BOUNDARY VALUE PROBLEMS IN MORREY SPACES FOR ELLIPTIC SYSTEMS ON LIPSCHITZ DOMAINS

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Abstract. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Let $\mathcal{L}$ be a second order elliptic system with constant coefficients satisfying the Legendre-Hadamard condition. We consider the Dirichlet problem $\mathcal{L}\mathbf{u} = 0$ in Ω, $\mathbf{u} = \mathbf{f}$ on $\partial\Omega$ with boundary data $\mathbf{f}$ in the Morrey space $L^{2,\lambda}(\partial\Omega)$. Assume that $0 \leq \lambda < 2 + \varepsilon$ for $n \geq 4$ where $\varepsilon > 0$ depends on Ω, and $0 \leq \lambda \leq 2$ for $n = 3$. We obtain existence and uniqueness results with nontangential maximal function estimate $\|(\mathbf{u})^*\|_{L^{2,\lambda}(\partial\Omega)} \leq C\,\|\mathbf{f}\|_{L^{2,\lambda}(\partial\Omega)}$. If $\mathcal{L}$ satisfies the strong elliptic condition and $0 \leq \lambda < \min(n-1, 2+\varepsilon)$, we show that the Neumann type problem $\mathcal{L}\mathbf{u} = 0$ in Ω, $\frac{\partial \mathbf{u}}{\partial \nu} = \mathbf{g} \in H^{2,\lambda}(\partial\Omega)$ on $\partial\Omega$, $\|(\nabla\mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} < \infty$ has a unique solution. Here $H^{2,\lambda}(\partial\Omega)$ is an atomic space with the property $(H^{2,\lambda}(\partial\Omega))^* = L^{2,\lambda}(\partial\Omega)$. The invertibility of layer potentials on $L^{2,\lambda}(\partial\Omega)$ and $H^{2,\lambda}(\partial\Omega)$ is also obtained. Finally we study the Dirichlet problem for the biharmonic equation. We establish a similar estimate in $L^{2,\lambda}$ for the biharmonic equation, in which case the range $0 \leq \lambda < 2 + \varepsilon$ is sharp for $n = 4$ or 5.

1. Introduction. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. The Dirichlet and Neumann problems with boundary data in $L^p(\partial\Omega)$ for the Laplace's equation in Ω are well understood due to the work of Dahlberg [5, 6], Jerison-Kenig [19], Verchota [31] and Dahlberg-Kenig [7]. For the stationary Stokes equations and the systems of elastostatics, the L^p boundary value problems were solved for $n \geq 3$ and p close to 2 in [14] and [11] respectively. Also see [17, 22] for the general elliptic systems. In [8] Dahlberg and Kenig were able to establish the L^p solvability for the optimal ranges of p in the case $n = 3$ for the systems of elastostatics. However the question of sharp ranges of p for which one can solve the L^p boundary value problems for elliptic systems remains open when $n \geq 4$.

In this paper we initiate the study of the Dirichlet problem for elliptic systems with boundary data in the Morrey space $L^{2,\lambda}(\partial\Omega)$ as well as the Neumann type problem with data in $H^{2,\lambda}(\partial\Omega)$. Here $H^{2,\lambda}(\partial\Omega)$ is a pre-dual space of $L^{2,\lambda}(\partial\Omega)$. We obtain existence and uniqueness results with dilation-invariant estimates in terms of the nontangential maximal functions for $0 \leq \lambda < 2 + \varepsilon$ in the case $n \geq 4$. These estimates may be regarded as the appropriate substitutes for the L^p estimates.

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More precisely, let $(\mathcal{L}(\mathbf{u}))^\alpha = -a_{ij}^{\alpha\beta} D_i D_j u^\beta$ where $D_i = \partial/\partial x_i$ and $\alpha, \beta = 1, \ldots, m$, we consider

$$\text{(1.1)} \qquad \begin{cases} \mathcal{L}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f} & \text{on } \partial\Omega \text{ in the sense of nontangential convergence,} \\ \|(\mathbf{u})^*\|_{L^{2,\lambda}(\partial\Omega)} < \infty \end{cases}$$

with boundary data $\mathbf{f}$ in the Morrey space $L^{2,\lambda}(\partial\Omega)$. In (1.1) and hereafter, $(\mathbf{u})^*$ denotes the nontangential maximal function of $\mathbf{u}$. We assume that the coefficients $a_{ij}^{\alpha\beta}$, $1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq m$ are real constants satisfying the symmetry condition $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ and the Legendre-Hadamard condition:

$$\text{(1.2)} \qquad \mu|\xi|^2|\eta|^2 \leq a_{ij}^{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \leq \frac{1}{\mu}|\xi|^2|\eta|^2$$

for some $\mu > 0$ and any $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$.

Definition 1.3. Let $0 \leq \lambda \leq n-1$. By $L^{2,\lambda}(\partial\Omega)$ we denote the linear space of functions $f \in L^2(\partial\Omega)$ such that

$$\text{(1.4)} \qquad \|f\|_{2,\lambda} \equiv \left\{ \sup_{\substack{Q \in \partial\Omega \\ 0<\rho<\operatorname{diam}(\partial\Omega)}} \rho^{-\lambda} \int_{B(Q,\rho)\cap\partial\Omega} |f|^2 \, d\sigma \right\}^{1/2} < \infty,$$

where $B(Q, \rho)$ denotes the ball centered at Q with radius ρ in $\mathbb{R}^n$.

With the norm in (1.4), $L^{2,\lambda}(\partial\Omega)$ becomes a Banach space. Clearly $L^{2,0}(\partial\Omega) = L^2(\partial\Omega)$ and $L^{2,n-1}(\partial\Omega) = L^\infty(\partial\Omega)$. The following is one of the main results of the paper.

THEOREM 1.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. Then there exists $\varepsilon > 0$ depending on n, m, μ and Ω such that, given any $\mathbf{f} \in L^{2,\lambda}(\partial\Omega)$ with $0 \leq \lambda < \min(n-1, 2+\varepsilon)$, the Dirichlet problem (1.1) has a unique solution. Moreover the solution* $\mathbf{u}$ *satisfies the estimate*

$$\text{(1.6)} \qquad \|(\mathbf{u})^*\|_{2,\lambda} + \|S(\mathbf{u})\|_{2,\lambda} \leq C\,\|\mathbf{f}\|_{2,\lambda}$$

where $S(\mathbf{u})$ denotes the square function of $\mathbf{u}$. *In the case $n = 3$, $\|(\mathbf{u})^*\|_{2,\lambda} \leq C\,\|\mathbf{f}\|_{2,\lambda}$ holds for $0 \leq \lambda \leq 2$.*

In this paper we also study the Neumann type problem as well as the regularity problem for the elliptic system. To this end we introduce a pre-dual space of $L^{2,\lambda}(\partial\Omega)$.

Definition 1.7. Let $0 \leq \lambda \leq n-1$. We say $a \in L^2(\partial\Omega)$ is an $(2,\lambda)$ atom on $\partial\Omega$ if there exist $Q \in \partial\Omega$ and $\rho > 0$ such that $\operatorname{supp} a \subset B(Q,\rho) \cap \partial\Omega$, $\|a\|_2 \leq \rho^{-\lambda/2}$, and $\int_{\partial\Omega} a\, d\sigma = 0$. By $H^{2,\lambda}(\partial\Omega)$ we denote the linear space

$$\left\{ g = C + \sum_{j=1}^{\infty} \mu_j a_j \colon \sum_{j=1}^{\infty} |\mu_j| < \infty, \right. \qquad \left. a_j \text{ is an } (2,\lambda) \text{ atom and } C \text{ is a constant.} \right\} \tag{1.8}$$

with norm

$$\|g\|_{H^{2,\lambda}(\partial\Omega)} \equiv \left| \int_{\partial\Omega} g\, d\sigma \right| + \inf \left\{ \sum_{j=1}^{\infty} |\mu_j| \colon g = C + \sum_{j=1}^{\infty} \mu_j a_j \right\}. \tag{1.9}$$

The space $H^{2,\lambda}$ was first introduced by C. Zorko [35] for bounded domains in $\mathbb{R}^n$. It is not hard to prove that $(H^{2,\lambda}(\partial\Omega))^* = L^{2,\lambda}(\partial\Omega)$ for $0 \leq \lambda < n-1$. Note that $H^{2,n-1}(\partial\Omega)$ is the atomic Hardy space $H^1(\partial\Omega)$. Thus $(H^{2,n-1}(\partial\Omega))^* = BMO(\partial\Omega)$.

Let

$$\left(\frac{\partial \mathbf{u}}{\partial \nu} \right)^{\alpha} = a_{ij}^{\alpha\beta} D_j u^{\beta} N_i, \quad \alpha = 1, \dots, m \tag{1.10}$$

be the conormal derivative, where $\mathbf{N} = (N_1, \dots, N_n)$ denotes the outward unit normal to Ω. Consider the Neumann problem

$$\begin{cases} \mathcal{L}(\mathbf{u}) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = \mathbf{g} \in H^{2,\lambda}(\partial\Omega) \qquad \text{on } \partial\Omega, \\ \|(\nabla \mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} < \infty. \end{cases} \tag{1.11}$$

THEOREM 1.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. Assume that operator $\mathcal{L}$ satisfies the symmetry condition and the strong ellipticity condition:*

$$\mu|\xi|^2 \leq a_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} \leq \frac{1}{\mu} |\xi|^2 \quad \textit{for some } \mu > 0 \textit{ and all } \xi = (\xi_i^{\alpha}) \in \mathbb{R}^{nm}. \tag{1.13}$$

Then there exists $\varepsilon > 0$ depending only on n, m, μ and Ω such that, given any $\mathbf{g} \in H^{2,\lambda}(\partial\Omega)$ with $0 \leq \lambda < \min(n-1, 2+\varepsilon)$ and $\int_{\partial\Omega} \mathbf{g}\, d\sigma = 0$, the Neumann problem (1.11) has a unique (up to a constant) solution. Moreover the solution $\mathbf{u}$

satisfies

$$\|(\nabla \mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} \leq C\,\|\mathbf{g}\|_{H^{2,\lambda}(\partial\Omega)}. \tag{1.14}$$

A few remarks are in order.

Remark 1.15. The proof of Theorems 1.5 and 1.12 is based on an approach developed by Dahlberg-Kenig [8] for the three-dimensional elliptic systems and by Pipher-Verchota [26, 27] for the biharmonic equation in $\mathbb{R}^3$. In the case $n = 3$, this approach yields the L^∞ estimate for solutions of the Dirichlet problem as well as an estimate in the Hardy space H^1 for the Neumann problem. The desired L^p estimates then follow by interpolation. Here we extend the approach to higher dimensions. Although it fails to get the L^∞ estimate for $n \geq 4$ (in fact, the L^∞ estimate does not hold in general for biharmonic functions in Lipschitz domains for $n \geq 4$ [26]), we are able to establish dilation-invariant estimates of solutions in terms of the nontangential maximal functions in the Morrey spaces and their pre-duals. Note that, by Hölder inequality,

$$L^p(\partial\Omega) \subset L^{2,\lambda}(\partial\Omega) \quad \text{and} \quad H^{2,\lambda}(\partial\Omega) \subset L^{p'}(\partial\Omega) \quad \text{for } p = \frac{2(n-1)}{n-1-\lambda}. \tag{1.16}$$

Thus our estimates (1.6) in $L^{2,\lambda}$ and (1.14) in $H^{2,\lambda}$ may be regarded as substitutes for the L^p estimates. Two main ingredients in this approach are the Caccioppoli inequality and L^p estimates for the Dirichlet problem with p close to 2. Since these techniques are basically L^2 type estimates, the L^2-based Morrey spaces seem to be a very natural choice.

Remark 1.17. For the Laplace equation, the estimate (1.6) holds for $0 \leq \lambda \leq n-1$. Indeed by [6], if $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$ and $\|(u)^*\|_2 < \infty$, one has

$$(u)^*(Q) \leq C\,\{M(|f|^p)(Q)\}^{1/p}, \qquad Q \in \partial\Omega$$

for some $p < 2$. It follows that

$$\|(u)^*\|_{2,\lambda} \leq C\,\|\{M(|f|^p)\}^{1/p}\|_{2,\lambda} \leq C\,\|f\|_{2,\lambda}$$

for any $0 \leq \lambda \leq n-1$. We point out that the estimate (1.14) with $0 \leq \lambda < n-1$ for harmonic functions follows by an inspection of the proof of Theorem 1.12 as well as proofs in [7]. Details are left to the reader.

Remark 1.18. In this paper we will also consider the Dirichlet problem with data $\mathbf{f}$ in $H_1^{2,\lambda}(\partial\Omega)$. Roughly speaking, $H_1^{2,\lambda}(\partial\Omega)$ is the space of functions in

$H^{2,\lambda}(\partial\Omega)$ whose first order derivatives are in $H^{2,\lambda}(\partial\Omega)$. We obtain the estimate

$$\|(\nabla\mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} \leq C\,\|\mathbf{f}\|_{H_1^{2,\lambda}(\partial\Omega)} \tag{1.19}$$

for $0 \leq \lambda < \min(n-1, 2+\varepsilon)$. See Theorem 3.7. Using estimates (1.14) and (1.19), we show that the conormal derivative of the single layer potential on $\partial\Omega$ for operator $\mathcal{L}$ is invertible on $H^{2,\lambda}(\partial\Omega)$ for $0 \leq \lambda < \min(n-1, 2+\varepsilon)$. By duality, the double layer potential is invertible on $L^{2,\lambda}(\partial\Omega)$. See Section 5 for details.

Remark 1.20. It is not known whether or not the conditions on λ in Theorems 1.5 and 1.12 are necessary for general elliptic systems. In the last section of this paper, we study the Dirichlet problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \mathbf{N}} = g & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

We establish a similar estimate in Morrey spaces for biharmonic functions:

$$\|(\nabla u)^*\|_{2,\lambda} \leq C\,\{\|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda}\} \tag{1.22}$$

where $0 \leq \lambda < 2+\varepsilon$ for $n \geq 4$, $0 \leq \lambda \leq 2$ for $n = 3$, and $\nabla_{\tan} f$ denotes the tangential derivatives of f. See Theorem 6.6. Examples constructed by Pipher-Verchota in [26] may be used to show that the range $0 \leq \lambda < 2+\varepsilon$ is sharp at least in dimensions $n = 4$, 5 for estimate (1.22).

Throughout this paper we use C and c to denote positive constants, which may be different from line to line, which depend only on n, m, the ellipticity constant μ, λ and Ω. We will use $\|\cdot\|_p$ to denote the norm in $L^p(\partial\Omega)$. For $P \in \partial\Omega$ and $r > 0$, we say $B(P,r) \cap \partial\Omega$ is a coordinate patch for $\partial\Omega$, if there exists a Lipschitz function φ: $\mathbb{R}^{n-1} \to \mathbb{R}$ such that, after a rotation of the coordinate system, we have

$$\Omega \cap B(P,r) = \left\{(X', x_n) \in \mathbb{R}^n:\ x_n > \varphi(X')\right\} \cap B(P,r). \tag{1.23}$$

In this new coordinate system, we let

$$\begin{aligned} \Delta(P,\rho) &= \left\{(X', \varphi(X')) \in \mathbb{R}^n:\ |X' - P'| < \rho\right\}, \\ D(P,\rho) &= \left\{(X', x_n) \in \mathbb{R}^n:\ |X' - P'| < \rho \text{ and } \varphi(X') < x_n < \varphi(X') + \rho\right\}. \end{aligned} \tag{1.24}$$

Recall that Ω is a Lipschitz domain if there exists $r_0 = r_0(\Omega) > 0$ such that $B(P, r_0) \cap \partial\Omega$ is a coordinate patch for any $P \in \partial\Omega$. Clearly, if $0 < \rho < c\,r_0$, we have $\Delta(P,\rho) \subset \partial\Omega$ and $D(P,\rho) \subset \Omega$. Finally we will make no effort to distinguish between the real valued Banach spaces like $L^p(\partial\Omega)$, $L^{2,\lambda}(\partial\Omega)$, $H^{2,\lambda}(\partial\Omega)$ and their $\mathbb{R}^m$-valued counterparts. It will be clear from the context.

