On Absolute Continuity of the Periodic Schrödinger Operators

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1 Introduction

Let $V$ be a real periodic function on $\mathbb{R}^d$; that is,

$$ V(x + e_j) = V(x), \quad j = 1, 2, \ldots, d, $$

for some basis $\{e_j\}_{j=1}^d$ of $\mathbb{R}^d$. We are interested in the spectral properties of the periodic Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^d$.

When $d = 3$, L. Thomas [20] proved that the spectrum of $-\Delta + V$ is purely absolutely continuous if $V \in L^2_{\text{loc}}(\mathbb{R}^3)$. Thomas’s approach plays an important role in the subsequent development. In the book by M. Reed and B. Simon [15], it was used to show that $-\Delta + V$ is absolutely continuous if $V \in L^r_{\text{loc}}(\mathbb{R}^d)$, where $r \geq d - 1$ if $d \geq 4$ and where $r = 2$ if $d = 2$ or $3$. In [4] the basic idea of Thomas was applied to the Dirac operator with periodic potential. Recently, the absolute continuity of the magnetic Schrödinger operator $(-i\nabla - a(x))^2 + V(x)$, with periodic potentials $a$ and $V$, was investigated by R. Hempel and I. Herbst [6], [7], M. Birman and T. Suslina [1], [2], A. Morame [13], and A. Sobolev [16]. In particular, the results in [2], pertaining to the case $a = 0$, give the absolute continuity of $-\Delta + V$ when $d = 2$ and $V \in L^r_{\text{loc}}(\mathbb{R}^2)$ for some $r > 1$. Very recently, Birman and Suslina [3] established the absolute continuity of $-\Delta + V$ for $d = 3$, $V \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$, and for $d \geq 4$, $V \in L^{d-2}_{\text{loc}}(\mathbb{R}^d)$.

In this paper we prove that the spectrum of $-\Delta + V$ is purely absolutely continuous if $d \geq 3$ and $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$. This improves the results in [3] when $d \geq 5$. In the context of
$L^p$ spaces, our result is the best possible in the sense that, with periodic condition (1.1), $L^{d/2}_\text{loc}$ is the largest space for which the self-adjoint operator $-\Delta + V$ may be defined by a quadratic form.

Let $D = -i\nabla$. For a $d \times d$ matrix $A = (a_{jk})$, we use $DAD^T$ to denote $\sum_{j,k} D_j a_{jk} D_k$. The following is the main result of this paper.

**Theorem 1.1.** Let $A = (a_{jk})_{d \times d}$ be a symmetric, positive definite matrix with real constant entries. Let $V$ be a real periodic function on $\mathbb{R}^d$. Suppose $d \geq 3$ and $V \in L^{d/2}_\text{loc} (\mathbb{R}^d)$. Then the spectrum of $DAD^T + V$ is purely absolutely continuous. $\square$

We now describe the main idea of the proof. First note that, by a change of variables, one may assume that $e_j = 2\pi(0, \ldots, 1, \ldots, 0)$ in (1.1); that is, $V$ is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. Let $\Omega = [0, 2\pi)^d$ be a cell of the lattice. Fix $a, b \in \mathbb{R}^d$. For $t \in \mathbb{R}$, we consider the self-adjoint operator

$$H_V(t) = (D + ta + b)A(D + ta + b)^T + V$$

(1.2)

on $L^2(\Omega)$, with periodic boundary conditions. It is known that $H_V(t)$ has a discrete spectrum.

Next, using the Floquet-Bloch decomposition, we reduce the proof of Theorem 1.1 to the problem of showing that none of the eigenvalues $E_j(t)$ of $H_V(t)$ is constant as a function of $t$. See Section 2 for details.

Now, following the Thomas approach, one argues by contradiction. Suppose that, for some $j$, $E_j(t) \equiv E$ is a constant. By the analytic perturbation theory, $E$ is an eigenvalue of $H_V(z)$ for all $z \in \mathbb{C}$. This is impossible if one can prove that

$$\|\{H_V(z)\}^{-1}\|_{L^p(\Omega) \to L^p(\Omega)} \to 0 \quad \text{as} \quad |\text{Im} z| \to \infty$$

(1.3)

for some $p \leq 2$. We remark that, in all previous work, only the case $p = 2$ was considered. Our main estimate in this paper states that, if $V \in L^{d/2}(\Omega)$ and $z = \delta + i\rho$ ($\delta$ depends on $a$ and $b$), then (1.3) holds for $p = 2d/(d + 2)$.

