow \int_{\mathbb{R}^m} H^{n-m}(B \cap f^{-1}(y))d\mathcal{H}^m(y) = 0$$

holds for every measurable $B \subset \mathbb{R}^n$.

Proof sketch (cont.)

Now recall the famous Eilenberg inequality:

Theorem (Eilenberg Inequality)

For $m \leq d \leq n$, $A \subset \mathbb{R}^n$ and $f \in W^{1,p}(A, \mathbb{R}^n)$ with $p > n$ we have:

$$(E) \qquad \int_{\mathbb{R}^m}^{*} \mathcal{H}^{d-m}(A \cap f^{-1}(y))dH^m(y) \leq c\mathcal{H}^d(A)^{1-\frac{m}{p}} \cdot (\int_{\mathbb{R}^n} |\nabla f|^p)^{\frac{m}{p}}.$$

Note: For Lipschitz f, $(\int_{\mathbb{R}^n} |\nabla f|^p)^{m/p} = (\mathrm{Lip}(f))^m$.

Proof sketch (cont.)

The Eilenberg inequality shows that f satisfies (N) and hence that both sides of the coarea formula will be zero on the set Z.

For the case of the positive level set A_+, we may use the fact that $Jf(x)$ is invertible and the following lemma to produce an arbitrarily close affine approximation to f:

Proof sketch (cont.)

Lemma

Let $t > 1$. Then there is a countable collection of Borel sets $\{A_k\}_{k=1}^{\infty}$ such that

1. $A_+ = \cup_{k=1}^{\infty} A_k$
2. $f|_{A_k}$ *is one-to-one*
3. *For all k there is a symmetric nonsingular linear map $T_k : \mathbb{R}^n \to \mathbb{R}^m$ so that*

$$\mathrm{Lip}(f|_{E_k} \circ f_k^{-1}) \leq t$$
$$\mathrm{Lip}(f_k \circ (f|_{E_k})^{-1}) \leq t$$
$$\frac{|\det T_k|}{t^n} \leq (Jf)|_{E_k} \leq t^n |\det T_k|.$$

Proof sketch (cont.)

For the case of the zero level set A_0, we can show that

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))dy = 0$$

by considering the map $f_\epsilon(x) = f(x) + \epsilon x$, noting that the coarea formula applies to Jf_ϵ by the previous case, and taking a limit. □

Proof of the Eilenberg inequality.

Theorem (Eilenberg Inequality)

For $m \leq d \leq n$, $A \subset R^n$ and $f \in W^{1,p}(A, \mathbb{R}^n)$ with $p > n$ we have:

$$(E) \qquad \int_{\mathbb{R}^m}^{*} \mathcal{H}^{d-m}(A \cap f^{-1}(y)) dH^m(y) \leq c\mathcal{H}^d(A)^{1-\frac{m}{p}} \cdot \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{\frac{m}{p}}.$$

Proof. Cover A by closed balls $\{B_{ij}\}$ so that $A \subset \cup_{i,j} B_{ij}$, $\mathrm{diam}(B_{ij}) \leq 1/j$, and $\sum_i \mathcal{L}^n(A) + 1/j$.

Eilenberg inequality proof. (cont.)

Let B be a ball in $\mathbb{R}^n$. A version of the Sobolev embedding theorem gives:

$$\mathrm{osc}_B(f) \leq C(\mathrm{diam}(B))^{1-\frac{n}{p}} \left(\int_B |\nabla f|^p\right)^{\frac{1}{p}}.$$

For each i, j, the isodiametric inequality implies:

$$\mathcal{L}^d(f(B_{ij})) \leq C[\mathrm{diam}(f(B))]^d$$

and so

$$\mathcal{L}^d(f(B_{ij})) \leq C[r_{ij}^{1-n/p}]^d \left(\int_{B_{ij}} |\nabla f|^p\right)^{\frac{d}{p}}$$

Eilenberg inequality proof. (cont.)

Let $g_{ij}(y) = C\mathrm{diam}(B_{ij})\chi_{f(B_{ij})}(y)$.