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2. Estimates in $L^{2,\lambda}$ for the Dirichlet problem.

The goal of this section is to give the proof of Theorem 1.5. Throughout this section we will assume that $\mathcal{L}$ is a second order elliptic operator with constant coefficients satisfying the symmetry property and the Legendre-Hadamard condition (1.2).

We begin by recalling the L^p estimates of solutions of the Dirichlet problem for p close to 2.

THEOREM 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. There exists $\delta > 0$ depending on n, m, μ and the Lipschitz character of Ω, such that if $|p-2| < \delta$ and $\mathbf{f} \in L^p(\partial\Omega)$, then there exists a unique solution $\mathbf{u}$ such that $\mathcal{L}\mathbf{u} = 0$ in Ω, $\mathbf{u} = \mathbf{f}$ on $\partial\Omega$ in the sense of nontangential convergence, and $\|(\mathbf{u})^*\|_p < \infty$. Moreover, the solution will satisfy*

$$\|(\mathbf{u})^*\|_p \leq C\,\|\mathbf{f}\|_p, \tag{2.2}$$

where C depends only on n, m, μ and the Lipschitz character of Ω. If, in addition, $\mathbf{f} \in W^p_1(\partial\Omega)$, i.e., $\mathbf{f}$ has the first order derivatives in $L^p(\partial\Omega)$, then

$$\|(\nabla\mathbf{u})^*\|_p \leq C\,\|\mathbf{f}\|_{W^p_1(\partial\Omega)}. \tag{2.3}$$

Remark 2.4. For a function $\mathbf{u}$ on Ω, its nontangential maximal function on $\partial\Omega$ is defined by

$$(\mathbf{u})^*(Q) = \sup\left\{|\mathbf{u}(X)|\colon\ X \in \Omega \text{ and } |X-Q| < 2\,\mathrm{dist}\,(X,\partial\Omega)\right\}.$$

We point out that Theorem 2.1 also holds if we replace Ω by the exterior domain $\mathbb{R}^n\backslash\overline{\Omega}$ and impose an additional condition $|\mathbf{u}(X)| = O(|X|^{2-n})$ as $|X| \to \infty$.

Theorem 2.1 was proved by W. Gao in his thesis [17]. For the boundary data in L^p, the proof was based on an idea of A. P. Calderón. Another proof of estimate (2.2) may be found in [28]. Also see [22] for the extension of Theorem 2.1 to Lipschitz domains in Riemannian manifolds. Regularity estimate (2.3) was established in [17] by showing that the single layer potential for operator $\mathcal{L}$ is invertible from $L^p(\partial\Omega)$ to $W^p_1(\partial\Omega)$. To do this, the main tool is the following

Rellich-Payne-Weinberger-Nečas identities for solutions of $\mathcal{L}\mathbf{u} = 0$ in $\mathbb{R}^n \backslash \partial\Omega$:

$$\begin{aligned}\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N}\rangle a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} &= 2\int_{\partial\Omega} h_i N_l\, a_{lj}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \\ &\quad \pm \int_{\Omega_\pm} \operatorname{div}(\mathbf{h})\, a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \\ &\quad \mp 2\int_{\Omega_\pm} \frac{\partial h_i}{\partial x_l} a_{lj}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N}\rangle a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} &= 2\int_{\partial\Omega} (h_l N_l a_{ij}^{\alpha\beta} - h_i N_l a_{lj}^{\alpha\beta}) \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \\ &\quad \mp \int_{\Omega_\pm} \operatorname{div}(\mathbf{h})\, a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \\ &\quad \pm 2\int_{\Omega_\pm} \frac{\partial h_i}{\partial x_l}\, a_{lj}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j},\end{aligned} \tag{2.6}$$

where $\mathbf{h} \in C^1(\mathbb{R}^n)$, $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^n \backslash \overline{\Omega}$. In the case of Ω_-, we also need to assume that $\mathbf{u}(X) = O(|X|^{2-n})$ as $|X| \to \infty$. These two identities will also be important in this paper.

The following Caccioppoli's inequality is rather standard. We omit its proof.

Lemma 2.7. *Let $P \in \partial\Omega$ and $0 < \rho < c\, r_0$. Assume that $\mathcal{L}\mathbf{u} = 0$ in $D(P, \rho)$ and $|\nabla \mathbf{u}| \in L^2(D(P, \rho))$. Also assume that on $\Delta(P, \rho)$, either $\mathbf{u} = 0$ or $\frac{\partial \mathbf{u}}{\partial \nu} = 0$. Then*

$$\int_{D(P, s\rho)} |\nabla \mathbf{u}|^2\, dX \leq \frac{C}{(t-s)^2 \rho^2} \int_{D(P, t\rho)} |\mathbf{u}|^2\, dX$$

for $0 < s < t < 1$.

Lemma 2.8. *Under the same assumption as in Lemma 2.7, we have*

$$\left(\frac{1}{\rho^n} \int_{D(P, \rho/2)} |\mathbf{u}|^2\, dX \right)^{1/2} \leq C_p \left(\frac{1}{\rho^n} \int_{D(P, \rho)} |\mathbf{u}|^p\, dX \right)^{1/p}$$

for any $p > 0$.

Proof. By rescaling, we may assume that $\rho = 1$. Let $D_s = D(P, s)$. By Sobolev inequality, we have

$$\left(\int_{D_s} |\mathbf{u}|^{\frac{2n}{n-2}}\, dY \right)^{\frac{n-2}{2n}} \leq C \left(\int_{D_s} |\nabla \mathbf{u}|^2\, dY \right)^{1/2} + \frac{C}{s} \left(\int_{D_s} |\mathbf{u}|^2\, dY \right)^{1/2} \tag{2.9}$$

$$\leq \frac{C\,t}{s(t-s)}\left(\int_{D_t}|\mathbf{u}|^2\,dY\right)^{1/2},$$

where $0 < s < t < 1$, and we have used Lemma 2.7 in the second inequality.

Now given any $0 < p < 2$, write

$$\frac{1}{2} = \frac{\alpha}{\bar{p}} + \frac{\beta}{p}$$

where $\bar{p} = 2n/(n-2)$, $0 < \alpha, \beta < 1$ and $\alpha+\beta = 1$. By Hölder inequality and (2.9),

$$\begin{aligned}\left(\frac{1}{|D_s|}\int_{D_s}|\mathbf{u}|^2\,dY\right)^{1/2} &\leq \left(\frac{1}{|D_s|}\int_{D_s}|\mathbf{u}|^{\bar{p}}\,dY\right)^{\frac{\alpha}{\bar{p}}}\left(\frac{1}{|D_s|}\int_{D_s}|\mathbf{u}|^{p}\,dY\right)^{\frac{\beta}{p}}\\ &\leq \frac{C}{s^{\frac{n}{2}+\alpha}(t-s)^{\alpha}}\left(\int_{D_t}|\mathbf{u}|^2\,dY\right)^{\frac{\alpha}{2}}\left(\int_{D_1}|\mathbf{u}|^{p}\,dY\right)^{\frac{\beta}{p}}.\end{aligned}$$

Thus, if we let

$$I(s) = \left(\frac{1}{|D_s|}\int_{D_s}|\mathbf{u}|^2\,dY\right)^{1/2},$$

we have

$$I(s) \leq \frac{C}{s^{\frac{n}{2}+\alpha}(t-s)^{\alpha}}\,I(t)^{\alpha}\left(\int_{D_1}|\mathbf{u}|^{p}\,dY\right)^{\frac{\beta}{p}}$$

for any $0 < s < t < 1$. It follows that

$$\log I(s) \leq C\log\frac{1}{s} + C\log\frac{1}{t-s} + \alpha\log I(t) + \beta\log\left(\int_{D_1}|\mathbf{u}|^{p}\,dY\right)^{1/p}. \tag{2.10}$$

Let $s = t^b$ and $b > 1$ to be chosen. We integrate both sides of (2.10) with respect to dt/t over $(\frac{1}{2}, 1)$ to obtain

$$\int_{1/2}^{1}\log I(t^b)\,\frac{dt}{t} \leq C + \alpha\int_{1/2}^{1} I(t)\frac{dt}{t} + C\log\left(\int_{D_1}|\mathbf{u}|^{p}\,dY\right)^{1/p}. \tag{2.11}$$

Note that

$$\int_{1/2}^{1}\log I(t^b)\,\frac{dt}{t} = \frac{1}{b}\int_{(1/2)^b}^{1}\log I(t)\,\frac{dt}{t} \geq \frac{1}{b}\int_{1/2}^{1}\log I(t)\,\frac{dt}{t}.$$

Inequality (2.11) then yields that

$$\left(\frac{1}{b}-\alpha\right)\int_{1/2}^{1}\log I(t)\,\frac{dt}{t}\leq C+C\,\log\left(\int_{D_1}|\mathbf{u}|^p\,dY\right)^{1/p}. \tag{2.12}$$

Choose $b>1$ so that $\frac{1}{b}-\alpha>0$. Since

$$I(t)\geq c\,\left(\int_{D_{1/2}}|\mathbf{u}|^2\,dY\right)^{1/2}$$

for $t\in(1/2,1)$, the estimate (2.12) implies that

$$\log\left(\int_{D_{1/2}}|\mathbf{u}|^2\,dY\right)^{1/2}\leq C+C\,\log\left(\int_{D_1}|\mathbf{u}|^p\,dY\right)^{1/p}.$$

From this, Lemma 2.8 follows. We remark that the argument given above may be found in [16, 15].

LEMMA 2.13. *Let* $P\in\partial\Omega$ *and* $0<\rho<c\,r_0$. *Suppose* $\mathcal{L}\mathbf{u}=0$ *in* $D(P,2\rho)$, $\mathbf{u}=0$ *on* $\Delta(P,2\rho)$, *and* $(\nabla\mathbf{u})^*\in L^2(\Delta(P,2\rho))$. *Then*

$$\int_{\Delta(P,\rho)}|\nabla\mathbf{u}|^2\,dQ\leq\frac{C}{\rho^3}\int_{D(P,2\rho)}|\mathbf{u}|^2\,dX.$$

Proof. Since $B(P,r_0)\cap\partial\Omega$ is a coordinate patch for $\partial\Omega$, there exists a constant vector $\mathbf{h}$ such that $\langle\mathbf{h},\mathbf{N}\rangle\geq c_0>0$ on $B(P,r_0)\cap\partial\Omega$. Let $D_{\tau\rho}=D(P,\tau\rho)$ for $\tau\in(1,3/2)$. We apply Rellich identity (2.6) on $D_{\tau\rho}$, which may be justified by using $(\nabla\mathbf{u})^*\in L^2(D(P,2\rho))$, to $\mathbf{u}$ and this constant vector $\mathbf{h}$. Since $\mathbf{u}=0$ on $\Delta(P,2\rho)$ and $|(h_lN_la_{ij}^{\alpha\beta}-h_iN_la_{lj}^{\alpha\beta})\frac{\partial u^\alpha}{\partial x_i}|\leq C\,|\nabla_{\tan}\mathbf{u}|$ for each j and β, where $\nabla_{\tan}\mathbf{u}$ denotes the tangential derivatives of $\mathbf{u}$ on $\partial\Omega$, we obtain

$$\int_{\Delta(P,\rho)}a_{ij}^{\alpha\beta}\,\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial u^\beta}{\partial x_j}\,dQ\leq C\int_{\Omega\cap\partial D_{\tau\rho}}|\nabla\mathbf{u}|^2\,dQ. \tag{2.14}$$

By an algebraic argument, one can show that

$$c\,|\nabla\mathbf{u}|^2\leq a_{ij}^{\alpha\beta}\,\frac{\partial u^\alpha}{\partial x_i}\cdot\frac{\partial u^\beta}{\partial x_j}+C|\nabla_{\tan}\mathbf{u}|^2 \tag{2.15}$$

(e.g. see [20, p. 168]). In view of (2.14)–(2.15), we have

$$\int_{\Delta(P,\rho)}|\nabla\mathbf{u}|^2\,dQ\leq C\int_{\Omega\cap\partial D_{\tau\rho}}|\nabla\mathbf{u}|^2\,dQ. \tag{2.16}$$

By integrating both sides of (2.16) with respect to τ over (1, 3/2), we get

$$\int_{\Delta(P,\rho)} |\nabla \mathbf{u}|^2 \, dQ \leq \frac{C}{\rho} \int_{D(P,3\rho/2)} |\nabla \mathbf{u}|^2 \, dX.$$

Lemma 2.13 now follows from Lemma 2.7.

Let $\Gamma(Y)$ denote the matrix of fundamental solutions in $\mathbb{R}^n$ for operator $\mathcal{L}$ with pole at 0. For $X \in \Omega$, let $v^X(Y)$ be the unique matrix valued solution to the L^2 Dirichlet problem with boundary data

$$v^X(Q) = \Gamma(X - Q) \qquad \text{on } \partial\Omega,$$

given by Theorem 2.1. Let

$$G^X(Y) = \Gamma(X - Y) - v^X(Y) \tag{2.17}$$

be the matrix Green's function for operator $\mathcal{L}$ in Ω. The following lemma was proved in [30, p. 241].

Lemma 2.18. *Let $X \in \Omega$, $P \in \partial\Omega$ and $\rho = |X - P|$. Suppose that $\rho < c\, r_0$ and $\rho \leq 2\,\mathrm{dist}\,(X, \partial\Omega)$. Then*

$$\int_{\partial\Omega \setminus \Delta(P,5\rho)} |(G^X)^*(Q)|^p \, dQ \leq C\, \rho^{(n-1)-(n-2)p}$$

where $|p - 2| < \delta$ and δ is the same as in Theorem 2.1.

Lemma 2.19. *Under the same assumption as in Lemma 2.18, we have*

$$\fint_{\Delta(P,32\rho)} \left| \frac{\partial G^X}{\partial \nu} \right|^2 dQ \leq \frac{C}{\rho^{n-1}}, \tag{2.20}$$

$$\fint_{\Delta(P,2^j\rho) \setminus \Delta(P,2^{j-1}\rho)} \left| \frac{\partial G^X}{\partial \nu} \right|^2 dQ \leq \frac{C\,(2^j)^{-\varepsilon}}{(2^j\rho)^2 \rho^{n-3}}, \tag{2.21}$$

$$\fint_{\partial\Omega \setminus \Delta(P,cr_0)} \left| \frac{\partial G^X}{\partial \nu} \right|^2 dQ \leq \frac{C}{\rho^{n-3-\varepsilon}}, \tag{2.22}$$

where $6 \leq j \leq J$, $2^J \sim r_0/\rho$, and $\varepsilon > 0$ depends only on n and δ in Theorem 2.1.

