CT Scans and an Introduction to Inverse Problems
(or what I did on my summer vacation)

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Inverse Problems: An Overview

- An exciting and expanding area of mathematics
- A rich source of theoretical and applied problems
- A direct problem: Know the cause; find the effect. Example: Know how X-rays waves attenuate as they pass through a body in response to size, shape, and material composition of the body. Knowing the cause, we can determine the attenuation of the radiation.
- An inverse problem: Know the effect; find the cause. Example: If the attenuated X-ray radiation is measured, what (if anything) can we say about the size, shape, and material composition of the body?

Acknowledgements

- I draw much of the theoretical background and intuition on the CT scan problem from a series of brilliant lectures delivered by Gunther Uhlmann, Peter Kuchment, and Leonid Kunyansky at the IPDE Summer School 2010 at the University of Washington.
- A comprehensive introduction to the mathematics of CT scans can be found in Charles Epstein’s book, *Introduction to the Mathematics of Medical Imaging* [1].
- A more advanced treatment can be found in the classic book of Frank Natter, *The Mathematics of Computed Tomography*, [2].
The CT Problem

- Application: The Computed Tomography problem: want to view the internal structure of something without cutting it open.
  1. Send radiation through the object and look at how it attenuates.
  2. Determine the density by looking at the attenuation patterns.
  3. Recover detailed image of internal structure from the density

- Example Problem: (Medical Computed Tomography (CT))
  Tumors and other abnormalities have specific densities, distinct from healthy tissues. Would like to look at a patient’s brain without cutting him open. Recover detailed images of the brain by knowing the densities at different points.

A model for X-ray Attenuation

- Goal: Image a 2D slice of the patient’s head. First, need to solve the direct problem.
- Let $I(x)$ be the flux of radiation at the point $x \in \mathbb{R}^2$, and let $I(x) = \|I(x)\|$ be the intensity of the radiation at $x$.
- Let $\mu(x)$ be the attenuation coefficient for X-ray radiation. This coefficient describes the energy loss for radiation passing through $x$ and is determined by the density and materials properties of the body.
A model for X-ray Attenuation

- For X-ray radiation traveling along a straight line we have Beer’s Law:

\[
\frac{dI}{dx} = -\mu(x)I
\]

- So if \( L \) is the line segment on the source and detector and the source emits initial intensity \( I_0 \), then the detector on the opposite side feels a radiation intensity

\[
I = I_0 e^{-\int_L \mu(x) \, dx}
\]

The Inverse Problem for CT

- We want to find \( \mu(x) \) which is essentially a proxy for the density of the patient’s head in the slice at the point \( x \).
- We know all the source intensities and we can compute all the detector intensities by computing all the integrals over all the lines on the sources and detectors.
- So solving the inverse problem is equivalent to resolving the following question: If we know all the line integrals of a function \( f(x) \) in a domain \( \Omega \), is this enough information to recover the function in \( \Omega \)?

The Radon Transform

Let \( L \) be a line in \( \mathbb{R}^2 \). Let \( t \) denote the perpendicular distance to the origin, \( \omega \) a unit vector perpendicular to \( L \), and \( \omega^\perp \) a unit vector perpendicular to \( \omega \). Then the points \( x \in L \) are those which satisfy

\[
\langle x, \omega \rangle = t
\]
The Radon Transform

The **Radon Transform** of \( f \in C_0^{\infty}(\mathbb{R}^2) \) is the map given by:

\[
Rf(t, \omega) = \int_{L_{t,\omega}} f(x) \, dx = \int_{-\infty}^{\infty} f(t \omega + s \omega^\perp) \, ds.
\]

- \( R \) maps functions in \( C_0^{\infty}(\mathbb{R}^2) \) into functions mapping lines in the plane into \( \mathbb{R} \).
- Even: \( Rf(-t, -\omega) = Rf(t, \omega) \)
- Domain? Range?

Sinograms

- The raw output of a CT scan procedure is called a **sinogram**.
- Typically, we visualize this data by a “heat” plot of the values of the Radon transform against \( t \) and \( \theta \).
- Darker and lighter areas of the sinogram plot correspond to different values of the Radon transform.
- We won’t usually have real X-ray data to use, but image files are a great substitute. Pixel locations correspond to points in space and gray scale colors correspond to densities.