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathcal{H}^{n-d}(A \cap f^{-1}(y)) dy &= \int_{\mathbb{R}^m} \lim_{j \to \infty} \mathcal{H}_{1/j}^{n-d}(A \cap f^{-1}(y)) dy \\
&\leq \int_{\mathbb{R}^m} \liminf_{j \to \infty} \sum_{i=1}^{\infty} g_{ij}(y) dy \\
&\leq \liminf_{j \to \infty} \sum_{i=1}^{\infty} \int_{\mathbb{R}^m} g_{ij}(y) dy \\
&= C \liminf_{j \to \infty} \sum_{i=1}^{\infty} [\mathrm{diam}(B_{ij})]^{n-d} \mathcal{L}^d(f(B_{ij}))
\end{aligned}$$

Eilenberg inequality proof. (cont.)

$$\begin{aligned}
&\leq C \liminf_{j \to \infty} \sum_{i=1}^{\infty} [\mathrm{diam}(B_{ij})]^{n-d} \cdot [\mathrm{diam}(B_{ij})]^{d-nd/p} \cdot \left(\int_{B_{ij}} |\nabla f|^p\right)^{\frac{d}{m}} \\
&= C \liminf_{j \to \infty} \sum_{i=1}^{\infty} [\mathrm{diam}(B_{ij})^n]^{1-d/p} \left(\int_{B_{ij}} |\nabla f|^p\right)^{\frac{d}{p}} \\
&\leq C\mathcal{L}^n(A)^{1-d/p} \cdot \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{\frac{d}{p}}.
\end{aligned}$$

□

What changes when the class of functions is more general than L^∞?

- **Differentiability.** If $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ for $p > n$, the Calderon theorem says that f is still differentiable a.e.. Beyond this class, we must modify our notion of differentiability.
- **Lusin.** Without the Eilenberg inequality (which holds only when $p > n$) property (N) may fail to hold.

Approximate differentiability

In the general context, we can replace a.e. differentiability with the weaker notion of approximate differentiability:

Definition

A measurable function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be approximately differentiable at a point x if for all $\epsilon > 0$

$$\lim_{r \downarrow 0} \frac{|\{y \in \mathbb{R}^n : |f(y) - f(x) - Df(x) \cdot (y - x)| > \epsilon |y - x|\} \cap B_r(x)|}{r^n} = 0.$$

A new decomposition result.

Now we will argue that we can decompose A as

$$A = A_0 \cup (\cup_{i=1}^{\infty} A_i)$$

where $|A_0| = 0$ and $f|_{A_i}$ is Lipschitz for each i.

We'll need two lemmas first.

A lemma.

Lemma

If $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^m)$, then there is F with $|F| = 0$ so that for all $x \in \mathbb{R}^n \backslash F$,

$$\frac{1}{r^n} \int_{B_r(x)} |u(y) - u(x) - Du(x) \cdot (y - x)| dy \leq o(r)$$

Proof of lemma.

Proof. Since $Du \in \mathcal{L}^1(\mathbb{R}^n)$ by the Lebesgue differentiation theorem, there exists an F with $|F| = 0$ so that

$$\lim_{r \downarrow 0} \fint_{B_r(x)} |Du(y) - Du(x)| dy = 0$$

and

$$\lim_{r \downarrow 0} \fint_{B_r(x)} |u(y) - u_{B_r(x)}| dy = 0$$

for all $x \in \mathbb{R}^n \backslash F$.

Proof of lemma (cont.)

By a variant of the Poincaré inequality:

$$\fint_{B_r(x)} |u(y) - u(x) - Du(x) \cdot (y - x)| dy \leq Cr \fint_{B_r(x)} |Du(y) - Du(x)| dy = o(r)$$

□

Another lemma.

Lemma

If $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^m)$ then there exists F with $|F| = 0$ such that f is approximately differentiable for all $x \in \mathbb{R}^n \backslash F$.

Proof of lemma.

Proof. Let F be the measure zero set of the previous lemma. Then for all $x \in \mathbb{R}^n \backslash F$

$$\frac{1}{r^n} \int_{B_r(x)} |u(y) - u(x) - Du(x) \cdot (y - x)| dy = o(1) r.$$

Now let $\epsilon > 0$ and

$$E_\epsilon = \{y : |u(y) - u(x) - Du(x)(y - x)| > \epsilon |y - x|\}$$

Proof of lemma.

$$o(1)r = \frac{1}{r^n}\int_{B_r(x)} |u(y) - u(x) - Du(x)\cdot(y-x)|dy$$
$$\geq \frac{1}{r^n}\int_{E_\epsilon \cap B_r(x)} \epsilon|y-x|dy$$
$$\geq Cr\frac{|E_\epsilon \cap B_r(x)|}{r^n}$$

Hence

$$\frac{|E_\epsilon \cap B_r(x)|}{r^n} \to 0 \text{ as } r \to 0$$

and so f is approximately differentiable. □

The new decomposition theorem.