To establish estimate (1.3), we write

$$\{H_V(z)\}^{-1} = \{H_0(z)\}^{-1} \left\{I + V\{H_0(z)\}^{-1}\right\}^{-1}.$$

(1.4)

We show that

$$\|\{H_0(\delta + i\rho)\}^{-1}\|_{L^p(\Omega) \to L^p(\Omega)} \to 0 \quad \text{as} \quad |\rho| \to \infty,$$

(1.5)
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(see Theorem 2.1) and that

\[ \left\| \{ \mathbb{H}_0(\delta + i\rho) \}^{-1} \right\|_{L^p(\Omega) \to L^q(\Omega)} \leq C \]  

(1.6)

with a constant C independent of \( \rho \) when \( |\rho| \geq 1 \), where \( p = 2d/(d+2) \), \( q = p' = 2d/(d-2) \) (see Theorem 2.2). Estimate (1.3) follows since \( V \in L^{d/2}(\Omega) \), (1.5) and (1.6), and the Hölder inequality yield

\[ \left\| V \{ \mathbb{H}_0(\delta + i\rho) \}^{-1} \right\|_{L^p(\Omega) \to L^p(\Omega)} \leq \frac{1}{2} \]  

when \( |\rho| \) is large.  

(1.7)

While the proof of (1.5) is relatively straightforward, the proof of (1.6) is very involved. Note that \( 1/q = (1/p) - (2/d) \) and that (1.6) is equivalent to

\[ \| \psi \|_{L^q(\Omega)} \leq C \left\| \mathbb{H}_0(\delta + i\rho) \psi \right\|_{L^p(\Omega)} \]  

(1.8)

for \( \psi \in C^\infty(\Omega) \). Estimate (1.8) is in fact a uniform Sobolev inequality on the d-torus for the second-order elliptic operator \( \mathbb{H}_0(\delta + i\rho) \). In the setting of \( \mathbb{R}^d \), similar inequalities were established by C. Kenig, A. Ruiz, and C. Sogge [10]. Such estimates play a key role in the study of unique continuation properties of differential operators (see, e.g., [9]). Our estimate (1.8) may be viewed as an analogue on the d-torus. It is well known that the unique continuation properties can be used to eliminate the possibility of certain eigenfunctions. In the study of the absolute continuity of periodic Schrödinger operators, it turns out that the main issue is also to show the absence of eigenvalues in the spectrum (see [7]), although this is not obvious in Thomas’s method. In this regard, the use of uniform Sobolev inequalities seems to be quite natural in our case.

To prove (1.8), we adapt the approach developed in [10] for \( \mathbb{R}^d \). We remark that certain forms of uniform Sobolev inequalities for second-order elliptic operators on compact manifolds were obtained by Sogge [17]. However, it is not clear how to apply directly the results in [17] to our case, although a general result on the spectral projection in [17] is used in the localization process.

With some modifications, the method outlined above also gives the optimal condition for the absolute continuity of \( -\Delta + V \) with periodic potentials in Lorentz spaces. Indeed, assuming \( V \in L^{d/2,\infty}(\Omega) \) where \( \Omega \) is a periodic cell for \( V \), we have the following result.

**Theorem 1.2.** Let \( A = (a_{jk})_{d \times d} \) be a symmetric, positive definite matrix with real constant entries. Let \( V \) be a real periodic function on \( \mathbb{R}^d \), \( d \geq 3 \). Then there exists \( \varepsilon_0 > 0 \)
depending on $d$, the shape of $\Omega$ and $A$, such that, if

$$\limsup_{t \to \infty} t \{ x \in \Omega : |V(x)| > t \}^{2/d} \leq \varepsilon_0,$$  

(1.9)

then the spectrum of $DA^T + V$ is purely absolutely continuous.  

