Photoacoustic tomography (PAT) and the related thermoacoustic tomography (TAT) are important emerging medical imaging technologies. PAT/TAT combines the high resolution of ultrasonic imaging with the high contrast afforded by imaging energy absorption rates of tissues. Very safe: No ionizing radiation used. PAT and TAT differ only in the type of energy source used in the imaging procedure. Terms are often used interchangeably in the mathematical literature.

Ideas of PAT modality formulated in early 1980s and there has been a major theoretical focus on PAT since 2000. Significant mathematical issues have been basically resolved in the past five years. Implementation issues remain. Some analogy to the classical computed tomography (CT) problem but the integral geometry is more complicated. Unlike CT, filtered back-projection (FBP) type formulas for PAT have been very hard won. Current work appears to be shifting to series and time reversal approaches over the more complicated FBP formulas.

In 1880, A.G. Bell published the interesting article *On the Production and Reproduction of Sound by Light* [1]. In this work, Bell proposed the photophone, a device which uses pulsed light to transmit sound. In his investigation of the photophone, Bell made the remarkable discovery that an intense, pulsed light source incident on a thin sheet of material can produce audible sound. This effect is called the photoacoustic effect. Moreover, he recognized that different materials produced different tones.
### Photoacoustic Effect and Imaging I

- In a medical photoacoustic scan, laser sources surround a slice of the patient and fire short pulses at the slice.
- The laser pulses cause thermal expansion of the tissue and this expansion propagates a pressure wave through the tissue.
- Detectors arranged around the boundary of slice record ultrasonic pressure data.
- The extent of the photoacoustic effect depends on the absorptivity of the tissue sample to the source energy.
- Different kinds of tissue vary significantly in their absorptivity to a particular energy type. For the rest of the talk, we will define $f(x)$ to be a function which characterizes how absorbing of the PA source energy the patient slice is at the location $x \in \mathbb{R}^d$ for $d = 2, 3$.

### Photoacoustic Effect and Imaging II

- The photoacoustic problem: given pressure data $p(y, t)$ recorded at the boundary $S$ of the patient slice, can we recover the absorptivity $f(x)$ of the slice?

### Schematic of a PAT Scan

![Schematic of a PAT Scan](image)

### Principles I

- Assume patient is inhomogeneous in her absorption of EM energy but relatively homogeneous in density.
- For a short EM pulse duration, the change in temperature is proportional to the applied thermal energy. Let the heating function $H(x, t)$ be thermal energy per unit time and $T(x, t)$ be the temperature:

\[
\rho C_p \frac{\partial T}{\partial t} = H(x, t)
\]

where $\rho$ is density and $C_p$ is specific heat.
- Acoustic pressure $p(x, t)$ is the deviation from the ambient pressure caused by the propagation of a sound wave. Let $u(x, t)$ be the acoustic displacement of the slice.
Principles II

- Pressure gradient drives displacement, so Newton’s law gives:
  \[ \rho \frac{\partial^2 u}{\partial t^2} = -\nabla p. \]
- Temperature rise drives displacement through thermal expansion:
  \[ \nabla \cdot u = -\frac{p}{\rho c(x)^2} + \beta T(x, t). \]

Here \( c(x) \) is the speed of sound in the medium and \( \beta \) is an isobaric volume expansion constant.

Principles III

- Differentiating twice in \( t \):
  \[ \nabla \cdot u_{tt} = -\frac{p_{tt}}{\rho c(x)^2} + \beta T_{tt}(x, t). \]

and substituting \( \rho u_{tt} = -\nabla p \) and \( \rho C_p \frac{\partial T}{\partial t} = H(x, t) \)

\[ \Delta p - \frac{p_{tt}}{c(x)^2} = -\frac{\beta}{C_p} H_t \]

- Now assume that \( H \) can be separated into a product of the spatial absorptivity function and a temporal illumination function:
  \[ H(x, t) = f(x) I(t). \]

Principles IV

- In PAT models, the heating pulse is idealized so that \( I(t) = \delta(t) \).
- The model is:
  \[ (\Delta - c(x)^2 \partial_{tt}) p(x, t) = f(x) \delta'(t) \]

where \( p(x, t) = 0 \) when \( t < 0 \).
The measured data in PAT is \( h(y, t) = p(y, t) \) where \( y \) is a point on the detection surface \( S \) (often a sphere or circle). Thus:

\[
\begin{cases}
  (\partial_{tt} - c(x)^2 \Delta) p = 0, & t \geq 0, x \in \mathbb{R}^n \\
  p(y, t) = h(y, t), & y \in S \\
  p(x, 0) = f(x) \\
  p_t(x, 0) = 0
\end{cases}
\]

Thus we seek to invert the mapping \( \Lambda \) defined by

\[
\Lambda f := h
\]

The map \( R_S \) is the **spherical mean operator**.

The collected data in a PAT scan is basically the average of \( f \) over all spheres with centers on the detection surface \( S \).