Theorem

If $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\operatorname{ap}\overline{\lim_{x\to a}}\frac{|f(x)-f(a)|}{|x-a|} < \infty$$

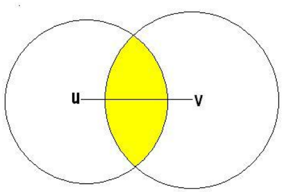
for all $a \in A$ then $A = \cup_{i=1}^{\infty} A_i$ of measurable sets such that $f|_{A_i}$ is Lipschitz.

Proof of theorem.

Proof. Set

$$\theta(u,v) := \frac{|B(u,|u-v|)\cap B(v,|u-v|)|}{|u-v|^n}$$

for $u \neq v \in \mathbb{R}^n$ and note that $0 < \theta < 1$.



Proof of theorem.

Then set:

$$Q(u,i,j) := B(u,r)\cap\{x : x \notin A \text{ or } |f(x)-f(u)| > j|x-y|\}$$
$$B_j := A \cap \{u : |Q(u,r,j)| < \frac{\theta r^n}{2}, 0 < r < \frac{1}{j}\}.$$

Then B_j is measurable and $A = \cup_{j=1}^{\infty} B_j$.

Proof of theorem.

Now we claim $f|_{B_j}$ is locally Lipschitz. To see why, let $u \neq v \in B_j$, $r = |u - v| < 1/j$. Note that

$$|Q(u,r,j) \cup Q(v,r,j)| < \frac{\theta r^n}{2} + \frac{\theta r^n}{2} = \theta r^n = |B(u,r) \cap B(v,r)|.$$

Hence, there is $x \in [B(u,r) \cap B(v,r)] \backslash [Q(u,r,j) \cup Q(v,r,j)]$ so that

$$|f(x) - f(u)| \leq j|x - y| \leq jr$$

$$|f(x) - f(v)| \leq j|x - y| \leq jr.$$

Proof of theorem.

By the triangle inequality:

$$|f(u) - f(v)| \leq |f(x) - f(u)| + |f(x) - f(v)| \leq 2jr = 2j|u - v|.$$

So if we further decompose each B_j into sets of diameter less than $1/j$ we have a decomposition of A so that f is locally Lipschitz on each set B_{ij}. □

The $W^{1,1}$ decomposition corollary.

Since the approximate differentiability of $f \in W^{1,1}(\Omega, \mathbb{R}^m)$ implies that for all x:

$$\mathrm{ap}\overline{\lim_{x \to a}} \frac{|f(x) - f(a)|}{|x - a|} < \infty$$

we have:

Corollary

If $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ there exists disjoint sets E_k in $\mathbb{R}^n$ so that

- $\mathcal{L}^n(\mathbb{R}^n \backslash \cup E_k) = 0$
- $f_k = f|_{E_k}$ *is Lipschitz on* E_k.

Lusin property $(\hat{N})$.

It turns out that a slightly different Lusin condition will facilitate a proof of the more general case of the coarea formula.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ will be said to satisfy the Lusin condition $(\hat{N})$ if

$$(\hat{N}) \qquad \mathcal{H}^n(E) = 0 \Rightarrow \mathcal{H}^n(\bar{f}(E)) = 0$$

Application of $(\hat{N})$.

Theorem

Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ with $1 \leq m \leq n$ and suppose that f satisfies the Lusin condition $(\hat{N})$. Then $f^{-1}(y)$ is countably $\mathcal{H}^{n-m}$ rectifiable for almost all $y \in \mathbb{R}^m$, the graph Γ_f is countably $\mathcal{H}^n$ rectifiable and the coarea formula holds for all measurable $A \subset \Omega$.

Proof. Decompose $A = F \cup (\cup A_k)$ where $f_k = f|_{A_k}$ is Lipschitz as in the $W^{1,1}$ decomposition corollary and F is $\mathcal{L}^n$ measure zero. Then the coarea formula will certainly hold for every $E \subset A_k$ since this is just the classical case.

Proof of the theorem.

Now we must argue that the coarea formula will hold on the $\mathcal{L}^n$ measure zero set F where $J\bar{f}(x)$ does not exist.