Proof. Using regularity estimate (2.3) for v^X and the well-known estimate for Γ

$$|\nabla^\gamma \Gamma(X)| \leq \frac{C_\gamma}{|X|^{n-2+|\gamma|}}$$

where γ is a multi-index, we have

$$\begin{aligned}\int_{\Delta(P,32\rho)} \left|\frac{\partial G^X}{\partial \nu}\right|^2 dQ &\leq C\int_{\Delta(P,32\rho)} |\nabla \Gamma(X-Q)|^2\, dQ + C\int_{\Delta(P,32\rho)} \left|\nabla v^X(Q)\right|^2 dQ \\ &\leq \frac{C}{\rho^{n-1}} + C\|\Gamma(X-\cdot)\|_{W_1^2(\partial\Omega)} \leq \frac{C}{\rho^{n-1}}.\end{aligned}$$

To prove (2.21) for $6 \leq j \leq J$, we cover $\Delta(P,2^j\rho)\backslash\Delta(P,2^{j-1}\rho)$ with $\{\Delta(P_i,2^{j-2}\rho)\}$ and apply Lemma 2.13 to $G^X(\cdot)$ on $\Delta(P_i,2^{j-2}\rho)$. We obtain

$$\text{(2.23)}\qquad \int_{\Delta(P,2^j\rho)\backslash\Delta(P,2^{j-1}\rho)} \left|\frac{\partial G^X}{\partial \nu}\right|^2 dQ \leq \frac{C}{(2^j\rho)^3}\int_{\Omega_{j+1}\backslash\Omega_{j-2}} \left|G^X(Y)\right|^2 dY,$$

where $\Omega_j = D(P,2^j\rho)$. Choose any $p \in (2-\delta, 2)$ where δ is given in Theorem 2.1. By Hölder inequality,

$$\begin{aligned}&\int_{\Omega_{j+1}\backslash\Omega_{j-2}} \left|G^X(Y)\right|^2 dY \\ &\leq C\,(2^j\rho)^{\frac{n+2}{2}-\frac{n}{p}}\left\{\int_{\Omega_{j+1}\backslash\Omega_{j-2}}\left|G^X\right|^p dY\right\}^{1/p}\left\{\int_{\Omega_{j+1}\backslash\Omega_{j-2}}\left|G^X\right|^{\frac{2n}{n-2}} dY\right\}^{\frac{n-2}{2n}} \\ &\leq C\,(2^j\rho)^{\frac{n}{2}-\frac{n}{p}}\left\{\int_{\Omega_{j+1}\backslash\Omega_{j-2}}\left|G^X\right|^p dY\right\}^{1/p}\left\{\int_{\Omega_{j+2}\backslash\Omega_{j-3}}\left|G^X\right|^2 dY\right\}^{1/2} \\ &\leq C\,(2^j\rho)^{\frac{n+1}{2}-\frac{n-1}{p}}\left\{\int_{\partial\Omega\backslash\Delta(P,5\rho)}\left|(G^X)^*\right|^p dQ\right\}^{1/p}\left\{\int_{\partial\Omega\backslash\Delta(P,5\rho)}\left|(G^X)^*\right|^2 dQ\right\}^{1/2} \\ &\leq C\,(2^j\rho)^{\frac{n+1}{2}-\frac{n-1}{p}}\cdot\rho^{\frac{n-1}{p}-(n-2)}\cdot\rho^{\frac{n-1}{2}-(n-2)} \\ &= C\,2^j\rho\cdot(2^j)^{(n-1)(\frac{1}{2}-\frac{1}{p})}\cdot\rho^{3-n},\end{aligned}$$

where we have used (2.9) in the second inequality and Lemma 2.18 in the last inequality. Estimate (2.21) now follows from this and (2.23) with $\varepsilon = (n-1)(\frac{1}{p}-\frac{1}{2}) > 0$. Estimate (2.22) may be proved in a similar manner.

We need one more lemma before we carry out the proof of Theorem 1.5.

LEMMA 2.24. *Let $\varepsilon > 0$ be given by Lemma 2.19. Suppose $\mathbf{f} \in L^{2,\lambda}(\partial\Omega)$, where $0 \leq \lambda < 2+\varepsilon$ for $n \geq 4$ and $0 \leq \lambda \leq 2$ for $n = 3$. Then the unique solution $\mathbf{u}$ for the Dirichlet problem with boundary data $\mathbf{f}$ and $\|(\mathbf{u})^*\|_2 < \infty$, given by Theorem 2.1, satisfies*

$$|\mathbf{u}(X)| \leq C\,\{\operatorname{dist}(X,\partial\Omega)\}^{\frac{\lambda+1-n}{2}}\,\|\mathbf{f}\|_{2,\lambda} \quad \textit{for any } X \in \Omega.$$

Proof. We may assume that $\operatorname{dist}(X, \partial\Omega) \le c\, r_0$ and $\|\mathbf{f}\|_{2,\lambda} = 1$. Fix $X \in \Omega$. Let $P \in \partial\Omega$ such that $\rho = |X - P| = \operatorname{dist}(X, \partial\Omega)$. Note that

$$\mathbf{u}(X) = -\int_{\partial\Omega} \frac{\partial G^X}{\partial \nu}\, \mathbf{f}(Q)\, dQ. \tag{2.25}$$

Write $\partial\Omega$ as the union of the sets $\Delta(P, 32\rho)$, $\partial\Omega\backslash\Delta(P, cr_0)$, and $\Delta(P, 2^j\rho)\backslash \Delta(P, 2^{j-1}\rho)$, $j = 6, 7, \dots, J$ where $2^J \sim c/\rho$. Using Cauchy inequality, Lemma 2.19 as well as the definition of $L^{2,\lambda}(\partial\Omega)$, we have

$$|\mathbf{u}(X)| \le C\rho^{\frac{\lambda+1-n}{2}} + C\rho^{\frac{3+\varepsilon-n}{2}} + C\sum_{j=6}^{\infty} \frac{(2^j)^{-\frac{\varepsilon}{2}}}{(2^j\rho)\,\rho^{\frac{n-3}{2}}} \cdot (2^j\rho)^{\frac{\lambda}{2}} \le C\rho^{\frac{\lambda+1-n}{2}}$$

if $0 \le \lambda < 2 + \varepsilon$ and $\lambda \le n - 1$. The lemma is proved.

We are now in a position to give the following:

Proof of Theorem 1.5. The uniqueness follows from the uniqueness of L^2 solutions in Theorem 2.1. To show the existence as well as estimate (1.6), let $\mathbf{f} \in L^{2,\lambda}(\partial\Omega)$ with $\|\mathbf{f}\|_{2,\lambda} = 1$, where $0 \le \lambda < 2 + \varepsilon$ for $n \ge 4$ and $0 \le \lambda \le 2$ for $n = 3$. Let $\mathbf{u}$ be the unique L^2 solution with boundary data $\mathbf{f}$.

Given $P \in \partial\Omega$ and $0 < \rho < c\, r_0$. Let $\Delta = \Delta(P, \rho)$ and $s\Delta = \Delta(P, s\rho)$. Write $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ where $\mathbf{f}_2 = \mathbf{f}\chi_{20\Delta}$. Then $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_i$ is the unique L^2 solution with boundary data $\mathbf{f}_i$, $i = 1, 2$. By estimate (2.2),

$$\int_{\partial\Omega} |(\mathbf{u}_2)^*|^2\, dQ \le C\int_{\partial\Omega} |\mathbf{f}_2|^2\, dQ \le C\rho^{\lambda}. \tag{2.26}$$

To estimate $(\mathbf{u}_1)^*$, we note that

$$(\mathbf{u}_1)^* \le \mathcal{M}_1(\mathbf{u}_1) + \mathcal{M}_2(\mathbf{u}_1)$$

where for $Q \in \partial\Omega$,

$$\begin{aligned} \mathcal{M}_1(v)(Q) &= \sup\{|v(X)|\colon\ X \in D(P, 2\rho) \text{ and } |X - Q| \le 2\operatorname{dist}(X, \partial\Omega)\}, \\ \mathcal{M}_2(v)(Q) &= \sup\{|v(X)|\colon\ X \in \Omega\backslash D(P, 2\rho) \text{ and } |X - Q| \le 2\operatorname{dist}(X, \partial\Omega)\}. \end{aligned} \tag{2.27}$$

We now apply estimate (2.2) to $\mathbf{u}_1$ on Lipschitz domain $D(P, t\rho)$ for each $t \in (4, 5)$. Since $\mathbf{u}_1 = 0$ on 5Δ, we obtain

$$\int_{\Delta} |\mathcal{M}_1(\mathbf{u}_1)|^2\, dQ \le C\int_{\Omega\cap\partial D(P,t\rho)} |\mathbf{u}_1|^2\, dQ. \tag{2.28}$$

Integrating both sides of (2.28) with respect to t over (4, 5) then yields

$$\begin{aligned}\int_\Delta |\mathcal{M}_1(\mathbf{u}_1)|^2\,dQ &\leq \frac{C}{\rho}\int_{D(P,5\rho)} |\mathbf{u}_1|^2\,dX\\ &\leq C_p\,\rho^{n-1-\frac{2n}{p}}\left\{\int_{D(P,10\rho)} |\mathbf{u}_1|^p\,dX\right\}^{2/p}\end{aligned}$$

for any $p > 0$, where we have used Lemma 2.8 in the last inequality. In view of Lemma 2.24, we choose $p > 0$ so that $(n - \lambda - 1)p < 2$. We have

$$\int_\Delta |\mathcal{M}_1(\mathbf{u}_1)|^2\,dQ \leq C\,\rho^{n-1-\frac{2n}{p}}\left\{\rho^{n-1}\int_0^{c\rho}\frac{dr}{r^{\frac{(n-\lambda-1)p}{2}}}\right\}^{2/p} \leq C\,\rho^{\lambda}.$$

Next note that, by Lemma 2.24,

$$\mathcal{M}_2(\mathbf{u}_1)(Q) \leq C\,\rho^{\frac{\lambda+1-n}{2}} \qquad \text{for any } Q \in \Delta.$$

The desired estimate for $\mathcal{M}_2(\mathbf{u}_1)$ follows from this easily. Thus we have proved that

$$\int_\Delta |(\mathbf{u})^*|^2\,dQ \leq C\,\rho^{\lambda}.$$

It remains to show the square function estimate

$$\int_\Delta |S(\mathbf{u})|^2\,dQ \leq C\,\rho^{\lambda} \tag{2.29}$$

for $0 \leq \lambda < \min(n-1, 2+\varepsilon)$. Our approach will be similar to that in the case of the nontangential maximal function estimate. Recall that the square function $S(\mathbf{u})$ on $\partial\Omega$ is defined by

$$S(\mathbf{u})(Q) = \left\{\int_{\gamma(Q)} |\nabla\mathbf{u}(X)|^2|X-Q|^{2-n}\,dX\right\}^{1/2} \tag{2.30}$$

where $\gamma(Q) = \{X \in \Omega\colon\ |X-Q| < 2\,\mathrm{dist}\,(X,\partial\Omega)\}$. It follows from a general result in [9] that for solutions of the system $\mathcal{L}\mathbf{u} = 0$ in Ω, one has

$$\|S(\mathbf{u})\|_p \leq C\,\|(\mathbf{u})^*\|_p \qquad \text{for any } 0 < p < \infty. \tag{2.31}$$

Given $\Delta = \Delta(P,\rho)$. Let $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ as before. The desired estimate for $S(\mathbf{u}_2)$ follows directly from (2.31) and Theorem 2.1 with $p = 2$. To estimate

$S(\mathbf{u}_1)$, we write $\{S(\mathbf{u}_1)\}^2 = \{S_1(\mathbf{u}_1)\}^2 + \{S_2(\mathbf{u}_1)\}^2$ where

$$\begin{aligned}\{S_1(\mathbf{u}_1)\}^2(Q) &= \int_{\gamma(Q)\cap D(P,2\rho)} |\nabla \mathbf{u}_1(X)|^2 |X-Q|^{2-n}\, dX,\\ \{S_2(\mathbf{u}_1)\}^2(Q) &= \int_{\gamma(Q)\cap(\mathbb{R}^n\setminus D(P,2\rho))} |\nabla \mathbf{u}_1(X)|^2 |X-Q|^{2-n}\, dX.\end{aligned} \tag{2.32}$$

For $S_1(\mathbf{u}_1)$, we apply estimate (2.31) and Theorem 2.1 on $D(P,t\rho)$ for each $t\in(4,5)$. We obtain

$$\int_\Delta |S_1(\mathbf{u}_1)(Q)|^2\, dQ \le C\int_{\Omega\cap\partial D(P,t\rho)} |\mathbf{u}_1|^2\, dQ,$$

since $\mathbf{u}_1 = 0$ on $\Delta(P,20\rho)$. Integrating above inequality in t over (4, 5), we have

$$\int_\Delta |S_1(\mathbf{u}_1)(Q)|^2\, dQ \le \frac{C}{\rho}\int_{D(P,5\rho)} |\mathbf{u}_1(X)|^2\, dX \le C\rho^\lambda,$$

as before.

Finally, we note that

$$\begin{aligned}&\int_\Delta |S_2(\mathbf{u}_1)(Q)|^2\, dQ\\ &\quad\le C\rho^{n-1}\int_{(\cup_{Q\in\Delta}\gamma(Q))\cap(\mathbb{R}^n\setminus D(P,2\rho))} |\nabla \mathbf{u}_1(X)|^2 |X-P|^{2-n}\, dX\\ &\quad\le C\rho^{n-1}\int_{(\cup_{Q\in\Delta}\gamma(Q))\cap(\mathbb{R}^n\setminus D(P,2\rho))} [\operatorname{dist}(X,\partial\Omega)]^{\lambda-1-n}\, |X-P|^{2-n}\, dX\\ &\quad\le C\rho^{n-1}\int_{\mathbb{R}^n\setminus D(P,2\rho)} |X-P|^{\lambda+1-2n}\, dX\\ &\quad\le C\rho^\lambda.\end{aligned}$$

We remark that in the second inequality above we have used the estimate

$$|\nabla \mathbf{u}_2(X)| \le C\, \{\operatorname{dist}(X,\partial\Omega)\}^{\frac{\lambda-1-n}{2}},$$

which is an easy consequence of Lemma 2.24 and the usual interior estimates. Estimate (2.29) is now proved and the proof of Theorem 1.5 is then complete.

Remark 2.33. It follows from the square function estimate in Theorem 1.5 that

$$\int_{D(P,\rho)} |\nabla \mathbf{u}(Y)|^2 \operatorname{dist}(Y,\partial\Omega)\, dY \le C\rho^\lambda\, \|\mathbf{f}\|_{2,\lambda}^2 \tag{2.34}$$

for any $P \in \partial\Omega$ and $0 < \rho < c\, r_0$. This implies that

$$\| |\nabla \mathbf{u}| \operatorname{dist}(\cdot, \partial\Omega)^{\frac{1}{2}} \|_{L^{2,\lambda}(\Omega)} \leq C \, \|\mathbf{f}\|_{L^{2,\lambda}(\partial\Omega)} \tag{2.35}$$

for $0 \leq \lambda < \min(n-1, 2+\varepsilon)$.

Remark 2.36. Theorem 1.5 also holds if we replace Ω by Ω_- and impose an additional condition $|\mathbf{u}(X)| = O(|X|^{2-n})$ as $|X| \to \infty$ in (1.1).

3. Regularity for the Dirichlet problem. In this section we study the Dirichlet problem with boundary data $\mathbf{f}$ in $H_1^{2,\lambda}(\partial\Omega)$. As in the last section, we assume that $\mathcal{L}$ is a second order elliptic operator with coefficients satisfying the symmetry property and the Legendre-Hadamard condition (1.2). We start with the definition of space $H_1^{2,\lambda}(\partial\Omega)$.