A Phantom:

![A Phantom Image](image_url)

Sinogram

![Sinogram Image](image_url)
The appearance of sine waves in sinograms is due to the fact that the Radon transform of a delta function has support on a trigonometric curve.

Why? Regard the point $x$ as a vector with angle $\theta_0$. By trigonometry the only lines containing $x$ will be those with angle $\theta$ and affine parameter $t$ satisfying

$$t = |x| \cos (\theta - \theta_0).$$

Since $R\delta_x(t, \theta) = 0$ whenever $x$ is not in $L_{t,\theta}$, the support of $\delta_x$ is a cosine curve in the $\theta$-$t$ plane.

We can show that the Radon transform is continuous and thus an object made up of many small, sharp-edged features will have a sinogram that is a combination of blurred sine curves.
**Back-Projection**

- Idea: Know the integrals of \( f \) over every line; reconstruct \( f(x) \) by averaging all the integrals of the lines passing through \( x \).

\[
\tilde{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} Rf(\langle x, \omega(\theta) \rangle, \theta) \, d\theta
\]

- Theory: Regarding the Radon transform \( R \) as a map between two appropriate function spaces, we can prove that back-projection (\( R^\# \)) is the adjoint of \( R \).

**Theory:**

For functions \( f(x) \) from a “good domain” space, recalling that 
\[
t = x \cdot \omega,
\]
compute:

\[
R^\# R f(x) = \int_{S^1} R f(x \cdot \omega, \theta) \, d\omega
\]

\[
= \int_{S^1} \int_{-\infty}^{\infty} f((x \cdot \omega) + s\omega^\perp) \, ds \, d\omega
\]

Basically, a polar integral. Rewrite:

\[
R^\# R f(x) = \int \frac{2f(y)}{|y - x|} \, dy = \frac{2}{|x|} \ast f(x)
\]

Thus, we should recover not \( f(x) \) but a blurred version of \( f(x) \) with back-projection.

**A Phantom:**

![Phantom Image](image.png)

**Back-projection: Blurring**

![Blurred Image](image2.png)
Our task is to “invert” the Radon transform. To do this, we will make use of the Fourier transform.

**Definition**

The Fourier Transform of a function $f(x)$ on $\mathbb{R}$ is

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx.$$ 

and on $\mathbb{R}^n$ as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx.$$

- Domains? Ranges?
- Roughly, the Fourier Transform says how much of each frequency $\xi$ is present in the function $f$.
- Extremely important transform with many interesting properties.
- Invertible (as a map from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$).

There is a simple relationship between the Fourier and Radon transforms

**Theorem**

$$\int_{-\infty}^{\infty} \mathcal{R}f(t, \omega) e^{-itr} \, dt = \hat{f}(r \omega)$$

- The 1D Fourier transform of $\mathcal{R}f$ in the affine parameter $t$ is the 2D Fourier transform of $f$ expressed in polar coordinates.

**Proof of the Projection-Slice Theorem**

Proof. By definition

$$\int_{-\infty}^{\infty} \mathcal{R}f(t, \omega) e^{-itr} \, dt = \int_{\mathbb{R}^2} f(t \omega + s \omega^\perp) e^{-itr} \, dsdt.$$

Change variables: $x = t \omega + s \omega^\perp$ and use the fact that $t = \langle x, \omega \rangle$ to obtain

$$\int_{\mathbb{R}^2} f(t \omega + s \omega^\perp) e^{-itr} \, dsdt = \int_{\mathbb{R}^2} f(x) e^{-i <x, \omega > r} \, dx = \hat{f}(r \omega).$$
Radon Inversion Formula

**Theorem**

Let \( f \) be a function such that \( f \in L^1(\mathbb{R}^2) \cap \{ \text{domain of the Radon transform} \} \) and \( \hat{f} \in L^1(\mathbb{R}^2) \), then

\[
f(x) = \frac{1}{(2\pi)^2} \int_0^\pi \int_{-\infty}^{\infty} e^{ir\langle x, \omega \rangle} F_t Rf(r, \omega) |r| dr d\omega.
\]

**Remarks on Radon Inversion Formula**

- Radial integral is a filter \(|r|\) applied to the Radon transform. Back-projection is the angular integral.
- Formula is often called filtered back-projection formula.
- Factor \(|r|\) suppresses low-frequency components and amplifies high frequency components.
- Extreme care is needed to be more specific about domains and ranges for Radon transform/Inversion formula. E.g. Fourier transform of a delta function is 1, and 1 is not integrable.