Suppose that $\mathcal{L}^n(E) = 0$. Let $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\rho : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be the projections:

$$\pi(x, x_{n+1}, ..., x_{n+m}) = x \quad \text{and} \quad \rho(x, x_{n+1}m, ..., x_{n+m}) = (x_{n+1}, ..., x_{n+m}).$$

Since projections are clearly Lipschitz, we apply the Eilenberg inequality:

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\bar{f}(E) \cap \rho^{-1}(y))d\mathcal{H}^m(y) \leq C \cdot (\mathrm{Lip}(\rho))^m \mathcal{H}^n(\bar{f}(E)) = 0$$

Proof of the theorem (cont.)

Claim:

$$\pi(\bar{f}(E) \cap \rho^{-1}(y)) = E \cap f^{-1}(y).$$

"$\supset$" $x \in E \cap f^{-1}(y) \Rightarrow f(x) = y \Rightarrow \rho(x, y) = x \Rightarrow x \in \rho^{-1}(y)$.

"$\subset$" $x \in \pi(\bar{f}(E) \cap \rho^{-1}(y)) \Rightarrow \rho(x) = y = f(x) \Rightarrow x \in f^{-1}(y) \cap E$.

Proof of the theorem (cont.)

Since Hausdorff measures do not increase on projection:

$$\begin{aligned}\int_{R^m} \mathcal{H}^{n-m}(E \cap f^{-1}(y))d\mathcal{H}^m(y) &= \int_{R^m} \mathcal{H}^{n-m}(\pi(\bar{f}(E) \cap \rho^{-1}(y)))d\mathcal{H}^m(y)\\ &\leq \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\bar{f}(E) \cap \rho^{-1}(y))dH^m(y)\\ &= 0.\end{aligned}$$

Hence the coarea formula holds for each component of the decomposition $A = F \cup (\cup_k A_k)$.

A general criterion for $(\hat{N})$.

We now show that $f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with $p > m$ (and more generally $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with $|\nabla f| \in L^{m,1}(\Omega)$) satisfies the Lusin property $(\hat{N})$. To do this, we provide a more general criterion for condition $(\hat{N})$.

A general criterion for $(\hat{N})$.

Theorem

Let $1 \leq m \leq n$, $f : \mathbb{R}^n \to \mathbb{R}^m$, and suppose $\theta \in L^1_{\text{loc}}(\mathbb{R}^n)$ is s.t.

$$\mathcal{H}^{n-m}_\infty(\pi(G_f \cap B(z,r))) \leq r^{-m} \int_{\pi(\Gamma_f \cap B(z,4r))} \theta(x)dx$$

$\forall z \in \mathbb{R}^{n+m}, r > 0$. Then $\exists C = C(n,m)$ s.t.

$$\mathcal{H}^n(\bar{f}(E)) \leq C \int_E \theta(x)dx$$

for all $\mathcal{L}^n$ measurable $E \subset \mathbb{R}^n$. In particular, f satisfies condition $(\hat{N})$.

Content lemma.

Lemma

Suppose $m \leq d \leq m+n$ and let $E \subset \mathbb{R}^{n+m}$. Then:

$$\mathcal{H}^d_\infty(E) \leq C(\operatorname{diam} E)^m \mathcal{H}^{d-m}_\infty(\pi(E))$$

where C is a constant depending on m, n and d.

Proof of general criterion for condition $(\hat{N})$.

Proof. Define a set function σ on $\mathbb{R}^{n+m}$ by

$$\sigma(E) = \int_{\pi(\Gamma_f \cap E)} \theta(x)dx.$$

Using the content lemma with $d = n$:

$$\begin{aligned}\mathcal{H}^n_\infty(\Gamma_f \cap B(z,r)) &\leq Cr^m \mathcal{H}^{n-m}_\infty(\pi(\Gamma_f \cap B(z,r))) \\ &\leq C \int_{\pi(\Gamma_f \cap B(z,4r))} \theta(x)dx \\ &= C\sigma(B(z,4r))\end{aligned}$$

for any $z \in \mathbb{R}^{n+m}$ and $r > 0$.

Proof of general criterion (cont.)

Recall that for $E \subset \mathbb{R}^n$ of finite H^q measure the q-dimensional upper density of $x \in A$ with respect to H^q_δ measure is bounded below:

$$\frac{1}{2^q} \leq \overline{\lim_{r\to 0}} \frac{\mathcal{H}^q_\infty(B(x,r)\cap A)}{\alpha(q)r^q} \leq 1.$$

Hence:

$$\overline{\lim_{r\to 0}} r^{-n}\mathcal{H}^n_\infty(\Gamma_f \cap B(z,r)) \geq C$$

for $\mathcal{H}^n$ a.e. $z \in \Gamma_f$. So

$$\overline{\lim_{r\to 0}} r^{-n}\sigma(B(z,r)) \geq C$$

for $\mathcal{H}^n$ a.e. $z \in \Gamma_f$.