Definition 3.1. Let $0 \leq \lambda \leq n-1$ and $a \in W_1^2(\partial\Omega)$. We say a is an $H_1^{2,\lambda}(\partial\Omega)$-atom if there exist $P \in \partial\Omega$, $0 < \rho < c\, r_0$ such that $\operatorname{supp} a \subset \Delta(P, \rho)$ and

$$\|\nabla_{\tan} a\|_2 \leq \rho^{-\lambda/2}. \tag{3.2}$$

Let $f \in H^{2,\lambda}(\partial\Omega)$. We say $f \in H_1^{2,\lambda}(\partial\Omega)$ if there exist $\mu_j \in \mathbb{R}$ and $H_1^{2,\lambda}(\partial\Omega)$-atom a_j such that $\sum |\mu_j| < \infty$ and $f = \sum \mu_j a_j$. The norm of f in $H_1^{2,\lambda}(\partial\Omega)$ is defined by

$$\|f\|_{H_1^{2,\lambda}(\partial\Omega)} = \inf \left\{ \sum_{j=1}^{\infty} |\mu_j| : \ f = \sum_{j=1}^{\infty} \mu_j a_j \right\}. \tag{3.3}$$

Remark 3.4. It is easy to see that $H_1^{2,0}(\partial\Omega) = W_1^2(\partial\Omega)$ and $H_1^{2,n-1}(\partial\Omega)$ is the atomic space $H^1_{1,at}(\partial\Omega)$ introduced in [7]. For $0 < \lambda < n-1$, by Hölder inequality, we have

$$H_1^{2,\lambda}(\partial\Omega) \subset W_1^p(\partial\Omega) \quad \text{for } p = \frac{2(n-1)}{n-1+\lambda}. \tag{3.5}$$

Also observe that, by Poincaré inequality on $\partial\Omega$ and (3.2),

$$\|a\|_2 \leq C\, \rho^{(2-\lambda)/2}. \tag{3.6}$$

The goal of this section is to establish the following theorem.

THEOREM 3.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. There exists $\varepsilon > 0$ depending only on n, m, the ellipticity constant μ and Ω such that, given any* $\mathbf{f} \in H_1^{2,\lambda}(\partial\Omega)$ *with* $0 \leq \lambda < \min(n-1, 2+\varepsilon)$*, there* *exists a unique* $\mathbf{u}$ *satisfying* $\mathcal{L}\mathbf{u} = 0$ *in* Ω, $\mathbf{u} = \mathbf{f}$ *on* $\partial\Omega$ *in the sense of nontangential convergence, and* $\|(\nabla\mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} < \infty$. *Moreover, we have*

$$\|(\nabla\mathbf{u})^*\|_{H^{2,\lambda}(\partial\Omega)} \leq C\,\|\mathbf{f}\|_{H_1^{2,\lambda}(\partial\Omega)}. \tag{3.8}$$

To establish the existence of solutions as well as estimate (3.8), we first introduce the notion of $(2,\lambda)$ molecules for the space $H^{2,\lambda}(\partial\Omega)$ with $0 \leq \lambda < n-1$ in a similar fashion to the case of the atomic Hardy space [4]. Also see [0].

Definition 3.9. Let $F \in L^2(\partial\Omega)$. F is called a $(2,\lambda)$ molecule if there exist constants $C > 0$, $\gamma > \lambda$ and $P \in \partial\Omega$, $\rho > 0$ such that

$$\int_{\partial\Omega} |F(Q)|^2\,dQ \leq C\,\rho^{-\lambda}, \tag{3.10}$$

$$\int_{\partial\Omega} |F(Q)|^2 |Q-P|^\gamma\,dQ \leq C\,\rho^{\gamma-\lambda}. \tag{3.11}$$

PROPOSITION 3.12. *Let* $0 \leq \lambda < n-1$. *If* F *is a* $(2,\lambda)$ *molecule, then* $F \in H^{2,\lambda}(\partial\Omega)$ *and*

$$\|F\|_{H^{2,\lambda}(\partial\Omega)} \leq C_0$$

where C_0 *depends only on* n, γ *and* C *in (3.10)–(3.11).*

The proof of Proposition 3.12, which is omitted here, may be carried out using the same argument as in the case of the atomic Hardy space [4]. Note that we only need $\gamma > \lambda$. Also, since $\partial\Omega$ is bounded, we do not need the mean zero property in the definition of molecules.

We are now ready to give the following:

Proof of Theorem 3.7. We begin with the uniqueness. Using the definition of nontangential maximal functions, it is not hard to show that

$$(\mathbf{u})^*(P) \leq \sup_{X\in K_\eta} |\mathbf{u}(X)| + C_\eta \int_{\partial\Omega} \frac{(\nabla\mathbf{u})^*(Q)}{|Q-P|^{n-2}}\,dQ, \tag{3.13}$$

where $K_\eta = \{X \in \Omega\colon \operatorname{dist}(X,\partial\Omega) \geq \eta\}$ and $\eta > 0$ is small. By the fractional integral estimates, if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n-1}$,

$$\|(\mathbf{u})^*\|_q \leq C \sup_{X\in K_\delta} |\mathbf{u}(X)| + C_\delta\,\|(\nabla\mathbf{u})^*\|_p. \tag{3.14}$$

Now suppose $(\nabla\mathbf{u})^* \in H^{2,\lambda}(\partial\Omega)$. Then $\|(\nabla\mathbf{u})^*\|_p < \infty$ for $p = 2(n-1)/(n-1+\lambda)$ by (1.16). It follows from (3.14) that $\|(\mathbf{u})^*\|_q < \infty$ for $q = 2(n-$

$1)/(n+1-\lambda)$. Note that $q > 2-\delta$ if $0 \le \lambda < \min(n-1, 2+\varepsilon)$ and ε is small. Thus the uniqueness in Theorem 3.7 follows from the uniqueness in Theorem 2.1.

To establish the existence of solutions, let $\mathbf{u}$ be the $W_1^2(\partial\Omega)$ solution of $\mathcal{L}(\mathbf{u}) = 0$ in Ω with $\mathbf{u} = \mathbf{a}$ on $\partial\Omega$, given in Theorem 2.1, where $\mathbf{a}$ is a vector-valued $H_1^{2,\lambda}(\partial\Omega)$-atom. We will show that, if $0 \le \lambda < \min(n-1, 2+\varepsilon)$, then $F = (\nabla\mathbf{u})^*$ is a $(2,\lambda)$ molecule with constants C and γ depending only on n, m, μ, λ and Ω. By Proposition 3.12. this gives the existence of solutions with boundary data in $H_1^{2,\lambda}(\partial\Omega)$ as well as estimate (3.8).

To this end, we suppose that supp $\mathbf{a} \subset \Delta(P_0, r)$ for some $0 < r < c\,r_0$, and $\|\nabla_{\tan}\mathbf{a}\|_2 \le r^{-\lambda/2}$. By estimate (2.3),

$$\int_{\partial\Omega} |F(Q)|^2\, dQ \le C\,\|\nabla_{\tan}\mathbf{a}\|_2^2 \le C\,r^{-\lambda}. \tag{3.15}$$

This gives (3.10).

Next, let $\Delta = \Delta(P,\rho)$ where $P \in \partial\Omega$ and $r \le \rho < c\,r_0$. We will show that, if $\mathbf{u} = 0$ on 20Δ, then

$$\int_{\Delta} |F(Q)|^2\, dQ \le C\left(\frac{r}{\rho}\right)^{\gamma} \cdot r^{-\lambda} \tag{3.16}$$

for some $\gamma > 2$. Since $0 \le \lambda < \min(n-1, 2+\varepsilon)$, it is not hard to see that estimate (3.16), together with (3.15), implies (3.11).

To show (3.16), note that $F = (\nabla\mathbf{u})^* \le \mathcal{M}_1(\nabla\mathbf{u}) + \mathcal{M}_2(\nabla\mathbf{u})$ where $\mathcal{M}_1$, $\mathcal{M}_2$ are defined in (2.27). First we apply estimate (2.3) to $\mathbf{u}$ in $D(P, t\rho)$ for $t \in (4,5)$ to obtain

$$\int_{\Delta} |\mathcal{M}_1(\nabla\mathbf{u})|^2\, dQ \le C\int_{\Omega\cap\partial D(P,t\rho)} |\nabla\mathbf{u}|^2\, dQ,$$

where we also used the fact that $\mathbf{u} = 0$ on 20Δ. Integrating the inequality above with respect to $t \in (4,5)$ then yields that

$$\int_{\Delta} |\mathcal{M}_1(\nabla\mathbf{u})|^2\, dQ \le \frac{C}{\rho}\int_{D(P,5\rho)} |\nabla\mathbf{u}|^2\, dX.$$

This, together with Lemmas 2.7 and 2.8, implies that

$$\int_{\Delta} |\mathcal{M}_1(\nabla\mathbf{u})|^2\, dQ \le C_p\,\rho^{n-3+\frac{2(1-n)}{p}}\,\|(\mathbf{u})^*\|_p^2 \tag{3.17}$$

for any $0 < p < 2$.

To estimate $\mathcal{M}_2(\nabla\mathbf{u})$, we note that, if $Q \in \Delta$, $|X - Q| \leq 2\,\mathrm{dist}\,(X, \partial\Omega)$ and $X \in \Omega \backslash D(P, 2\rho)$, then $\mathrm{dist}\,(X, \partial\Omega) > 2c\rho$. Thus, by interior estimates,

$$\begin{aligned} |\nabla\mathbf{u}(X)|^2 &\leq \frac{C}{\rho^{n+2}} \int_{B(X,c\rho)} |\mathbf{u}(Y)|^2\, dY \\ &\leq C_p\, \rho^{-2-\frac{2n}{p}} \left\{ \int_{B(X,2c\rho)} |\mathbf{u}(Y)|^p\, dY \right\}^{2/p} \\ &\leq C_p\, \rho^{-2+\frac{2(1-n)}{p}} \|(\mathbf{u})^*\|_p^2. \end{aligned}$$

It follows that estimate (3.17) also holds for $\mathcal{M}_2(\nabla\mathbf{u})$. Thus we have proved that

$$\int_\Delta |F(Q)|^2\, dQ \leq C_p\, \rho^{n-3+\frac{2(1-n)}{p}} \|(\mathbf{u})^*\|_p^2. \tag{3.18}$$

Finally, choose $p \in (2 - \delta, 2)$. By Theorem 2.1 and (3.6), we have

$$\|(\mathbf{u})^*\|_p \leq C\|\mathbf{a}\|_p \leq C r^{(n-1)(\frac{1}{p}-\frac{1}{2})+1-\frac{\lambda}{2}}. \tag{3.19}$$

In view of (3.18) and (3.19), we obtain

$$\int_\Delta |F(Q)|^2\, dQ \leq C \left(\frac{r}{\rho}\right)^{2+(n-1)(\frac{2}{p}-1)} \cdot r^{-\lambda}.$$

Estimate (3.16) is then proved with $\gamma = 2 + (n - 1)(\frac{2}{p} - 1) > 2$. The proof of Theorem 3.7 is now complete.

Remark 3.20. We mention in the Introduction that $(H^{2,\lambda}(\partial\Omega))^* = L^{2,\lambda}(\partial\Omega)$ for $0 \leq \lambda < n - 1$. Indeed, given any $f \in L^{2,\lambda}(\partial\Omega)$,

$$\ell_f(g) = \int_{\partial\Omega} f\, g\, d\sigma, \qquad g \in H^{2,\lambda}(\partial\Omega) \tag{3.21}$$

defines a bounded linear functional on $H^{2,\lambda}(\partial\Omega)$. Moreover, $\|\ell_f\| \approx \|f\|_{2,\lambda}$. Also, any element ℓ in $(H^{2,\lambda}(\partial\Omega))^*$ is given by (3.21) with some $f \in L^{2,\lambda}(\partial\Omega)$. The proof is left to the reader. It follows from the definition of $L^{2,\lambda}(\partial\Omega)$ that $\|\varphi f\|_{2,\lambda} \leq \|\varphi\|_\infty \|f\|_{2,\lambda}$. By duality, we obtain

$$\|\varphi\, g\|_{H^{2,\lambda}(\partial\Omega)} \leq C\, \|\varphi\|_\infty \|g\|_{H^{2,\lambda}(\partial\Omega)}. \tag{3.22}$$

In view of (3.8), this implies that the unique solution $\mathbf{u}$ in Theorem 3.7 satisfies

$$\left\|\frac{\partial\mathbf{u}}{\partial\nu}\right\|_{H^{2,\lambda}(\partial\Omega)} \leq C\, \|\nabla\mathbf{u}\|_{H^{2,\lambda}(\partial\Omega)} \leq C\|\mathbf{u}\|_{H_1^{2,\lambda}(\partial\Omega)}. \tag{3.23}$$

Remark 3.24. Theorem 3.7 also hold if we replace Ω by Ω_- and impose an additional condition

$$|\mathbf{u}(X)| = O(|X|^{2-n}) \quad \text{as } |X| \to \infty. \tag{3.25}$$

4. The Neumann problem in $H^{2,\lambda}$. In this section we study the Neumann problem (1.11) with data in $H^{2,\lambda}(\partial\Omega)$ and give the proof of Theorem 1.12. Throughout this section we will assume that the operator $\mathcal{L}$ satisfies the strong ellipticity condition (1.13). For general elliptic systems satisfying Legendre-Hadamard condition (1.2), Verchota [32] points out that the L^2 Neumann problem needs not be uniquely solvable modulo finitely many linear conditions.

We begin with the L^p estimate for p close to 2.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. There exists $\delta > 0$ depending only on n, m, the ellipticity constant μ in (1.13) and the Lipschitz character of Ω such that, given any $\mathbf{g} \in L^p(\partial\Omega)$ with $2 - \delta < p < 2 + \delta$ and $\int_{\partial\Omega} \mathbf{g}\, d\sigma = 0$, there exists a unique (up to a constant) $\mathbf{u}$ satisfying $\mathcal{L}\mathbf{u} = 0$ in Ω, $\frac{\partial \mathbf{u}}{\partial \nu} = \mathbf{g}$ on $\partial\Omega$ in the sense of nontangential convergence, and $\|(\nabla \mathbf{u})^*\|_p < \infty$. Moreover, the solution $\mathbf{u}$ will satisfy*

$$\|(\nabla \mathbf{u})^*\|_p \leq C\, \|\mathbf{g}\|_p. \tag{4.2}$$

Proof. It was pointed out in [11] that the same argument there also yields the result in Theorem 4.1. We sketch the main ideas here.

First, the uniqueness follows easily from (3.14) and the identity

$$\int_\Omega a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j}\, dX = \int_{\partial\Omega} \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial \nu}\, dQ. \tag{4.3}$$

To establish the existence as well as estimate (4.2), we consider the single layer potential $\mathcal{S}(\mathbf{f})$ defined by

$$(\mathcal{S}(\mathbf{f}))^\alpha(X) = \int_{\partial\Omega} \Gamma_{\alpha\beta}(X - Q) f_\beta(Q)\, dQ \tag{4.4}$$

where $\mathbf{f} \in L^p(\partial\Omega)$, $p > 1$ and $\Gamma(X) = (\Gamma_{\alpha\beta}(X))_{m\times m}$ denotes the matrix of fundamental solutions in $\mathbb{R}^n$ for $\mathcal{L}$ with pole at 0. On $\partial\Omega$, the traces of $\nabla \mathbf{u}$ are given by

$$D_i u_\pm^\alpha(P) = \pm\frac{1}{2} N_i(P) b_{\alpha\beta}(P) f_\beta(P) + \text{p.v.} \int_{\partial\Omega} D_i \Gamma_{\alpha\beta}(P - Q) f_\beta(Q)\, dQ, \tag{4.5}$$

(see [17]), where $(b_{\alpha\beta})_{m\times m}$ is the inverse of the matrix $(a_{ij}^{\alpha\beta} N_i N_j)_{m\times m}$ and $\pm$ indicates that the nontangential limit of $D_i u^\alpha$ is taken from $\Omega_+ = \Omega$ and $\Omega_- =$

$\mathbb{R}^n \backslash \overline{\Omega}$ respectively. It follows from (4.5) that

$$\nabla_{\tan}\mathbf{u}_+ = \nabla_{\tan}\mathbf{u}_- \quad \text{and} \quad \frac{\partial \mathbf{u}_\pm}{\partial \nu} = \pm\frac{1}{2}\mathbf{f} + K_\nu \mathbf{f} \quad \text{on } \partial\Omega, \tag{4.6}$$

where K_ν is a bounded operator on $L^p(\partial\Omega)$ for $p > 1$ [3]. Thus the existence of solution $\mathbf{u}$ with estimate (4.2) will follow if one can show that $(1/2)I + K_\nu$ is invertible from $L^p_0(\partial\Omega)$ to itself for p close to 2, where $L^p_0(\partial\Omega)$ denotes the space of functions $\mathbf{f}$ in $L^p(\partial\Omega)$ such that $\int_{\partial\Omega} \mathbf{f}\, d\sigma = 0$. In fact it suffices to show that $(1/2)I + K_\nu$ is invertible on $L^2_0(\partial\Omega)$. The L^p case follows from a stability theorem on interpolation scales of Banach spaces (see e.g. [34, Proposition 4.1]).