We remark that, if $V \in L^{d/2,q}(\Omega)$ for some $q < \infty$, then $V$ satisfies (1.9) with $\varepsilon_0 = 0$.

To prove Theorem 1.2, we show that

$$\left\| \{ \mathbb{H}_V(z) \}^{-1} \right\|_{L^{p,2}(\Omega) \to L^{p,2}(\Omega)} \to 0 \quad \text{as } |\rho| \to \infty,$$  

(1.10)

in the place of (1.3), where $p = 2d/(d + 2)$ and $z = \delta + i\rho$ as before (see Section 6). We remark that, in the case $d = 3$ or 4, the absolute continuity of $-\Delta + V$ with condition (1.9) for $\varepsilon_0 = 0$ was obtained in [3].

This paper is organized as follows. In Section 2, we provide the details of the Thomas approach that reduces Theorem 1.1 to estimates (1.5) and (1.6). Section 3 contains the proof of estimate (1.5). Sections 4 and 5 are devoted to the proof of the key estimate (1.6). In Section 6 we give the proof of Theorem 1.2.

Throughout the rest of the paper, we assume that $d \geq 3$, $\Omega = [0,2\pi)^d$, and $V \in L^{d/2}(\Omega)$ (except in Section 6, where we assume that $V \in L^{d/2,\infty}(\Omega)$) and is periodic with respect to the lattice $(2\pi\mathbb{Z})^d$. We use $\| \cdot \|_p$ to denote the norm in $L^p(\Omega)$. Finally we use $C$ and $c$ to denote positive constants that may depend on $d$ and the matrix $A$, which are not necessarily the same at each occurrence.

2 The Thomas approach

The materials in Sections 2.1–2.3 are standard. We include them here for the reader’s convenience.

2.1 The definition of $DA^T + V$ on $\mathbb{R}^d$

Let $\Omega = [0,2\pi)^d$. Given $\varepsilon > 0$, since $V \in L^{d/2}(\Omega)$, we may write $V = V_1 + V_2$ such that

$$\|V_1\|_{d/2} \leq \varepsilon \quad \text{and} \quad \|V_2\|_{\infty} \leq C \varepsilon.$$  

(2.1)
Let \( f \in H^1(\mathbb{R}^d) \); that is, \( f \in L^2(\mathbb{R}^d) \) and \( |\nabla f| \in L^2(\mathbb{R}^d) \). By the Sobolev embedding, we have

\[
\int_{\Omega} |V| |f|^2 \, dx \leq \int_{\Omega} |V_1| |f|^2 \, dx + \int_{\Omega} |V_2| |f|^2 \, dx \\
\leq \left\{ \int_{\Omega} |V_1|^{d/2} \, dx \right\}^{d/2} \left\{ \int_{\Omega} |f|^{2d/(d-2)} \, dx \right\}^{(d-2)/d} + C_\varepsilon \int_{\Omega} |f|^2 \, dx
\]

\[
\leq \varepsilon C \int_{\Omega} |\nabla f|^2 \, dx + \int_{\Omega} |f|^2 \, dx + C_\varepsilon \int_{\Omega} |f|^2 \, dx
\]

\[
\leq \varepsilon C \int_{\Omega} |\nabla f|^2 \, dx + (\varepsilon C + C_\varepsilon) \int_{\Omega} |f|^2 \, dx
\]

Since \( V \) is periodic with respect to \((2\pi \mathbb{Z})^d\), it follows from (2.2) by summation that

\[
\int_{\mathbb{R}^d} |V| |f|^2 \, dx \leq \varepsilon C \int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^2 \, dx
\]

(2.3)

for any \( \varepsilon > 0 \) and \( f \in H^1(\mathbb{R}^d) \).