Proof of general criterion (cont.)

- Can easily show the projection $\pi(\Gamma_f \cap E)$ is Lebesgue measurable for every Borel E.

$\Rightarrow$ σ is a Borel measure on the Σ-algebra of $\mathbb{R}^{n+m}$.

$\Rightarrow$ We can then extend it to be a regular Borel outer measure on all of $\mathbb{R}^{n+m}$:

$$\hat{\sigma}(E) = \inf\{\sigma(B) : E \subset B, B \text{ is a borel set}\}.$$

$\Rightarrow$ Since the integrand function θ is locally integrable, $\hat{\sigma}$ will be finite on compact subsets of $\mathbb{R}^{n+m}$ which makes $\hat{\sigma}$ a Radon measure.

Proof of general criterion (cont.)

Use the Besicovitch lemma to cover of E with a family of balls $\{B_i\}$ so that

$$\hat{\sigma}(E \backslash \cup B_i) = 0.$$

Then:

$$\begin{aligned}\hat{\sigma}(E) &= \sum_{i=1}^{\infty} \sigma(B_i) \\ &\geq \sum_{i=1}^{\infty} C r_i^n \\ &\geq C\mathcal{H}^n(E).\end{aligned}$$

Proof of general criterion (cont.)

Now let G be a Borel set containing E. Then $\bar{f}(E) \subset G \times \mathbb{R}^m$ is Borel and

$$\begin{aligned}\mathcal{H}^n(\bar{f}(E)) &\leq C\hat{\sigma}(\bar{f}(E)) \\ &\leq C\sigma(G \times \mathbb{R}^m) \\ &= C\int_G \theta(x)dx\end{aligned}$$

Take the infimum over all such G to obtain:

$$\mathcal{H}^n_\infty(E) \leq C\int_E \theta(x)dx.$$

□

Capacity.

To proceed, we need a capacitary lemma.

Definition

For $1 \leq p < \infty$, the p-capacity $\gamma_p(E)$ of a set $E \subset \mathbb{R}^N$ is:

$$\gamma_p(E) = \inf_{u \in W_0^{1,p}(\mathbb{R}^n)} \{ \int_{\mathbb{R}^n} |\nabla u|^p dx : u \geq 1 \text{ on a neighborhood of } E\}$$

Idea: Gauge the size of a set with test functions rather than set coverings.

Content and capacity.

We now relate Hausdorff content and capacity through the following theorem.

Theorem

Suppose that $1 \leq m < p$ and $E \subset \mathbb{R}^N$. Then

$$\mathcal{H}_\infty^{n-m}(E) \leq C\gamma_p(E)$$

where $C = C(n, m, p)$.

A content estimate.

From this result, we get the following corollary.

Corollary

Suppose that $1 \leq m < p$, $E \subset \mathbb{R}^n$, and $u \in W^{1,p}(\mathbb{R}^n)$ with $u \geq 1$ on E. Then

$$\mathcal{H}_\infty^{n-m}(E) \leq C \int_{\mathbb{R}^n} (|\nabla u|^p + |u|^p) dx$$

The $W^{1,p}$, $p > m$ case.

Theorem

The coarea formula holds for $f \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^m)$ when $p > m$.

Proof.

We will show that $f \in W^{1,p}(\Omega, \mathbb{R}^m)$ satisfies the Lusin $(\hat{N})$ property. If this is the case then the coarea formula will hold by our theorem.

We show that f satisfies the Lusin $(\hat{N})$ property by verifying that f satisfies the conditions of the general criterion. Need to prove: $\exists \theta \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ so that

$$\mathcal{H}^{n-m}_\infty(\pi(\Gamma_f \cap B(z,r))) \leq r^{-m} \int_{\pi(\Gamma_f \cap B(z,4r))} \theta(x)dx.$$

Proof.