Using (4.3) and a similar identity on Ω_-, it is easy to see that $(1/2)I + K_\nu$ is one-to-one on $L^2_0(\partial\Omega)$. Next, choose $\mathbf{h} \in C^1_0(\mathbb{R}^n)$ such that $< \mathbf{h}, \mathbf{N} > \geq c_0 > 0$ on $\partial\Omega$. Using Rellich identities (2.5)–(2.6) with this $\mathbf{h}$ and $\mathbf{u} = \mathcal{S}(\mathbf{f})$, as well as the strong ellipticity condition (1.13), one can show that, if $\mathbf{u} = \mathcal{S}(\mathbf{f})$ and $\mathbf{f} \in L^2(\partial\Omega)$, then

$$\begin{aligned} \|\nabla \mathbf{u}_\pm\|_2 &\leq C\left\{\|\nabla_{\tan}\mathbf{u}\|_2 + \left|\int_{\partial\Omega} \mathbf{u}\right|\right\}, \\ \|\nabla \mathbf{u}_\pm\|_2 &\leq C\left\{\left\|\frac{\partial \mathbf{u}_\pm}{\partial \nu}\right\|_2 + \left|\int_{\partial\Omega} \mathbf{u}\right|\right\}. \end{aligned} \tag{4.7}$$

See e.g. [13]. Since $\mathbf{f} = \frac{\partial \mathbf{u}_+}{\partial \nu} - \frac{\partial \mathbf{u}_-}{\partial \nu}$, we obtain

$$\|\mathbf{f}\|_2 \leq C\left\{\left\|\left(\pm\frac{1}{2}I + K_\nu\right)\mathbf{f}\right\|_2 + \left|\int_{\partial\Omega} \mathcal{S}(\mathbf{f})\, dQ\right|\right\}. \tag{4.8}$$

This implies that the range of $(1/2)I + K_\nu$: $L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$ is closed. Consequently, $(1/2)I + K_\nu$ is a semi-Fredholm operator on $L^2_0(\partial\Omega)$.

Next we show that the index of $(1/2)I + K_\nu$ is zero. We will prove that for $\gamma > 1/2$,

$$\|\mathbf{f}\|_2 \leq C_\gamma \left\{\|(\gamma I + K_\nu)\|_2 + \|T(\mathbf{f})\|_2 + \|\mathcal{S}(\mathbf{f})\|_2\right\} \tag{4.9}$$

where T: $L^2(\partial\Omega) \to L^2(\partial\Omega)$ is a compact operator. Estimate (4.9), together with (4.8), implies that the operator $\gamma I + K_\nu$: $L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$ is semi-Fredholm for all $\gamma \geq 1/2$. Since $\gamma I + K_\nu$ is clearly invertible on $L^2_0(\partial\Omega)$ for γ large, by the continuity of the index, we deduce that the index of $(1/2)I + K_\nu$ is zero. Since $(1/2)I + K_\nu$ is one-to-one, we conclude that it is invertible on $L^2_0(\partial\Omega)$.

It remains to prove (4.9). To do this, we apply Rellich identity (2.5) in Ω to $\mathbf{u} = \mathcal{S}(\mathbf{f})$. Using (4.3), $\|\frac{\partial \mathbf{u}_+}{\partial \nu}\|_2 \leq C\|\mathbf{f}\|_2$ and $a^{\alpha\beta}_{ij} D_i u^\alpha_+ D_j u^\beta_+ \geq \mu\, |\nabla \mathbf{u}_+|^2 \geq 0$, we obtain

$$0 \leq \int_{\partial\Omega} h_i \frac{\partial \mathbf{u}_+}{\partial x_i} \cdot \frac{\partial \mathbf{u}_+}{\partial \nu} + C\|\mathbf{u}\|_2\|\mathbf{f}\|_2. \tag{4.10}$$

Note that by (4.5),

$$h_i \frac{\partial u_+^\alpha}{\partial x_i} = \frac{1}{2}\langle \mathbf{h}, \mathbf{N}\rangle b_{\alpha\beta} f_\beta + (K_{\mathbf{h}}(\mathbf{f}))^\alpha \tag{4.11}$$

where

$$(K_{\mathbf{h}}(\mathbf{f}))^\alpha (P) = \text{p.v.} \int_{\partial\Omega} h_i(P) D_i \Gamma_{\alpha\beta}(P-Q) f_\beta(Q)\, dQ. \tag{4.12}$$

Write $\frac{\partial \mathbf{u}_+}{\partial \nu} = (\frac{1}{2} - \gamma)\mathbf{f} + (\gamma I + K_\nu)(\mathbf{f})$. Plug this and (4.11) into (4.10) and use $\gamma > 1/2$, we get

$$\begin{aligned} &\int_{\partial\Omega} \langle \mathbf{h}, \mathbf{N}\rangle b_{\alpha\beta} f_\alpha f_\beta \, dQ \\ &\quad \le -2 \int_{\partial\Omega} \langle K_{\mathbf{h}}(\mathbf{f}), \mathbf{f}\rangle \, dQ + C_\gamma \left\{ \|(\gamma I + K_{\mathbf{h}})\mathbf{f}\|_2 + \|\mathbf{u}\|_2 \right\} \|\mathbf{f}\|_2 \\ &\quad = -\int_{\partial\Omega} \left\langle \left(K_{\mathbf{h}} + K_{\mathbf{h}}^*\right)(\mathbf{f}), \mathbf{f}\right\rangle \, dQ + C_\gamma \left\{ \|(\gamma I + K_{\mathbf{h}})\mathbf{f}\|_2 + \|\mathbf{u}\|_2 \right\} \|\mathbf{f}\|_2 \end{aligned} \tag{4.13}$$

where $K_{\mathbf{h}}^*$ denotes the adjoint operator of $K_{\mathbf{h}}$ on $L^2(\partial\Omega)$. Since $\mathbf{h} \in C^1$, $K_{\mathbf{h}} + K_{\mathbf{h}}^*$ is compact on $L^2(\partial\Omega)$. Also note that $\langle \mathbf{h}, \mathbf{N}\rangle b_{\alpha\beta} f_\alpha f_\beta \ge c|\mathbf{f}|^2$. The desired estimate (4.9) follows from (4.13) by Hölder inequality with an ε. We remark that our method of proving (4.9), as well as the idea of using it to show the invertibility of $(1/2)I + K_\nu$, is taken from [12] and [23].

Remark 4.14. Theorem 4.1 also holds if we replace Ω by Ω_- and impose an additional condition (3.25). In this case, the mean zero condition on $\mathbf{g}$ is not needed. Indeed we may use the same argument as in the proof of Theorem 4.1 to show that $-(1/2)I + K_\nu$ is invertible on $L^p(\partial\Omega)$ for p close to 2. In fact it is not hard to see that $-(1/2)I + K_\nu$: $L^2(\partial\Omega) \to L^2(\partial\Omega)$ is one-to-one. Estimate (4.8) implies that the range of $-(1/2)I + K_\nu$ is closed. To show that the index of the operator is zero, we note that (4.9) also holds for all $\gamma < -1/2$. This follows by using Rellich identity (2.5) in Ω_- to obtain, in the place of (4.10),

$$0 \le \int_{\partial\Omega} h_i \frac{\partial \mathbf{u}_-}{\partial x_i} \cdot \frac{\partial \mathbf{u}_-}{\partial \nu} + C\, \|\mathbf{u}\|_2 \|\mathbf{f}\|_2. \tag{4.15}$$

The rest is the same.

We now give the

Proof of Theorem 1.12. For the uniqueness, we construct a matrix of Neumann functions. To do this, we fix $X \in \Omega$. Let $W = W^X = (w_{\alpha\ell})_{m\times m}$ be a matrix-valued function such that, for each $\ell = 1, \dots, m$, $(w_{1\ell}, \dots, w_{m\ell})$ is a solution of

$$(4.16)\qquad \begin{cases} a_{ij}^{\alpha\beta} D_i D_j w_{\beta\ell} = 0 \quad \text{in } \Omega, \quad \alpha = 1, \dots, m, \\ a_{ij}^{\alpha\beta} D_j w_{\beta\ell} N_i = a_{ij}^{\alpha\beta} D_j \left(\Gamma_{\beta\ell}(X - \cdot) \right) N_i + \dfrac{\delta_{\alpha\ell}}{|\partial\Omega|} \qquad \text{on } \partial\Omega, \\ \|(\nabla w_{\alpha\ell})^*\|_2 < \infty. \end{cases}$$

By Theorem 4.1, such solution exists and is unique (up to a constant). Moreover, we have $\|(\nabla W)^*\|_{\bar{p}} < \infty$ for $2 < \bar{p} < 2 + \delta$. Define

$$(4.17)\qquad G_\nu^X(Y) = \Gamma(X - Y) - W^X(Y).$$

Now suppose $\mathbf{u}$ is a solution of (1.11). Since $(\nabla \mathbf{u})^* \in H^{2,\lambda}(\partial\Omega)$, we have $\|(\nabla \mathbf{u})^*\|_p < \infty$ for $p = 2(n-1)/(n-1+\lambda)$. Also by (3.14), $\|(\mathbf{u})^*\|_q < \infty$, $\|(W)^*\|_{\bar{q}} < \infty$ for $q = 2(n-1)/(n+1-\lambda)$ and $\bar{q} = \bar{p}(n-1)/(n-1-\bar{p})$ $(1 < \bar{q} < \infty$ if $n = 3)$. Thus we have $(W^X)^* \cdot (\nabla \mathbf{u})^* \in L^1(\partial\Omega)$, $(\nabla W^X)^* \cdot (\mathbf{u})^* \in L^1(\partial\Omega)$ if $0 \le \lambda < \min(n-1, 2+\varepsilon)$ and ε is small. This is enough to justify the formula

$$(4.18)\qquad \int_{\partial\Omega} W^X \frac{\partial \mathbf{u}}{\partial \nu}\, dQ - \int_{\partial\Omega} \frac{\partial W^X}{\partial \nu}\, \mathbf{u}\, dQ = 0.$$

Same argument also gives

$$(4.19)\qquad \mathbf{u}(X) = \int_{\partial\Omega} \Gamma(X - Q) \frac{\partial \mathbf{u}}{\partial \nu}\, dQ - \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left\{ \Gamma(X - Q) \right\} \mathbf{u}(Q)\, dQ.$$

Combining (4.18) and (4.19), we obtain

$$\begin{aligned} \mathbf{u}(X) &= \int_{\partial\Omega} G^X(Q) \frac{\partial \mathbf{u}}{\partial \nu}\, dQ - \int_{\partial\Omega} \frac{\partial G^X}{\partial \nu}\, \mathbf{u}(Q)\, dQ \\ &= \int_{\partial\Omega} G^X(Q) \frac{\partial \mathbf{u}}{\partial \nu}\, dQ + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{u}(Q)\, dQ. \end{aligned}$$

This implies that if $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on $\partial\Omega$, then $\mathbf{u}$ is constant. The uniqueness is now proved.

To show the existence as well as estimate (1.14), it suffices to show that, if $\mathbf{u}$ is an L^2 solution of (1.11) with data $\mathbf{g} = \mathbf{a}$, and $\mathbf{a}$ is an $(2, \lambda)$ atom, then $F = (\nabla \mathbf{u})^*$ is a $(2, \lambda)$ molecule with constant C and γ depending only on n, m, μ, λ and Ω. By subtracting a constant from $\mathbf{u}$, we may assume that

$$(4.20)\qquad \int_{\partial\Omega} \mathbf{u}\, d\sigma = 0.$$

Suppose that supp $\mathbf{a} \subset \Delta(P_0, r)$ for some $0 < r < c\,r_0$, $\|\mathbf{a}\|_2 \leq r^{-\lambda/2}$, and $\int_{\partial\Omega} \mathbf{a}\, d\sigma = 0$. By Theorem 4.1,

$$\int_{\partial\Omega} |F(Q)|^2 dQ \leq C\,\|\mathbf{a}\|_2^2 \leq C\,r^{-\lambda}, \tag{4.21}$$

which is (3.10). To see (3.11), as in the proof of Theorem 3.7, it suffices to show that

$$\int_{\Delta} |F(Q)|^2\, dQ \leq C\,\left(\frac{r}{\rho}\right)^{\gamma} r^{-\lambda}, \tag{4.22}$$

for some $\gamma > 2$, where $\Delta = \Delta(P, \rho)$ for some $P \in \partial\Omega$ and $0 < \rho < c\,r_0$, and $\frac{\partial \mathbf{u}}{\partial \nu} = 0$ on 20Δ.

To this end, we use Theorem 4.1 on the Lipschitz domain $D(P, t\rho)$ for $t \in (4, 5)$. By an inspection of the proof of Theorem 3.7, we see that

$$\int_{\Delta} |F(Q)|^2\, dQ \leq C_p\, \rho^{n-3+\frac{3(1-n)}{p}}\, \|\mathbf{u}\|_p^2 \tag{4.23}$$

for $2-\delta < p < 2$. Finally, to estimate $\|\mathbf{u}\|_p$, we let $X \to Q \in \partial\Omega$ nontangentially in (4.19). We obtain

$$\left(\frac{1}{2}I + K_\nu^*\right) \mathbf{u}(Q) = \mathcal{S}(\mathbf{a})(Q) \qquad \text{on } \partial\Omega, \tag{4.24}$$

where K_ν^* denotes the adjoint of K_ν on $L^2(\partial\Omega)$. Recall that for q close to 2, $(1/2)I + K_\nu$ is invertible on $L_0^q(\partial\Omega)$ and the range of $(1/2)I + K_\nu$: $L^q(\partial\Omega) \to L^q(\partial\Omega)$ is $L_0^q(\partial\Omega)$. By duality, this implies that $(1/2)I + K_\nu^*$ is invertible from $L_0^p(\partial\Omega)$ to a subspace of $L^p(\partial\Omega)$ of codimension n. In view of (4.20) and (4.24), we have

$$\|\mathbf{u}\|_p \leq C\,\left\|\left(\frac{1}{2}I + K_\nu^*\right)(\mathbf{u})\right\|_p = C\,\|\mathcal{S}(\mathbf{a})\|_p. \tag{4.25}$$

By the usual estimates on Γ and the mean zero condition of $\mathbf{a}$, we have

$$|\mathcal{S}(\mathbf{a})(Q)| \leq \begin{cases} C\,\displaystyle\int_{Z\in\Delta} \frac{|\mathbf{a}(Z)|}{|Q-Z|^{n-2}}\, dZ, & Q \in 3\Delta, \\ C\,\rho\,\displaystyle\int_{Z\in\Delta} \frac{|\mathbf{a}(Z)|}{|Q-Z|^{n-1}}\, dZ, & Q \in \partial\Omega\backslash 3\Delta. \end{cases} \tag{4.26}$$

It follows that

$$\|\mathcal{S}(\mathbf{a})\|_p \leq C\,\rho^{(\frac{1}{p}-\frac{1}{2})(n-1)+1-\frac{\lambda}{2}}.$$

This, together with (4.23) and (4.25), gives the desired estimate (4.22) with

$$\gamma = 2 + (n-1)\left(\frac{2}{p} - 1\right) > 2.$$

The proof is complete.