Let \( \langle \cdot, \cdot \rangle \) denote the usual inner product in \( \mathbb{C} \) or in \( \mathbb{C}^d \). We define the quadratic form

\[
q[f, g] = \int_{\mathbb{R}^d} \left\{ \langle (Df)A, Dg \rangle + \langle Vf, g \rangle \right\} \, dx
\]

(2.4)

for \( f, g \in H^1(\mathbb{R}^d) \). Clearly, by (2.3), the symmetric quadratic form \( q \) is semibounded and closed. Thus there exists a unique self-adjoint operator, which we denote by \( DAD^T + V \), such that

\[
q[f, g] = \int_{\mathbb{R}^d} \langle (DAD^T + V)f, g \rangle \, dx
\]

(2.5)

for \( f \in \text{Domain}(DAD^T + V) \) and \( g \in H^1(\mathbb{R}^d) \). Also,

\[
\text{Domain}(DAD^T + V) = \{ f \in H^1(\mathbb{R}^d) : (DAD^T + V)f \in L^2(\mathbb{R}^d) \}.
\]

(2.6)

### 2.2 Definition of \((D + k)A(D + k)^T + V\) on \( \Omega \)

Let

\[
H^1_{\text{per}}(\Omega) = \left\{ \psi \in L^2(\Omega) : \psi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{inx} \text{ and } \sum_{n \in \mathbb{Z}^d} |n|^2 |a_n|^2 < \infty \right\}
\]

(2.7)
be the Sobolev space on $\Omega$ of index one with periodic boundary conditions. For $k \in \mathbb{C}^d$, we introduce the quadratic form

$$q(k)[\phi, \psi] = \int_{\Omega} \left\langle \left( (D + k)\phi \right)A, (D + \bar{k})\psi \right\rangle + \langle V\phi, \psi \rangle \right\rangle \ dx$$  \hspace{1cm} (2.8)$$

for $\phi, \psi \in H^1_{\text{per}}(\Omega)$, where $\bar{k}$ denotes the conjugate of $k$. Using (2.2), we see that the form $q(k)$ is strictly $m$-sectorial. It follows that there exists a unique closed operator, which we denote by $(D + k)A(D + k)^T + V$, such that

$$q(k)[\phi, \psi] = \int_{\Omega} \left\langle \left( (D + k)A(D + k)^T + V \right)\phi, \psi \right\rangle \ dx$$  \hspace{1cm} (2.9)$$

for any $\phi \in \text{Domain}((D + k)A(D + k)^T + V)$ and $\psi \in H^1_{\text{per}}(\Omega)$. Furthermore, we have

$$\text{Domain}((D + k)A(D + k)^T + V) = \{ \phi \in H^1_{\text{per}}(\Omega) : ((D + k)A(D + k)^T + V)\phi \in L^2(\Omega) \}$$
$$= \{ \phi \in H^1_{\text{per}}(\Omega) : (DAD^T + V)\phi \in L^2(\Omega) \}, \hspace{1cm} (2.10)$$

and

$$\{(D + k)A(D + k)^T + V\}^* = (D + \bar{k})A(D + \bar{k})^T + V. \hspace{1cm} (2.11)$$

2.3 The Floquet-Bloch decomposition

Let $Q = [0, 1)^d$, and let

$$\mathcal{H} = \int_{Q} L^2(\Omega) \ dk = \left( Q, L^2(\Omega) \right). \hspace{1cm} (2.12)$$

For $f \in C_0^\infty(\mathbb{R}^d)$, we define

$$Uf(k, x) = \sum_{n \in \mathbb{Z}^d} e^{-i k(x + 2\pi n)} f(x + 2\pi n), \hspace{0.5cm} x \in \Omega, \ k \in Q. \hspace{1cm} (2.13)$$

It is easy to check that

$$\|Uf\|_{\mathcal{H}} = \|f\|_{L^2(\mathbb{R}^d)}. \hspace{1cm} (2.14)$$

Furthermore, one may extend $U$ to a unitary operator from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ by continuity.
A direct computation shows, for \( f, g \in C_0^\infty(\mathbb{R}^d) \), that

\[
q[f, g] = \int_{\Omega} q(k) [Uf(k, \cdot), Ug(k, \cdot)] \, dk.
\] (2.15)

In particular,

\[
\int_{\mathbb{R}^d} |\nabla f|^2 \, dx = \int_{\mathbb{Q}} \int_{\Omega} |(D + k)Uf|^2 \, dx \, dk.
\] (2.16)

**Proposition 2.1.** We have that \( U : H^1(\mathbb{R}^d) \rightarrow \int_{\mathbb{Q}} H^1_{\text{per}}(\Omega) \, dk \equiv L^2(\mathbb{Q}, H^1_{\text{per}}(\Omega)) \) is unitary. \( \square \)