Proof. Let $z \in \mathbb{R}^{n+m}$ and $r > 0$. Write $z = (x_0, y_0)$. Then:

$$\Gamma_f \cap B(z,r) \subset \Gamma_f \cap [B(x_0,r) \times B(y_0,r)]$$

and

$$\pi(\Gamma_f \cap B(z,r)) \subset B(x_0,r) \cap f^{-1}(B(y_0,r)).$$

Let $E := B(x_0,r) \cap f^{-1}(B(y_0,r))$. Define

$$\hat{E} = \frac{1}{2r}(E - x_0) = \{x \in \mathbb{R}^n : x_0 + 2rx \in E\}.$$

Proof (cont.)

By properties of Hausdorff measure:

$$\mathcal{H}^{n-m}_\infty(\hat{E}) = (2r)^{m-n}\mathcal{H}^{n-m}_\infty(E)$$

and so

$$\mathcal{H}^{n-m}_\infty(\pi(\Gamma_f \cap B(z,r))) \leq Cr^{n-m}\mathcal{H}^{n-m}_\infty(\hat{E}).$$

Now if $\xi \in \hat{E}$ we have:

$$|\xi| \leq \frac{1}{2} \quad \text{and} \quad \frac{|f(x_0+2r\xi) - f(x_0)|}{2r} \leq \frac{1}{2}.$$

Proof (cont.)

Consider the test function $u\eta \in W^{1,p}(\mathbb{R}^n)$ where:

$$u(\xi) = 2\left(1 - \frac{|f(x_0+2r\xi) - f(x_0)|}{2r}\right)^+$$

and η is a smooth cutoff function so that

$$\chi_{B(0,1/2)} \leq \eta \leq \chi_{B(0,1)}$$

Proof (cont.)

We notice that $u\eta \geq 1$ on $\hat{E}$ and so by the content-capacity corollary:

$$\begin{aligned}\mathcal{H}^{n-m}_\infty(\hat{E}) &\leq C\int_{\mathbb{R}^n}(|u\eta|^p + |\nabla u\eta|^p)dx \\ &\leq C\int_{B(0,1)\cap\{u>0\}}(1+|\nabla u|^p)dx\end{aligned}$$

Now perform the change variable $x = x_0 + 2r\xi$:

$$\mathcal{H}^{n-m}_\infty(\hat{E}) \leq Cr^{-n}\int_{B(x_0,2r)\cap f^{-1}(B(y_0,2r))}(1+|\nabla u|^p)dx$$

Proof (cont.)

Now notice:

$$B(x_0,2r)\cap f^{-1}(B(y_0,2r)) \subset \pi(\Gamma_f \cap B(z,4r))$$

So:

$$\begin{aligned}\mathcal{H}^{n-m}_\infty(\pi(\Gamma_f\cap B(z,r))) &\leq Cr^{n-m}\mathcal{H}^{n-m}_\infty(\hat{E}) \\ &\leq Cr^{n-m}r^{-n}\int_{B(x_0,2r)\cap f^{-1}(B(y_0,2r))}(1+|\nabla u|^p)dx \\ &\leq Cr^{-m}\int_{\pi(\Gamma_f\cap B(z,4r))}(1+|\nabla u|^p)dx\end{aligned}$$

Proof (cont.)

Now, set $\theta = C(1+|\nabla u|^p)$. Then θ verifies the hypotheses of general criterion theorem. □

The $L^{m,1}$ case.

In the content-capacity corollary above, we related the Hausdorff content of E to the p-power integral of u and ∇u where $u \in W^{1,p}$, $p > m$.

It turns out that a more general result will hold but we will need some facts about Young functions.

Definition

A convex, nonnegative function $F : [0,\infty) \to \mathbb{R}$ *is a Young function if it satisfies*

$$F(t) = 0 \Leftrightarrow t = 0.$$

Young functions and capacity.

Theorem

Let $m > 1$, $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ with $|\nabla u| \in L^{m,1}$ and $u \geq 1$ on E. Then there is a unique Young function F so that:

$$\mathcal{H}^{n-m}_\infty(E) \leq C \int_{\mathbb{R}^n} (F(|\nabla u|) + F(|u|))dx.$$

How to modify the proof of the $p > m$ case.

Remark

For $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ and $|\nabla f| \in L^{m,1}$, we can modify the proof above by simply replacing:

$$\mathcal{H}^{n-m}_\infty(\hat{E}) \leq \boxed{C \int_{\mathbb{R}^n} (|u\eta|^p + |\nabla u\eta|^p)dx}$$

with

$$\mathcal{H}^{n-m}_\infty(\hat{E}) \leq \boxed{C \int_{\mathbb{R}^n} (F(|\nabla u|) + F(|u|))dx.}$$

The rest of the proof follows in the same way as before.

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