**Proof of Radon Inversion Formula**

**Proof.** Trace through definitions to see that \( Rf \) is even

\[
F_t Rf(-r, -\omega) = F_t Rf(r, \omega).
\]

Then by the Fourier inversion theorem:

\[
f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\langle x, \xi \rangle} \hat{f}(\xi) d\xi
\]

\[
= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} e^{ir\langle x, \omega \rangle} \hat{f}(r, \omega) r dr d\omega
\]

\[
= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} e^{ir\langle x, \omega \rangle} F_t Rf(r, \omega) r dr d\omega
\]

\[
= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} e^{ir\langle x, \omega \rangle} F_t Rf(r, \omega) |r| dr d\omega.
\]

**A second look at filtration**

- It is very interesting to note that if we replace \(|r|\) with just \( r \) in the formula

\[
f(x) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} e^{ir\langle x, \omega \rangle} F_t Rf(r, \omega) r dr d\omega
\]

we could write

\[
f(x) = \frac{1}{2\pi i} \int_0^{\pi} \int_{-\infty}^{\infty} e^{ir\langle x, \omega \rangle} F_t (\partial_t Rf)(r, \omega) dr d\omega
\]

\[
= \frac{1}{2\pi i} \int_0^{\pi} \int_{-\infty}^{\infty} \partial_t Rf(t, \omega) d\omega
\]

- We can derive a new, useful back-projection formula by introducing the Hilbert transformation \( \mathcal{H}g = F^{-1}(\text{sgn}\hat{g}) \):

\[
f = \frac{1}{4\pi} \mathcal{R}^\# \mathcal{H} \frac{d}{dt}(Rf).
\]

Radial integral is a filter \(|r|\) applied to the Radon transform.
Back-projection is the angular integral.
Formula is often called filtered back-projection formula.
Factor \(|r|\) suppresses low-frequency components and amplifies high frequency components.
Extreme care is needed to be more specific about domains and ranges for Radon transform/Inversion formula. E.g. Fourier transform of a delta function is 1, and 1 is not integrable.
Given: real or simulated Radon transform data over a discrete range of affine parameter values and angles.

Our samples are approximately 500 pixels x 500 pixels. We sample 200 evenly spaced angles in $[0, 2\pi]$. For each angle sample we approximate the line integral through the angle at 130 different values of $t$. This gives about 26,000 sample data points.

Use the Hilbert transform back-projection formula:

$$ f = \frac{1}{4\pi} \mathcal{R} \# \mathcal{H} \frac{dt}{dt}(\mathcal{R}f) . $$

Use differencing to approximate $\frac{dt}{dt} \mathcal{R}f$.

Possible to approximate $\mathcal{H} \frac{dt}{dt}(\mathcal{R}f)$ as a convolution and integrate approximately to back-project.

Domains: For what types of functions will the Radon transform be defined?

- Need $f(x, y)$ to be such that the restriction to any line will be locally integrable.
- Need some decay at infinity for convergence of integrals.
- Smooth functions of compact support will certainly work. What else?

Stability: Do small errors in our measurement of $\mathcal{R}f$ lead to small errors in the reconstruction of $f$?

$\mathcal{R}$ turns out to be a smoothing operator: $\mathcal{R}f$ has 1/2 more derivatives than $f$. 
Important details: Range

- Range: What functions can be Radon transforms of other functions?
  - Harder question.
  - For a good choice of domain, $\mathcal{R}$ has zero kernel. In general, this means that many choices for $\mathcal{R}^{-1}$, the left inverse of $\mathcal{R}$, exist.
  - Need useful range conditions which produce uniqueness of the Radon inversion procedure.

Practical details

- Stability!
- What angular range do we really need to use? Can we exploit symmetries and take fewer measurements or view over smaller range?
- Resolution: What is the scale of detail in our reconstructions?
- Contrast: Usually more interested in regions of sharp transition vs smoother textures. Can we emphasize these in our reconstruction?

References

Charles L. Epstein.
*Introduction to the mathematics of medical imaging.*

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