**Proof.** It follows easily from (2.14) and (2.16) by a standard approximation argument that

\[
U(H^1(\mathbb{R}^d)) \subset \int_{\mathbb{Q}} H^1_{\text{per}}(\Omega) \, dk.
\]

To show that the operator \( U \) is surjective, suppose that there exists \( \eta \in \int_{\mathbb{Q}} H^1_{\text{per}}(\Omega) \, dk \) such that

\[
\int_{\mathbb{Q}} \int_{\Omega} \left\{ \langle Duf, D\eta \rangle + \langle Uf, \eta \rangle \right\} \, dx \, dk = 0
\]

for any \( f \in C_0^\infty(\mathbb{R}^d) \). Let \( g \in C_0^\infty(\Omega) \), and let \( f(x) = g(x - 2\pi n_0) \) for some \( n_0 \in \mathbb{Z}^d \). Then

\[
Uf(k, x) = e^{-ik(x+2\pi n_0)} g(x)
\]

and

\[
Duf(k, x) = e^{-ik(x+2\pi n_0)} (D - k)g(x).
\]

Hence,

\[
\int_{\mathbb{Q}} e^{-2\pi i n_0 \cdot k} \int_{\Omega} e^{-ikx} \left\{ \langle (D - k)g, D\eta \rangle + \langle g, \eta \rangle \right\} \, dx \, dk = 0.
\]

Since \( n_0 \in \mathbb{Z}^d \) is arbitrary, we have
Finally, since \( g \in C_0^\infty(\Omega) \) is arbitrary, we conclude that

\[
e^{ikx}(-\Delta + 1)\eta = 0 \quad \text{in } \Omega
\]

(in the sense of distributions). Thus \( \eta = 0 \), and the proof is complete. \( \blacksquare \)

Remark 2.1. It follows from Proposition 2.1 that (2.15) holds for any \( f, g \in H^1(\mathbb{R}^d) \).

**Proposition 2.2.** We have the following Floquet-Bloch decomposition of \( DAD^T + V \) in a direct integral:

\[
U(DAD^T + V)U^{-1} = \int_Q \{ (D + k)A(D + k)^T + V \} \, dk.
\]

That is,

\[
\text{Domain} \left( U(DAD^T + V)U^{-1} \right) = \text{Domain} \left( \int_Q \{ (D + k)A(D + k)^T + V \} \, dk \right)
\]

\[
= \left\{ \phi \in \mathcal{H} : \phi(k, \cdot) \in \text{Domain} \left( (D + k)A(D + k)^T + V \right) \text{ for a.e. } k \in Q \text{ and } \int_Q \| \{ (D + k)A(D + k)^T + V \} \phi \|^2 \, dk < \infty \right\},
\]

and, for any \( \phi \in \text{Domain}(U(DAD^T + V)U^{-1}) \),

\[
U(DAD^T + V)U^{-1} \phi(k, x) = \{ (D + k)A(D + k)^T + V \} \phi(k, x).
\]

**Proof.** First note that, by (2.6),
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Domain \((\mathcal{U}(\mathcal{D}\mathcal{A}\mathcal{D}^T + V)\mathcal{U}^{-1})\)
\[= \left\{ \phi \in \mathcal{H} : U^{-1} \phi \in \text{Domain} \left( \mathcal{D}\mathcal{A}\mathcal{D}^T + V \right) \right\} \]
\[= \left\{ \phi \in \mathcal{H} : U^{-1} \phi \in H^1(\mathbb{R}^d) \text{ and } (\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1} \phi \in L^2(\mathbb{R}^d) \right\} \]
\[= \left\{ \phi \in \bigoplus_Q H^1_{\text{per}}(\Omega) \text{ d}k : (\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1} \phi \in L^2(\mathbb{R}^d) \right\}, \]
where we have used Proposition 2.1 in the last equality.