Remark 4.27. Theorem 1.12 also holds if we replace Ω by Ω_- and impose the condition (3.25). The mean zero condition on $\mathbf{g}$ is not needed as in the L^2 case.

Remark 4.28. We claim that the solutions $\mathbf{u}$ in Theorem 1.12 satisfy

$$\left\| \mathbf{u} - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \mathbf{u}\, d\sigma \right\|_{H_1^{2,\lambda}(\partial\Omega)} \le C \left\| \frac{\partial \mathbf{u}}{\partial \nu} \right\|_{H^{2,\lambda}(\partial\Omega)}. \tag{4.29}$$

To prove (4.29), we may assume that $\frac{\partial \mathbf{u}}{\partial \nu} = \mathbf{a}$ is an $(2,\lambda)$ atom and $\int_{\partial\Omega} \mathbf{u}\, d\sigma = 0$. Suppose $\mathbf{a}$ is an $(2,\lambda)$ atom with support $\Delta(P,\rho)$ where $0 < \rho < cr_0$. By the proof of Theorem 1.12, we have

$$\begin{aligned} \int_{\Delta(P,2\rho)} |\nabla_{\tan}\mathbf{u}|^2\, dQ \ &\le\ C\rho^{-\lambda}, \\ \int_{|Q-P|\sim 2^j\rho} |\nabla_{\tan}\mathbf{u}|^2\, dQ \ &\le\ C(2^{-j})^{\gamma}\rho^{-\lambda} \end{aligned} \tag{4.30}$$

for some $\gamma > 2$. Also

$$\|\mathbf{u}\|_2 \le C\rho^{(2-\lambda)/2}. \tag{4.31}$$

Let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on $B(P, r_0/2)$ and $\varphi = 0$ outside of $B(P, r_0)$. We will show that

$$\|\mathbf{u}\varphi\|_{H_1^{2,\lambda}(\partial\Omega)} \le C. \tag{4.32}$$

The remaining part $\mathbf{u}(1-\varphi)$ is easy to handle.

To show (4.32), we choose a partition of unity of $\mathbb{R}^n$

$$1 = \psi_0 + \sum_{j=1}^{\infty} \psi_j$$

so that supp $\psi_0 \subset B(P, 2\rho)$, supp $\psi_j \subset B(P, 2^{j+1}\rho)\backslash B(P, 2^{j-1}\rho)$ for $j = 1, 2, \dots$ and $|\nabla\psi_j| \le C/(2^j\rho)$ for $j = 0, 1, \dots$ We write

$$\mathbf{u}\varphi = \sum_{j=0}^{J} \mathbf{u}\varphi\psi_j.$$

Note that, by estimates (4.30) and (4.31),

$$\|\nabla_{\tan}(\mathbf{u}\varphi\psi_j)\|_2 \le C\left\{(2^{-j})^{\gamma/2}\rho^{-\lambda/2} + \rho^{(2-\lambda)/2}\cdot\frac{1}{2^j\rho}\right\} \le C\,2^{-j}\rho^{-\lambda/2}.$$

This implies that $C^{-1}2^j(\mathbf{u}\varphi\psi_j)$ is an $H_1^{2,\lambda}(\partial\Omega)$-atom. It follows that

$$\|\mathbf{u}\varphi\|_{H_1^{2,\lambda}(\partial\Omega)} \le C\sum_{j=0}^{J} 2^{-j} \le C.$$

Estimate (4.32) is then proved.

5. The invertibilty of layer potentials on $L^{2,\lambda}(\partial\Omega)$ and $H^{2,\lambda}(\partial\Omega)$.

Let

$$\mathcal{S}(\mathbf{f})(X) = \int_{\partial\Omega}\Gamma(X-Q)\mathbf{f}(Q)\,dQ, \tag{5.1}$$

$$\mathcal{D}(\mathbf{f})(X) = \int_{\partial\Omega}\frac{\partial}{\partial\nu(Q)}\{\Gamma(X-Q)\}\,\mathbf{f}(Q)\,dQ \tag{5.2}$$

denote the single layer potential and double layer potential respectively with density $\mathbf{f}$. Both $\mathcal{S}(\mathbf{f})$ and $\mathcal{D}(\mathbf{f})$ are solutions of $\mathcal{L}(\mathbf{u}) = 0$ in $\mathbb{R}^n\backslash\partial\Omega$. On $\partial\Omega$, one has

$$\begin{aligned}\frac{\partial}{\partial\nu}\mathcal{S}(\mathbf{f})_\pm &= \left(\pm\frac{1}{2}I + K_\nu\right)\mathbf{f},\\ \mathcal{D}(\mathbf{f})_\pm &= \left(\mp\frac{1}{2}I + K_\nu^*\right)\mathbf{f}.\end{aligned} \tag{5.3}$$

In the last section we show that $(1/2)I+K_\nu$ is invertible on $L_0^p(\partial\Omega)$ and $-(1/2)I+K_\nu$ is invertible on $L^p(\partial\Omega)$ for p close to 2. By duality, $-(1/2)I+K_\nu^*$ is invertible on $L^p(\partial\Omega)$, while $(1/2)I+K_\nu^*$ is invertible from $L_0^p(\partial\Omega)$ to a subspace of $L^p(\partial\Omega)$ of codimention n for p close to 2. In this section we study the invertibility of $\pm(1/2)I+K_\nu$ on $H^{2,\lambda}(\partial\Omega)$. By duality, this will give the invertibilty of $\pm(1/2)I+K_\nu^*$ on $L^{2,\lambda}(\partial\Omega)$.

PROPOSITION 5.4. *Let* $0 \le \lambda < n-1$. *Then* K_ν *is bounded on* $H^{2,\lambda}(\partial\Omega)$. *Consequently,* K_ν^* *is bounded on* $L^{2,\lambda}(\partial\Omega)$.

To prove Proposition 5.4, it suffices to show that if $\mathbf{a}$ is an $(2,\lambda)$ atom, then $K_\nu(\mathbf{a})$ is a $(2,\lambda)$ molecule. This can be done easily by using the boundedness of K_ν on $L^2(\partial\Omega)$ and well-known estimates on $\Gamma(X)$. The same argument also shows that, if $0 \le \lambda < n-1$,

$$\|(\nabla\mathcal{S}(\mathbf{f}))^*\|_{H^{2,\lambda}(\partial\Omega)} \le C\,\|\mathbf{f}\|_{H^{2,\lambda}(\partial\Omega)}.$$

We omit the details.

Define

$$H_0^{2,\lambda}(\partial\Omega) = \left\{ \mathbf{g} \in H^{2,\lambda}(\partial\Omega)\colon \int_{\partial\Omega} \mathbf{g}\, d\sigma = 0 \right\}, \tag{5.5}$$

$$L_0^{2,\lambda}(\partial\Omega) = \left\{ \mathbf{f} \in L^{2,\lambda}(\partial\Omega)\colon \int_{\partial\Omega} \mathbf{f}\, d\sigma = 0 \right\}.$$

THEOREM 5.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$ with connected boundary. Suppose that the operator $\mathcal{L}$ satisfies the strong ellipticity condition (1.13). There exists $\varepsilon > 0$ depending only on n, m, μ and Ω such that, if $0 \leq \lambda < \min(n-1, 2+\varepsilon)$, then $(1/2)I + K_\nu$ is invertible on $H_0^{2,\lambda}(\partial\Omega)$ and $-(1/2)I + K_\nu$ is invertible on $H^{2,\lambda}(\partial\Omega)$. Consequently, $-(1/2)I + K_\nu^*$ is invertible on $L^{2,\lambda}(\partial\Omega)$ and $(1/2)I + K_\nu^*$ is invertible from $L_0^{2,\lambda}(\partial\Omega)$ to a subspace of $L^{2,\lambda}(\partial\Omega)$ of codimension n.*

Proof. We give the proof of the invertibility of $(1/2)I + K_\nu$ on $H_0^{2,\lambda}(\partial\Omega)$. The invertibilty of $-(1/2)I + K_\nu$ on $H^{2,\lambda}(\partial\Omega)$ may be established in a similar manner.

Let $\varepsilon > 0$ be the same as in Theorem 1.12 and $0 \leq \lambda < \min(n-1, 2+\varepsilon)$. We first show that $(1/2)I + K_\nu\colon H_0^{2,\lambda}(\partial\Omega) \to H_0^{2,\lambda}(\partial\Omega)$ is one-to-one. To do this, suppose $\mathbf{g} \in H_0^{2,\lambda}(\partial\Omega)$ and $((1/2)I + K_\nu)\mathbf{g} = 0$. Let $\mathbf{u} = \mathcal{S}(\mathbf{g})$. Then $\frac{\partial \mathbf{u}_+}{\partial\nu} = 0$. Since $(\nabla\mathbf{u})^* \in H^{2,\lambda}(\partial\Omega)$, by the uniqueness result in Theorem 1.12, $\mathbf{u}$ is constant in Ω. It follows that $\mathbf{u} = \mathbf{k}$ is constant on $\partial\Omega$. Note that $\mathbf{u}$ is also a solution in Ω_- of the regularity problem considered in Section 3. Let $(\,\cdot\,)_e^*$ denote the nontangential maximal function with respect to Ω_-. Since $(\nabla\mathbf{u})_e^* \in H^{2,\lambda}(\partial\Omega)$ and $\mathbf{u}$ satisfies condition (3.25), by Remark 3.24, $\|(\nabla\mathbf{u})_e^*\|_2 < \infty$. It follows that

$$\begin{aligned}\int_{\Omega_-} |\nabla\mathbf{u}(X)|^2\, dX &= -\int_{\partial\Omega} \mathbf{u}\cdot\frac{\partial\mathbf{u}_-}{\partial\nu}\, dQ = -\mathbf{k}\cdot\int_{\partial\Omega}\frac{\partial\mathbf{u}_-}{\partial\nu}\, dQ \\ &= \mathbf{k}\cdot\int_{\partial\Omega}\mathbf{g}\, dQ = 0.\end{aligned}$$

Thus $\mathbf{u}$ is constant in Ω_-. We obtain $\mathbf{g} = \frac{\partial\mathbf{u}_+}{\partial\nu} - \frac{\partial\mathbf{u}_-}{\partial\nu} = 0$.

Next, we note that, by Remark 3.24 and (3.23),

$$\begin{aligned}\|\mathbf{g}\|_{H^{2,\lambda}(\partial\Omega)} &\leq \left\|\frac{\partial\mathbf{u}_+}{\partial\nu}\right\|_{H^{2,\lambda}(\partial\Omega)} + \left\|\frac{\partial\mathbf{u}_-}{\partial\nu}\right\|_{H^{2,\lambda}(\partial\Omega)} \\ &\leq \left\|\frac{\partial\mathbf{u}_+}{\partial\nu}\right\|_{H^{2,\lambda}(\partial\Omega)} + C\,\|\mathbf{u}\|_{H_1^{2,\lambda}(\partial\Omega)} \\ &\leq C\left\|\frac{\partial\mathbf{u}_+}{\partial\nu}\right\|_{H^{2,\lambda}(\partial\Omega)} + C\left|\int_{\partial\Omega}\mathbf{u}\, d\sigma\right|\end{aligned}$$

where we used Remark 4.28 in the last inequality. This implies that the range of $(1/2)I + K_\nu$: $H^{2,\lambda}(\partial\Omega) \to H^{2,\lambda}(\partial\Omega)$ is closed.

Finally since $(1/2)I + K_\nu$: $L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$ is onto and $L^2_0(\partial\Omega)$ is clearly dense in $H^{2,\lambda}_0(\partial\Omega)$, we conclude that $(1/2)I + K_\nu$: $H^{2,\lambda}_0(\partial\Omega) \to H^{2,\lambda}_0(\partial\Omega)$ is onto. It follows that $(1/2)I + K_\nu$ is invertible on $H^{2,\lambda}_0(\partial\Omega)$.

COROLLARY 5.7. *Under the same assumption as in Theorem 5.6, there exists* $\varepsilon > 0$ *depending only n, m,* μ *and* Ω *such that:*

(a) *if* $\mathbf{f} \in L^{2,\lambda}(\partial\Omega)$ *and* $0 \le \lambda < \min(n-1, 2+\varepsilon)$*, the unique solution of the Dirichlet problem (1.1) in* Ω *with data* $\mathbf{f}$*, given in Theorem 1.5, may be represented by a double layer potential with density function* $(-(1/2)I + K^*_\nu)^{-1}\mathbf{f}$*,*

(b) *if* $\mathbf{g} \in H^{2,\lambda}_0(\partial\Omega)$ *and* $0 \le \lambda < \min(n-1, 2+\varepsilon)$*, the unique (up to a constant) solution of the Neumann problem (1.11) in* Ω *with data* $\mathbf{g}$*, given in Theorem 1.12, may be represented by a single layer potential with density function* $((1/2)I + K_\nu)^{-1}\mathbf{g}$.

6. The biharmonic equation. Consider the Dirichlet problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \mathbf{N}} = g & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Let

$$WA^p(\partial\Omega) = \text{the closure of } \{(F|_{\partial\Omega}, \nabla F|_{\partial\Omega}) \colon F \in C^\infty(\mathbb{R}^n)\} \text{ under the norm in } W^p_1(\partial\Omega). \tag{6.2}$$

THEOREM 6.3. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n \ge 3$ *with connected boundary. There exists* $\delta > 0$ *depending only on n and the Lipschitz character of* Ω *such that, given* $f \in W^p_1(\partial\Omega)$, $g \in L^p(\partial\Omega)$ *with* $2-\delta < p < 2+\delta$*, there exists a unique u satisfying (6.1) and* $(\nabla u)^* \in L^p(\partial\Omega)$*. In fact, u satisfies*

$$\|(\nabla u)^*\|_p \le C\,\{\|\nabla_{\tan} f\|_p + \|g\|_p\}. \tag{6.4}$$

If, in addition, $(u|_{\partial\Omega}, \nabla u|_{\partial\Omega})$ *(which is completely determined by f and g) is in* $WA^p(\partial\Omega)$*, then*

$$\|(\nabla\nabla u)^*\|_p \le C\|\nabla_{\tan}(\nabla u)\|_p. \tag{6.5}$$

We remark that the Dirichlet problem (6.1) with $(W^p_1(\partial\Omega), L^p(\partial\Omega))$ data for $2-\delta < p < 2+\delta$ was solved in [10], while the regularity estimate (6.5) was established in [33]. In [26, 27], Pipher and Verchota obtain the estimate (6.4) for

p in the optimal range $2-\delta < p \leq \infty$ in the case $n=3$. They also construct examples which show that estimate (6.4) fails in general for $p > 6$ and $n=4$ or $p>4$ and $n \geq 5$.

In this section we establish an estimate in Morrey spaces for the solutions of the Dirichlet problem (6.1).

THEOREM 6.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 4$ with connected boundary. There exists $\varepsilon > 0$ depending only on n and Ω such that, given any $f \in W_1^2(\partial\Omega)$, $g \in L^{2,\lambda}(\partial\Omega)$ with $|\nabla_{\tan} f| \in L^{2,\lambda}(\partial\Omega)$ and $0 \leq \lambda < 2+\varepsilon$, there exists a unique u satisfying (6.1) and $\|(\nabla u)^*\|_{2,\lambda} < \infty$. Moreover, we have*

$$\|(\nabla u)^*\|_{2,\lambda} \leq C \left\{ \|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda} \right\}. \tag{6.7}$$

If $n=3$, estimate (6.7) holds for $0 \leq \lambda \leq 2$.