So solving the inverse problem is basically inverting \( R_S \).
The methods of CT have motivated many of the attempts to solve the PAT inverse problem, so it is worthwhile to review the filtered-back projection approach to the Radon transform.

In CT, the task is to reconstruct a function \( f \) from knowledge of its Radon transform. The Radon Transform of \( f \in S(\mathbb{R}^2) \) is the map given by:

\[
Rf(t, \omega) = \int_{L_{t,\omega}} f(x) \, dx = \int_{-\infty}^{\infty} f(t \omega + s \omega^\perp) \, ds.
\]

Here the parameter \( t \) and the unit vector \( \omega \) uniquely determine the line \( L_{t,\omega} \) that is perpendicular to \( \omega \) and at distance \( t \) from the origin.

The basic CT problem is: knowing the value of every line integral of a function \( f \) over the domain \( \Omega \subset \mathbb{R}^2 \), can we recover \( f \)?

**Back-Projection I**

- Considering the geometry of the problem, it seems reasonable that to recover \( f \) at a point \( x \) from its Radon data, we should compute the average of \( Rf(\omega, t) \) over all \( L_{\omega, t} \) passing through \( x \).
- Define the back-projection operator \( R^\#g \) for \( g(t, \omega) \) by
  \[
  R^\#g(x) = \frac{1}{2\pi} \int_{S^1} g(x \cdot \omega, \omega) \, d\omega.
  \]
  This is precisely the operator which averages \( g \) over every line passing through \( x \). It also happens to be the formal adjoint of \( R \).

**Back-Projection II**

- For functions \( f \) from a "good domain" space, recalling that \( t = x \cdot \omega \), compute:
  \[
  R^\#_t Rf(x) = \int_{|\omega|=1} Rf(x \cdot \omega, \omega) \, dS(\omega)
  = \int_{|\omega|=1} \int_{-\infty}^{\infty} f((x \cdot \omega) \omega + s \omega^\perp) \, dsdS(\omega)
  \]
  Basically, a polar integral. Rewrite:
  \[
  R^\#_t Rf(x) = \int_{\mathbb{R}^2} \frac{2f(y)}{|y-x|} \, dy = \frac{2}{|x|} \ast f(x).
  \]
  Not \( f(x) \) but a blurred version of \( f(x) \)!
Now define the operator $(-\Delta)^{1/2}$ by

$$\mathcal{F}((-\Delta)^{1/2}f)(\xi) = |2\pi\xi|\mathcal{F}f(\xi)$$

then by Fourier transform properties:

$$f(x) = (-\Delta)^{1/2}R^\#Rf(x).$$

We note that the kernel $K(x,y) = 1/|y-x|$ appearing in

$$R^\#Rf(x) = \int_{\mathbb{R}^2} \frac{2f(y)}{|y-x|} dy$$

suggested what operator should be applied to recover $f$. 

### Back-Projection for CT Problem

#### Radon Data:

#### Back-Projection:

### A Phantom
Filtered Back-Projection for CT Problem

Filtered Back-Projection:

Brief History of Spherical Mean Inversion I

Can we recover \( f \) from the data

\[
R_S f(y, t) = \frac{1}{\omega_n} \int_{|\theta|=1} f(y + t\theta) \, dS(\theta)
\]

for \( y \) in the detection surface \( S \)?

  - Found a series reconstruction for the case of a circular detection surface in 2D.
  - Proof is à la Allan Cormack [2], expanding \( f \) and the collected data \( h \) in spherical harmonic series and establishing a relation on the coefficients.
  - Impractical formulas which involve division by Bessel functions with infinitely many zeros.

Brief History of Spherical Mean Inversion II

- Also proved that the Radon transform is the limit of the spherical mean transform for the circular detection surface as the radius \( \rho \to \infty \).
- First instance of a more general approach to the inversion: series expansion in a useful basis.
- Formulas in various special cases found throughout the 80s and 90s.
- Many researchers obtained very good approximate inversion with parametrices (e.g. [8]), providing numerical evidence that there should be some kind of FBP analog for PAT reconstruction.

Brief History of Spherical Mean Inversion III

- Finally, in 2004 Finch, Patch, and Rakesh [4] produced exact filtered back-projection formulas for the spherical detection surface in odd dimensions. Even dimensional analogs took several more years and were obtained in 2007 [3].
Overview
Model of Photoacoustic Generation
Classical FBP Approach
Recent Approaches
References

Spherical Mean Transform
CT Analogy
FBP Formulas

Sketch of Finch et. all $n = 3$ FBP I

For the spherical detection geometry in $\mathbb{R}^3$ and $f$ smooth and compactly supported on $B(0, \rho)$ with detection surface $S = \partial B(0, \rho)$

$$f(x) = c(-\Delta)(R_S^2 t^2 R)f(x).$$

- As before, $R_S^2$ is the formal adjoint of $R_S$ and is easy to compute. In particular, for $g(y, t)$ it is given by

$$R_S^2 g(x) = \frac{1}{\omega_n} \int_{|y|=\rho} g(y, |y-x|) dS(\theta).$$

For $g = Rf$, this operator forms a weighted average of all the integrals of $f$ over spheres with centers on the detection surface and which pass through $x$.