Now let \(\phi \in \text{Domain}(\mathcal{U}(\mathcal{D}\mathcal{A}\mathcal{D}^T + V)\mathcal{U}^{-1})\). By Remark 2.1, for any \(\psi \in \bigoplus_Q H^1_{\text{per}}(\Omega)\),
\[\int_Q \int_{\Omega} \left\langle \left( (\mathcal{D} + k)\phi \right) \Lambda (\mathcal{D} + k)\psi + \langle V\phi, \psi \rangle \right\rangle \text{ d}x \text{ d}k \]
\[= \int_Q q(k) \left[ \phi(\cdot, k), \psi(\cdot, k) \right] \text{ d}k \]
\[= q \left[ U^{-1} \phi, U^{-1} \psi \right] \]
\[= \int_{\mathbb{R}^d} \langle (\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1} \phi, U^{-1} \psi \rangle \text{ d}x \]
\[= \int_Q \int_{\Omega} \langle \mathcal{U}(\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1} \phi, \psi \rangle \text{ d}x \text{ d}k. \]

Since \(\bigoplus_Q H^1_{\text{per}}(\Omega)\) is dense in \(\mathcal{H}\), we obtain
\[\left\{ (\mathcal{D} + k)\Lambda (\mathcal{D} + k)^T + V \right\} \phi = \mathcal{U}(\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1} \phi \in \mathcal{H}.\]

This shows that the domain of \(\mathcal{U}(\mathcal{D}\mathcal{A}\mathcal{D}^T + V)U^{-1}\) is contained in the domain of \(\int_Q [(\mathcal{D} + k)\Lambda (\mathcal{D} + k)^T + V] \text{ d}k\). The other side of the inclusion can be shown in the same manner.

Choose \(a \in \mathbb{R}^d\) such that
\[|a| = 1, \quad aA = (s_0, 0, \ldots, 0), \quad \text{and } s_0 > 0. \tag{2.17} \]

Then any \(x \in \mathbb{R}^d\) can be written as \(x = ta + b\) with \((a, b) = 0\). Note that, if \(x \in Q = [0, 1)^d\), then \(|x|^2 = t^2 + |b|^2 \leq d\). Hence, \(|t| \leq \sqrt{d}\) and \(|b| \leq \sqrt{d}\).

Next let
\[L = \{ b \in \mathbb{R}^d : |b| \leq \sqrt{d} \text{ and } (b, a) = 0 \}. \tag{2.18} \]

For each \(b \in L\), we define
\[J(b) = \{ t \in \mathbb{R} : ta + b \in Q \}. \tag{2.19} \]
Since $Q$ is convex, $J(b)$ must be an interval of $\mathbb{R}$, if not empty.

We may now write

\[
U(DAD^T + V)U^{-1} = \int_Q \{ (D + k)A(D + k)^T + V \} \, dk = \int_{b \in L} \left\{ \int_{t \in J(b)} \left[ (D + ta + b)A(D + ta + b)^T + V \right] \, dt \right\} \, db.
\]

By [15, Theorem XIII.85, p. 284], we have the following proposition.

**Proposition 2.3.** If, for each $b \in L$,

\[
\int_{t \in J(b)} \left[ (D + ta + b)A(D + ta + b)^T + V \right] \, dt
\]

has purely absolutely continuous spectrum, then so does $DAD^T + V$. \[\square\]

### 2.4 The Thomas approach

Fix $b \in L$. We consider the family of operators

\[
\mathbb{H}_V(z) = (D + za + b)A(D + za + b)^T + V, \quad z \in \mathbb{C},
\]

defined by the quadratic form (2.9). Note that, by (2.10), the domain of $\mathbb{H}_V(z)$ is independent of $z$. Thus it is easy to check that $\{\mathbb{H}_V(z) : z \in \mathbb{C}\}$ is an analytic family of type $(A)$ (see [8, p. 375] for definition). Moreover, by (2.11), the family is self-adjoint in the sense that $\mathbb{H}_V(z)^* = \mathbb{H}_V(z)$.