Remark 6.8. In [26], Pipher and Verchota construct a biharmonic function of form $u = u(r,\theta) = r^\alpha v(\theta)$ on $\mathbb{R}_+ \times \omega$ such that $\nabla \mathbf{u} = 0$ on $\mathbb{R}_+ \times \partial\omega$, where ω is a smooth domain on the unit sphere $\mathbb{S}^{n-1}$. They show that, given any $\delta > 0$, one may find ω and u such that $(1/2) < \alpha = \alpha(\delta) < (1/2)+\delta$ for $n=4$, and $0 < \alpha < \delta$ for $n \geq 5$. Using this example, it is not hard to see that our condition $0 \leq \lambda < 2+\varepsilon$ in Theorem 6.6 is sharp for $n=4$ or 5. Same example also shows that estimate (6.7) can not hold in general for $\lambda > n-3$ and $n \geq 6$.

To prove Theorem 6.6, we follow the same line of arguments as in Section 2. We begin with a lemma on harmonic functions.

LEMMA 6.9. *Let $\varphi \in C^\infty(\mathbb{R}^n)$. Suppose that $\Delta\psi = 0$ in Ω and $\|(\nabla\psi)^*\|_2 < \infty$. Then*

$$\left| \int_{\partial\Omega} \varphi \frac{\partial\psi}{\partial \mathbf{N}}\, dQ \right| \leq C \|\nabla_{\tan}\varphi\|_2 \|\psi\|_2. \tag{6.10}$$

Proof. Using a partition of unity as well as estimate $\|(\psi)^*\|_2 \leq C\|\psi\|_2$ for harmonic functions [5, 6], we may assume that Ω is a star-like bounded Lipschitz domain with respect to the origin. We may also assume that $\int_{\partial\Omega} \varphi\, dQ = 0$ and $\psi(0) = 0$. In this case, estimate (6.10) follows directly from formula (2.14) in [27, p. 384].

Remark 6.11. By continuity, one may extend estimate (6.10) to functions φ, ψ satisfying $\varphi \in W_1^2(\partial\Omega)$, $\Delta\psi = 0$ in Ω and $(\psi)^* \in L^2(\partial\Omega)$.

The next lemma is a Caccioppoli inequality for biharmonic functions.

LEMMA 6.12. *Let $P \in \partial\Omega$ and $0 < \rho < c\,\rho$. Suppose $\Delta^2 u = 0$ in $D(P, 2\rho)$ and $(\nabla\nabla u)^* \in L^2(\Delta(P, 2\rho))$. Also assume that $u = 0$ and $\nabla u = 0$ on $\Delta(P, 2\rho)$. Then, for*

$0 < s < t < 1$,

$$\int_{D(P,s\rho)} |\nabla\nabla u|^2 \, dX \le \frac{C}{(t-s)^4\rho^2} \int_{D(P,t\rho)} |\nabla u|^2 \, dX. \tag{6.13}$$

Proof. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on $D(P, s\rho)$, $\varphi = 0$ on $\Omega \backslash D(P, t\rho)$ and $|\nabla\varphi| \le C/((t-s)\rho)$, $|\nabla\nabla\varphi| \le C/((t-s)\rho)^2$. A direct computation shows that

$$\begin{aligned} D_i(\varphi^2 D_j u) \cdot D_i(\varphi^2 D_j u) \;\le\; & D_i D_j u \cdot D_i D_j(u\varphi^4) \\ & + C\left\{|\nabla(\varphi^2 \nabla u)| + |\nabla u|\,|\nabla\varphi|\right\} \\ & \times \left\{|\nabla u|\,|\nabla\varphi| + |u|\,|\nabla\varphi|^2 + |u|\,|\nabla\nabla\varphi|\right\}. \end{aligned}$$

By Hölder inequality with an ε, we obtain

$$\begin{aligned} c\,|\nabla(\varphi^2\nabla u)| \;\le\; & D_i D_j u \cdot D_i D_j(u\varphi^4) \\ & + C\left\{|\nabla u|^2|\nabla\varphi|^2 + |u|^2|\nabla\varphi|^4 + |u|^2|\nabla\nabla\varphi|^2\right\}. \end{aligned} \tag{6.14}$$

Since $\Delta^2 u = 0$ in $D(P, 2\rho)$, an integration by parts yields, at least formally,

$$\begin{aligned} \int_{D(P,\rho)} D_i D_j u \cdot D_i D_j(u\varphi^4)\, dX \;=\; & \int_{\partial D(P,\rho)} N_i D_i D_j u \cdot D_j(u\varphi^4)\, dQ \\ & - \int_{\partial D(P,\rho)} N_j D_j \Delta u \cdot u\varphi^4 \, dQ \\ =\; & 0. \end{aligned} \tag{6.15}$$

To justify (6.15), we approximate $D(P, \rho)$ by a sequence of smooth domains from inside. Using assumption $(\nabla\nabla u)^* \in L^2(\Delta(P, 2\rho))$, $u = 0$, $\nabla u = 0$ on $\Delta(P, 2\rho)$, it is easy to see that the first boundary integral in (6.15) goes to zero in the approximation. Since Δu is harmonic, we may use Lemma 6.9 to conclude that the second boundary integral in (6.15) also goes to zero in the approximation.

Finally note that, since $u = 0$ on $\Delta(P, 2\rho)$, we have

$$\int_{D(P,t\rho)} |u|^2 \, dX \le C\rho^2 \int_{D(P,t\rho)} |\nabla u|^2 \, dX. \tag{6.16}$$

In view of (6.15)–(6.16), estimate (6.13) follows by integrating (6.14) over $D(P, \rho)$.

The following lemma may be proved by the same arguments as in the proof of Lemma 2.8.

LEMMA 6.17. *Under the same assumption as in Lemma 6.12, we have*

$$\left(\frac{1}{\rho^n}\int_{D(P,\rho/2)} |\nabla u|^2\,dX\right)^{1/2} \le C_p \left(\frac{1}{\rho^n}\int_{D(P,\rho)} |\nabla u|^p\,dX\right)^{1/p}$$

for any $p > 0$.

Let B denote the fundamental solution for the operator Δ^2 in $\mathbb{R}^n$ with pole at 0, i.e., $B(Y) = c_3|Y|$ for $n = 3$, $B(Y) = c_4 \log|Y|$ for $n = 4$, and $B(Y) = c_n|Y|^{4-n}$ for $n \ge 5$. Fix $X \in \Omega$. Let W^X be the solution of the Dirichlet problem (6.1) given by Theorem 6.3 with $f = B^X$, $g = \langle \nabla B^X, \mathbf{N}\rangle$, where $B^X(Y) = B(X - Y)$. Let

$$G^X(Y) = B^X(Y) - W^X(Y) \tag{6.18}$$

be the Green's function for Δ^2 on Ω. For solutions of the Dirichlet problem (6.1) with $(W_1^2(\partial\Omega), L^2(\partial\Omega))$ data and $(\nabla u)^* \in L^2(\partial\Omega)$, we have

$$u(X) = \int_{\partial\Omega} f(Q)\frac{\partial}{\partial \mathbf{N}(Q)}\Delta_Q G^X(Q)\,dQ - \int_{\partial\Omega} g(Q)\Delta_Q G^X(Q)\,dQ. \tag{6.19}$$

(see [27]).

LEMMA 6.20. *Let $X \in \Omega$, $P \in \partial\Omega$ and $\rho = |X - P|$. Suppose $\rho < c\,r_0$ and $\rho \le 2\,\mathrm{dist}\,(X, \partial\Omega)$. Then*

$$\int_{\partial\Omega\setminus\Delta(P,5\rho)} |(\nabla G^X)^*|^p\,dQ \le C\,\rho^{n-1-(n-3)p}$$

where $p \in (2 - \delta, 2 + \delta)$ and δ is the same as in Theorem 6.3.

Proof. We apply estimate (6.4) to G^X on the Lipschitz domain $\Omega\setminus D(P, 3\rho)$. We obtain

$$\begin{aligned}\int_{\partial\Omega\setminus\Delta(P,5\rho)} |(\nabla G^X)^*|^p\,dQ &\le C\int_{\Omega\cap\partial D(P,3\rho)} |\nabla G^X|^p\,dQ\\ &\le C\int_{\Omega\cap\partial D(P,3\rho)} |\nabla B^X|^p + C\int_{\Omega\cap\partial D(P,3\rho)} |\nabla W^X|^p\,dQ.\end{aligned} \tag{6.21}$$

It is easy to see that the integral of $|\nabla B^X|^p$ is bounded by $C\,\rho^{n-1-(n-3)p}$. To handle the integral of $|\nabla W^X|^p$, we choose $q \ge p$ such that $q < 2 + \delta$ and $q > (n - 1)/(n - 2)$. Then

$$\int_{\Omega\cap\partial D(P,3\rho)} |\nabla W^X|^p\,dQ \le C\int_{\Delta(P,4\rho)} |\nabla B^X|^p + C\,\rho^p\int_{\Delta(P,4\rho)} |(\nabla\nabla W^X)^*|^p\,dQ$$

$$
\begin{aligned}
&\le C\,\rho^{n-1-(n-3)p} + C\,\rho^{p+n-1-\frac{p(n-1)}{q}} \left\{ \int_{\Delta(P,4\rho)} |(\nabla\nabla W^X)^*|^q \right\}^{p/q} \\
&\le C\,\rho^{n-1-(n-3)p} + C\,\rho^{p+n-1-\frac{p(n-1)}{q}} \left\{ \int_{\partial\Omega} |\nabla\nabla B^X|^q \, dQ \right\}^{p/q} \\
&\le C\rho^{n-1-(n-3)p},
\end{aligned}
$$

where we have used estimate (6.5) in the third inequality. The proof is complete.

LEMMA 6.22. *Under the same assumption as in Lemma 6.20, we have*

$$\int_{\Delta(P,32\rho)} |\nabla_Q \nabla_Q G^X|^2 \, dQ \le \frac{C}{\rho^{n-3}}, \tag{6.23}$$

$$\int_{\Delta(P,2^j\rho)\setminus\Delta(P,2^{j-1}\rho)} |\nabla_Q \nabla_Q G^X|^2 \, dQ \le \frac{C(2^j)^{-\varepsilon}}{(2^j\rho)^2\rho^{n-5}}, \tag{6.24}$$

$$\int_{\partial\Omega\setminus\Delta(P,cr_0)} |\nabla_Q \nabla_Q G^X|^2 \, dQ \le \frac{C}{\rho^{n-5-\varepsilon}}, \tag{6.25}$$

where $6 \le j \le J$, $2^J \sim c/\rho$, *and* $\varepsilon > 0$ *depends only on n and* δ *in Theorem 6.3.*

Proof. To see (6.23), we choose $q \in (2, 2+\delta)$. By (6.18) and Hölder's inequality, we have

$$
\begin{aligned}
&\int_{\Delta(P,32\rho)} |\nabla\nabla G^X|^2 \, dQ \\
&\qquad \le C \int_{\Delta(P,32\rho)} |\nabla\nabla B^X|^2 \, dQ + C\,\rho^{n-1-\frac{2(n-1)}{q}} \left\{ \int_{\Delta(P,32\rho)} |\nabla\nabla W^X|^q \, dQ \right\}^{2/q} \\
&\qquad \le \frac{C}{\rho^{n-3}} + C\,\rho^{n-1-\frac{2(n-1)}{q}} \left\{ \int_{\partial\Omega} |\nabla\nabla B^X|^q \, dQ \right\}^{2/q} \\
&\qquad \le \frac{C}{\rho^{n-3}},
\end{aligned}
$$

where we have used estimate (6.5) in the second inequality.

To show (6.24) and (6.25), it suffices to prove that, if $\rho \le R \le c\,r_0$ and $|P_0 - P| \ge 32R$, then

$$\int_{\Delta(P_0,R)} |\nabla\nabla G^X|^2 \, dQ \le C \left(\frac{\rho}{R}\right)^{\varepsilon} \cdot \frac{1}{R^2\rho^{n-5}}. \tag{6.26}$$

To this end, we apply estimate (6.5) to G^X in the domain $D(P_0, tR)$ with $t \in (1, 3/2)$. Since $G^X = 0$, $\nabla G^X = 0$ on $\partial\Omega$, we obtain

$$\int_{\Delta(P_0,R)} |\nabla\nabla G^X|^2 \, dQ \le C \int_{\Omega\cap\partial D(P_0,tR)} |\nabla\nabla G^X|^2 \, dQ.$$

We then integrate both sides of the inequality above with respect to t and use Lemma 6.12. This gives

$$\int_{\Delta(P_0,R)} |\nabla\nabla G^X|^2 \, dQ \le \frac{C}{R^3} \int_{D(P_0,2R)} |\nabla G^X(Y)|^2 \, dY. \tag{6.27}$$

Choose $p \in (2-\delta, 2)$. Using Hölder inequality and Sobolev inequality as in the proof of Lemma 2.19, we have

$$\begin{aligned}
&\int_{D(P_0,2R)} |\nabla G^X(Y)|^2 \, dY \\
&\le C R^{\frac{n}{2}-\frac{n}{p}-1} \left\{ \int_{D(P_0,2R)} |\nabla G^X|^p \, dY \right\}^{1/p} \left\{ \int_{D(P_0,2R)} |\nabla G^X|^{\frac{2n}{n-2}} \, dY \right\}^{(n-2)/(2n)} \\
&\le C R^{\frac{n+1}{2}-\frac{n-1}{p}} \left\{ \int_{\partial\Omega\setminus\Delta(P,5\rho)} |(\nabla G^X)^*|^p \, dQ \right\}^{1/p} \left\{ \int_{\partial\Omega\setminus\Delta(P,5\rho)} |(\nabla G^X)^*|^2 \, dQ \right\}^{1/2} \\
&\le C R^{\frac{n+1}{2}-\frac{n-1}{p}} \cdot \rho^{\frac{n-1}{p}-(n-3)} \cdot \rho^{\frac{n-1}{2}-(n-3)} \\
&= C \left(\frac{\rho}{R} \right)^{(n-1)(\frac{1}{p}-\frac{1}{2})} \cdot \frac{R}{\rho^{n-5}},
\end{aligned}$$

where we have used Lemma 6.20 in the last inequality. This, together with (6.27), gives (6.26) with $\varepsilon = (n-1)(\frac{1}{p}-\frac{1}{2}) > 0$. The proof is complete.

Lemma 6.28. *Let* $\mathbf{f} \in W_1^2(\partial\Omega)$. *Suppose* $|\nabla_{\tan} f| \in L^{2,\lambda}(\partial\Omega)$ *where* $0 \le \lambda \le n-1$. *Then, for* $P \in \partial\Omega$ *and* $0 < \rho < c\, r_0$, *we have*

(a) *if* $\lambda < n-3$,

$$\int_{\Delta(P,\rho)} \left| f - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f \right|^2 dQ \le C \, \|\nabla_{\tan} f\|_{2,\lambda}^2 \, \rho^{2+\lambda}, \tag{6.29}$$

(b) *if* $\lambda = n-3$,

$$\int_{\Delta(P,2^j\rho)} \left| f - \frac{1}{|\Delta(P,\rho)|} \int_{\Delta(P,\rho)} f \right|^2 dQ \le C \, \|\nabla_{\tan} f\|_{2,\lambda}^2 (2^j\rho)^{2+\lambda} j^2, \tag{6.30}$$

(c) *if* $\lambda > n-3$,

$$\int_{\Delta(P,2^j\rho)} \left| f - \frac{1}{|\Delta(P,\rho)|}\int_{\Delta(P,\rho)} f \right|^2 dQ \le C\,\|\nabla_{\tan} f\|_{2,\lambda}^2 (2^j\rho)^{2+\lambda}, \tag{6.31}$$

where $0 \le j \le J$ *and* $2^J \sim c/\rho$.