Sketch of Finch et. all $n = 3$ FBP II

Disadvantages of FBP

- FBP approaches are currently restricted to spherical detection geometries, are difficult to formulate, and less intuitive than their counterparts in Radon theory.
- No direct way to incorporate variable sound speed.
- Only work when the support of $f$ is completely enclosed by the detection surface $S$. Interestingly, different exact inversion formulas lead to different incorrect results when part of the support of $f$ is outside the detection surface [6].

These considerations suggest that a different approach will probably be needed to solve more general PAT inversion problems.
Stefanov & Uhlmann (2009) [9] provides a surprisingly simple and extremely complete answer to PAT inversion. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $f$ be supported in $\bar{\Omega}$. Suppose we run the PAT experiment for only a finite time $T > \text{diam}(\Omega)$. It turns out that the pressure is approximated by solution to the problem

$$
\begin{cases}
(\partial_t^2 - c(x)^{-2} \Delta) p = 0, & (t, x) \in (0, T) \times \Omega \\
p(y, t) = h(y, t), & y \in \partial \Omega, t \in [0, T] \\
p(x, T) = 0, & x \in \bar{\Omega} \\
\partial_t p(x, T) = 0, & x \in \bar{\Omega}
\end{cases}
$$

Here $\Lambda f = h$ is the measured data we collect at the detection surface $\partial \Omega$.

If $T = \infty$, then this is the exact equation for pressure and the Cauchy data at $t = \infty$ is justified by finite energy. Recover $f$ by $p(x, 0)$.

For finite $T$, this is approximate and error in approximating $f$ by $p(x, 0)$ is $O(e^{-T/C})$.
A Neumann Series Approach I

For an approximate inverse $A$ of $\Lambda$ we define the error operator by

$$K = I - A\Lambda$$

Insight: Define $A$ in such a way that $K$ is compact and contractive ($\|K\| < 1$). Rearranging:

$$A\Lambda f = (I - K)f$$

and so

$$f = (A\Lambda - I)^{-1}f = \sum_{n=0}^{\infty} K^n A\Lambda f.$$ 

The Neumann series converges since $\|K\| < 1$.

Sketch of the Proof II

To define $A$ in the right way, we form the new backward problem

$$\begin{cases}
(\partial_t - c(x)^{-2}\Delta)v(x, t) = 0 \text{ in } (0, T) \times \Omega \\
v(y, t) = h(y, t) \text{ for } y \in \partial\Omega, t \in [0, T] \\
v(x, T) = 0 \text{ for } x \in \Omega \\
\partial_t v(x, T) = 0 \text{ for } x \in \partial\Omega \\
\n-\Delta \phi(x) = 0, x \in \Omega \quad \phi(y) = h(y, T).
\end{cases}$$

We then define $A$ by

$$Ah := v(y, 0).$$

Sketch of the Proof III

Note that on $[0, T] \times \Omega$, $p$ and $v + w$ satisfy the same problem! So by uniqueness

$$p = v + w$$

and restricting to $t = 0$:

$$f(x) = Ah + w(x, 0) = A\Lambda f(x) + w(x, 0).$$

Since

$$I = A\Lambda + K$$

we should have

$$Kf(x) = w(x, 0).$$
Sketch of the Proof IV

Defining the Dirichlet norm

\[ \|f\|_{H_D}^2 := \int_\Omega (|Df|^2 + c(x)^{-2}|f|^2) \, dx \]

it is not difficult to prove that for \( f \neq 0 \)

\[ \|Kf\|_{H_D}(\Omega) \leq \|f\|_{H_D}(\Omega) \]

A more tricky analysis with a version of Holmgren’s theorem is needed to show that this inequality is actually strict. The proof that \( K \) is compact will then follow by considering the various smooth mappings of \( f \) into data at time \( T \) defined by the problems for \( p, v, \) and \( w \) and then expressing \( K \) as a composition of these maps.

References I


Sketch of the Proof V

For the compactness of \( K \):

- Since \( T > \text{diam}(\Omega) \), all singularities starting in \( \bar{\Omega} \) have left \( \Omega \) by time \( T \). In particular, the mappings
  \[
  f \mapsto p(\cdot, T) - \phi \\
  f \mapsto p_t(\cdot, T)
  \]
  are compact on \( H_D(\Omega) \oplus L^2(\omega) \).

- The solution operator from \( t = T \) to \( t = 0 \) of the \( w \) equation:
  \[
  (u(\cdot, T) - \phi, u_T(\cdot, T)) \mapsto w(\cdot, 0)
  \]
  is unitary in \( H_D(\omega) \oplus L^2(\Omega) \).

- The mapping \( K \) can be expressed as the composition of these two maps, the first compact and the second bounded, so \( K \) is itself compact.

References II


References III


References IV