Proof. We may assume that $\|\nabla_{\tan} f\|_{2,\lambda} = 1$ and $\int_{\partial\Omega} f\, dQ = 0$. By Poincaré inequality,

$$\int_{\Delta(P,\rho)} \left| f - \frac{1}{|\Delta(P,\rho)|}\int_{\Delta(P,\rho)} f \right|^2 dQ \le C\rho^2 \int_{\Delta(P,\rho)} |\nabla_{\tan} f|^2\, dQ \le C\rho^{2+\lambda}. \tag{6.32}$$

Thus, to show (6.29), it suffices to estimate

$$f_{\Delta(P,\rho)} = \frac{1}{|\Delta(P,\rho)|}\int_{\Delta(P,\rho)} f(Q)\, dQ.$$

To this end, let $\Delta_j = \Delta(P, 2^j\rho)$. By Hölder and Poincaré inequalities,

$$\begin{aligned} |f_{\Delta_j} - f_{\Delta_{j+1}}| &\le C\left\{\frac{1}{|\Delta_{j+1}|}\int_{\Delta_{j+1}} |f - f_{\Delta_{j+1}}|^2\, dQ\right\}^{1/2} \\ &\le C\left\{(2^j\rho)^{3-n}\int_{\Delta_{j+1}} |\nabla_{\tan} f|^2\, dQ\right\}^{1/2} \\ &\le C\,(2^j\rho)^{\frac{3+\lambda-n}{2}}. \end{aligned} \tag{6.33}$$

Since $\int_{\partial\Omega} f\, dQ = 0$, we obtain by summation that

$$|f_{\Delta(P,\rho)}| \le C\rho^{\frac{3+\lambda-n}{2}} \quad \text{if } 3+\lambda-n < 0.$$

This, together with (6.32), gives (6.29). Estimates (6.30) and (6.31) follow in a similar manner.

Remark 6.34. It follows from (6.33) that, if $0 < \rho < R < c\,r_0$,

$$|f_{\Delta(P,R)} - f_{\Delta(P,\rho)}| \le \begin{cases} C\rho^{\frac{3+\lambda-n}{2}}\, \|\nabla_{\tan} f\|_{2,\lambda}, & \text{if } \lambda < n-3, \\ C\log\left(\dfrac{R}{\rho}\right)\|\nabla_{\tan} f\|_{2,\lambda}, & \text{if } \lambda = n-3, \\ CR^{\frac{3+\lambda-n}{2}}\, \|\nabla_{\tan} f\|_{2,\lambda}, & \text{if } \lambda > n-3. \end{cases}$$

LEMMA 6.35. *Let* $\varepsilon > 0$ *be given by Lemma 6.22. Let* $f \in W_1^2(\partial\Omega)$, $g \in L^{2,\lambda}(\partial\Omega)$ *with* $|\nabla_{\tan} f| \in L^{2,\lambda}(\partial\Omega)$ *where* $0 \le \lambda < 2+\varepsilon$ *for* $n \ge 4$, *and* $0 \le \lambda \le 2$

for $n = 3$. *Then the unique solution* u *of the Dirichlet problem (6.1), given by Theorem 6.3, satisfies*

$$(6.36)\quad |\nabla u(X)| \le C\,\{\operatorname{dist}(X,\partial\Omega)\}^{\frac{\lambda+1-n}{2}}\,\{\|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda}\}\quad \textit{for any } X \in \Omega.$$

Proof. We may assume that $\|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda} = 1$. Also it suffices to consider two cases. In the first case, we assume $f = 0$. In the second case, assume $g = 0$.

Case I. We use the representation formula (6.19) and Lemma 6.22. We obtain

$$|u(X)| \le C\,\{\operatorname{dist}(X,\partial\Omega)\}^{\frac{\lambda+3-n}{2}}\qquad \text{for any } X \in \Omega$$

by the same argument as in the proof of Lemma 2.24. The desired estimate (6.36) then follows by the interior estimates for biharmonic functions.

Case II and $\lambda < n-3$. We may assume $\int_{\partial\Omega} f\,dQ = 0$. Given $X \in \Omega$. Let $\rho = \operatorname{dist}(X,\partial\Omega)$. We may assume that $\rho < c\,r_0$. Let $P \in \partial\Omega$ and $\rho < R < c\,r_0$. Suppose $X \notin D(P,10R)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on $D(P,R/2)$, $\varphi = 0$ on $\Omega\backslash D(P,R)$, and $|\nabla\varphi| \le C/R$. We will show that

$$(6.37)\qquad \left|\int_{\partial\Omega} \varphi f \frac{\partial}{\partial \mathbf{N}(Q)} \Delta_Q G^X(Q)\,dQ\right| \le C\left(\frac{\rho}{R}\right)^{\frac{2+\varepsilon-\lambda}{2}} \rho^{\frac{\lambda+3-n}{2}}$$

where ε is the same as in Lemma 6.22. This, together with (6.19) and a partition of unity, implies that

$$(6.38)\qquad |u(X)| \le C\,\rho^{\frac{\lambda+3-n}{2}} = C\,[\operatorname{dist}(X,\partial\Omega)]^{\frac{\lambda+3-n}{2}}$$

if $0 \le \lambda < 2+\varepsilon$ for $n \ge 4$ and $0 \le \lambda \le 2$ for $n = 3$. Estimate (6.36) follows from (6.38) by the interior estimates as in Case I.

To see (6.37), we use Remark 6.11 to obtain

$$(6.39)\qquad \begin{aligned} \left|\int_{\partial\Omega} \varphi f \frac{\partial}{\partial \mathbf{N}(Q)} \Delta_Q G^X(Q)\,dQ\right| &= \left|\int_{\partial D(P,R)} \varphi u \frac{\partial}{\partial \mathbf{N}(Q)} \Delta_Q G^X(Q)\,dQ\right| \\ &\le C\,\|\nabla_{\tan}(\varphi u)\|_{L^2(\partial D(P,R))} \\ &\quad\times \|\Delta_Q G^X\|_{L^2(\partial D(P,R))}. \end{aligned}$$

In view of (6.29) and $\int_{\partial\Omega} f\,dQ = 0$, we have

$$(6.40)\qquad \|\nabla_{\tan}(\varphi u)\|_{L^2(\partial D(P,R))} = \|\nabla_{\tan}(\varphi f)\|_{L^2(\Delta(P,R))} \le C R^{\lambda/2}.$$

By the proof of Lemma 6.22, we have

$$\|\Delta_Q G^X\|_{L^2(\partial D(P,R))} \le C\left(\frac{\rho}{R}\right)^{\varepsilon/2}\frac{1}{R\,\rho^{\frac{n-5}{2}}}. \tag{6.41}$$

(see the proof of (6.26)). Estimate (6.37) now follows from (6.39)–(6.41).

Case II and $\lambda \ge n-3$. Fix $X \in \Omega$. Let $P \in \partial\Omega$ such that $\rho = |X-P| = \operatorname{dist}(X, \partial\Omega)$. By formula (6.19), for $Y \in \Omega$ and $\alpha \in \mathbb{R}$, we may write

$$u(Y) - \alpha = \int_{\partial\Omega} (f-\alpha)\frac{\partial}{\partial \mathbf{N}(Q)}\Delta_Q G^Y(Q)\,dQ. \tag{6.42}$$

Choose

$$\alpha = \frac{1}{|\Delta(P,\rho)|}\int_{\Delta(P,\rho)} f\,dQ.$$

Using the same argument as in the proof of (6.38), but with (6.29) replaced by (6.30) and (6.31), one may show that, if $Y \in B(X, \rho/4)$,

$$|u(Y)-\alpha| \le C\,\rho^{\frac{\lambda+3-n}{2}}. \tag{6.43}$$

By the interior estimates, this implies that

$$|\nabla u(X)| \le C\,\rho^{\frac{\lambda+1-n}{2}} = C\,[\operatorname{dist}(X,\partial\Omega)]^{\frac{\lambda+1-n}{2}}.$$

The proof of Lemma 6.34 is complete.

Finally we are in a position to give the following:

Proof of Theorem 6.6. The uniqueness follows from Theorem 6.3 since $L^{2,\lambda}(\partial\Omega) \subset L^2(\partial\Omega)$. To establish the existence, we may assume that $\|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda} = 1$. Let u be the solution of (6.1) given by Theorem 6.3. Fix $P \in \partial\Omega$ and $0 < \rho < c\,r_0$. We need to show

$$\int_{\Delta(P,\rho)} |(\nabla u)^*|^2\,dQ \le C\,\rho^{\lambda}. \tag{6.44}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on $D(P, 10\rho)$, $\varphi = 0$ on $\Omega \backslash D(P, 20\rho)$, and $|\nabla\varphi| \le C/\rho$. Let $u = u_1 + u_2 + \beta$ where

$$\beta = \frac{1}{|\Delta(P,20\rho)|}\int_{\Delta(P,20\rho)} f\,dQ$$

and u_2 is the solution of (6.1) with data $u_2 = (f - \beta)\varphi$, $\frac{\partial u_2}{\partial \mathbf{N}} = g\varphi$ on $\partial\Omega$. By estimate (6.4) with $p = 2$,

$$\begin{aligned}\int_{\Delta(P,\rho)} |(\nabla u_2)^*|^2\, dQ &\le \int_{\partial\Omega} |(\nabla u_2)^*|^2\, dQ \le C\left\{\|\nabla_{\tan}((f-\beta)\varphi)\|_2 + \|g\varphi\|_2\right\}\\ &\le C\rho^{\lambda}\end{aligned}$$

where we have used Poincaré inequality in the last inequality.

To estimate $(\nabla u_1)^*$, we note that $u_1 = 0$, $\frac{\partial u_1}{\partial \mathbf{N}} = 0$ on $\Delta(P, 10\rho)$. With Lemmas 6.17 and 6.35 at our disposal, we may use the same argument as in the proof of Theorem 1.5 to obtain

$$\begin{aligned}\int_{\Delta(P,\rho)} |(\nabla u_1)^*|^2\, dQ &\le C\rho^{\lambda}\left\{\|\nabla_{\tan}[(f-\beta)(1-\varphi)]\|_{2,\lambda} + \|g(1-\varphi)\|_{2,\lambda}\right\}^2\\ &\le C\rho^{\lambda}\left\{\|\nabla_{\tan} f\|_{2,\lambda} + \|g\|_{2,\lambda} + \|(f-\beta)\nabla_{\tan}\varphi\|_{2,\lambda}\right\}^2.\end{aligned}$$

Finally, by Remark 6.34, it is not hard to see that $\|(f-\beta)\nabla_{\tan}\varphi\|_{2,\lambda} \le C\,\|\nabla_{\tan} f\|_{2,\lambda}$. Estimate (6.44) is then proved.

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REFERENCES

[0] J. Alvarez, Continuity of Calderón-Zygmund type operators on the predual of a Morrey space, *Clifford Algebras in Analysis and Related Topics, Stud. Adv. Math.*, CRC, Boca Raton, FL, 1996, pp. 309–319.

[1] R. Brown and Z. Shen, Estimates for the Stokes operator in Lipschitz domains, *Indiana Univ. Math. J.* **44** (1995), 1183–1206.

[2] R. Brown, P. Perry, and Z. Shen, On the dimension of the attractor for the non-homogeneous Navier-Stokes equations in non-smooth domains, *Indiana Univ. Math. J.* **49** (2000), 81–112.

[3] R. Coifman, A. McIntosh, and Y. Meyer, L'integrale de Cauchy definit un operateur borne sur L^2 pour les courbes Lipschitziennes, *Ann. of Math.* **116** (1982), 361–388.

[4] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–646.

[5] B. Dahlberg, On estimates for harmonic measure, *Arch. Rat. Mech. Anal.* **65** (1977), 273–288.

[6] ———, On the Poisson integral for Lipschitz and C^1 domains, *Studia Math.* **66** (1979), 13–24.

[7] B. Dahlberg and C. Kenig, Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, *Ann. of Math.* **125** (1987), 437–466.

[8] ———, L^p estimates for the three-dimensional systems of elastostatics on Lipschitz domains, *Lecture Notes in Pure and Appl. Math.*, vol. 122, Dekker, NY, 1990, pp. 621–634.

[9] B. Dahlberg, C. Kenig, J. Pipher, and G. Verchota, Area integral estimates for higher order elliptic equations and systems, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 1425–1461.

[10] B. Dahlberg, C. Kenig, and G. Verchota, The Dirichlet problem for the biharmonic equation in a Lipschitz domain, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 109–135.

[11] ————, Boundary value problems for the systems of elastostatics in Lipschitz domains, *Duke Math. J*. **57** (1988), 795–818.

[12] L. Escauriaza, E. Fabes, and G. Verchota, On a regularity method for weak solutions to transmission problems with internal Lipschitz boundary, *Proc. Amer. Math. Soc*. **115** (1992), 1069–1076.

[13] E. Fabes, Layer potential methods for boundary value problems on Lipschitz domains, *Lecture Notes in Math*., vol. 1344, 1988, Springer-Verlag, NY, pp. 55–80.

[14] E. Fabes, C. Kenig, and G. Verchota, Boundary value problems for the Stokes system on Lipschitz domains, *Duke Math. J*. **57** (1988), 769–793.

[15] E. Fabes and D. Stroock, The L^p integrability of Green's functions and fundamental solutions for elliptic and parabolic equations, *Duke Math. J*. **51** (1984), 977–1016.

[16] C. Fefferman and E. Stein, H^p spaces of several variables, *Acta Math*. **129** (1972), 137–193.

[17] W. Gao, Boundary value problems on Lipschitz domains for general elliptic systems, *J. Funct. Anal*. (1991), 377–399.

[18] M. Giaquinta, *Multiple Integrals in the Calculus of Variation and Nonlinear Elliptic Systems, Ann. of Math. Stud*. vol. 105, Princeton Univ. Press, 1983.

[19] D. Jerison and C. Kenig, The Neumann problem in Lipschitz domains, *Bull. Amer. Math. Soc*. **4** (1981), 203–207.

[20] C. Kenig, Elliptic boundary value problems on Lipschitz domains, *Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud*., vol. 112, 1986, Princeton Univ. Press, pp. 131–183.

[21] ————, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Reg. Conf. Ser. Math*., vol. 83, AMS, Providence, RI, 1994.

[22] D. Mitrea, M. Mitrea, and M. Taylor, Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds, *Mem. Amer. Math. Soc*. **150** (2001).

[23] M. Mitrea, The method of layer potentials in electro-magnetic scattering theory on non-smooth domains, *Duke Math. J*. **77** (1995), 111–133.

[24] J. Necas, *Les Méthodes Directes en Théorie des Equations Elliptique*, Academia, Prague, 1967.

[25] J. Peetre, On the theory of $L^{p,\lambda}$ spaces, *J. Funct. Anal*. **4** (1964), 71–87.

[26] J. Pipher and G. Verchota, The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains, *Amer. J. Math*. **114** (1992), 923–972.

[27] ————, A maximum principle for biharmonic functions in Lipschitz and C^1 domains, *Comment. Math. Helv*. **68** (1993), 385–414.

[28] ————, Dilation invariant estimates and the boundary Garding inequality for higher order elliptic operators, *Ann. of Math*. **142** (1995), 1–38.

[29] Z. Shen, A note on the Dirichlet problem for the Stokes system in Lipschitz domains, *Proc. Amer. Math. Soc*. **123** (1995), 801–811.

[30] ————, Resolvent estimates in L^p for elliptic systems in Lipschitz domains, *J. Funct. Anal*. **133** (1995), 224–251.

[31] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation, *J. Funct. Anal*. **59** (1984), 572–611.

[32] ————, Remarks on second order elliptic systems in Lipschitz domains, *Proc. Centre Math. Anal. Austral. Nat. Univ*., vol. 14, Austral. Nat. Univ., Canberra, 1986, pp. 303–325.

[33] ————, The Dirichlet problem for the polyharmonic equation in Lipschitz domains, *Indiana Univ. Math. J*. **39** (1990), 671–702.

[34] T. Vignati and M. Vignati, Spectral theory and complex interpolation, *J. Funct. Anal*. **80** (1988), 383–397.

[35] C. Zorko, Morrey space, *Proc. Amer. Math. Soc*. **98** (1986), 586–592.