The resolvent of $\mathbb{H}_V(z)$ is not empty for any $z \in \mathbb{C}$. This follows easily from (2.2). Also, since $\text{Domain}(\mathbb{H}_V(z)) \subset H^1_{\text{per}}(\Omega)$, $\mathbb{H}_V(z)$ has compact resolvent. Now consider the family of self-adjoint operators $\{\mathbb{H}_V(t) : t \in \mathbb{R}\}$. By the analytic perturbation theory (see [8, Theorem 3.9, p. 392]), there exist real analytic functions $E_j(t)$ and corresponding $L^2(\Omega)$-valued functions $\eta_j(t)$ so that, for each $t \in \mathbb{R}$, $[\eta_j(t)]_{j=1}^\infty$ is an orthonormal basis for $L^2(\Omega)$ and

\[
\mathbb{H}_V(t)\eta_j(t) = E_j(t)\eta_j(t).
\]

**Proposition 2.4.** If, as a function of $t \in \mathbb{R}$, no $E_j(t)$ is constant, then $DAD^T + V$ has purely absolutely continuous spectrum. \[\square\]
Proposition 2.4 follows directly from Proposition 2.3 and [15, Theorem XIII.86 p. 286].

Let \( a = (a_1, \ldots, a_d) \). For \( b = (b_1, \ldots, b_d) \), choose

\[
\delta = \frac{1}{a_1} \left( \frac{1}{2} - b_1 \right).
\]

(2.22)

Note that, since \( a_1 s_0 = \langle a A, a \rangle \) and \( |a| = 1, a_1 \neq 0 \).

We now state the main estimates of this paper.

**Theorem 2.1.** Let \( \delta \) be given by (2.22). Then, for \( \rho \in \mathbb{R}, |\rho| \geq 1 \),

(i) \( \| \{ \mathbb{H}_0 (\delta + i\rho) \}^{-1} \|_{L^p(\Omega) \to L^2(\Omega)} \to 0 \) as \( |\rho| \to \infty \), where \( 1 \leq p \leq 2 \) if \( d = 3 \) and \( 2(d-2)/d < p \leq 2 \) if \( d \geq 4 \);

(ii) there exists \( C > 0 \) depending only on the matrix \( A \) and \( d \) such that, for \( |\rho| \geq 1 \),

\[
\| \{ \mathbb{H}_0 (\delta + i\rho) \}^{-1} \|_{L^p(\Omega) \to L^q(\Omega)} \leq C.
\]

where \( 1 \leq p \leq 2 \) if \( d = 3 \), \( 1 < p \leq 2 \) if \( d = 4 \), and \( 2(d-2)/d \leq p \leq 2 \) if \( d \geq 5 \). \( \square \)

**Theorem 2.2.** Let \( p = 2d/(d+2) \), and let \( q = 2d/(d-2) \). Then there exists \( C > 0 \) depending on \( A \) and \( d \) such that, for \( |\rho| \geq 1 \),

\[
\| \{ \mathbb{H}_0 (\delta + i\rho) \}^{-1} \|_{L^p(\Omega) \to L^q(\Omega)} \leq C.
\]

\( \square \)

Finally in this section, assuming Theorems 2.1 and 2.2, we give the following proof.

**Proof of Theorem 1.1.** In view of Proposition 2.4, it suffices to show that \( E_j(t) \) is not a constant function. We argue by contradiction.

Suppose that \( E_j(t) = E \) is a constant for some \( j \). By the analytic perturbation theory (see [8, Theorem 1.10, p. 371]), \( E \) is an eigenvalue of \( \mathbb{H}_V(z) \) for any \( z \in \mathbb{C} \). In particular, for \( \rho \in \mathbb{R} \), there exists \( \phi_\rho \in \text{Domain}(\mathbb{H}_V(\delta + i\rho)) \) such that \( \|\phi_\rho\|_2 = 1 \) and

\[
\mathbb{H}_V(\delta + i\rho)\phi_\rho = E\phi_\rho.
\]

(2.23)

Since \( \phi_\rho \in \mathbb{H}^1_{\text{per}}(\Omega) \), by the Sobolev embedding, \( \phi_\rho \in L^q(\Omega) \), where \( q = 2d/(d-2) \).

Let \( p = 2d/(d+2) \). Since \( (1/p) = (2/d) + (1/q) \), by Hölder’s inequality, \( V\phi_\rho \in L^p(\Omega) \). This implies that

\[
\mathbb{H}_0(\delta + i\rho)\phi_\rho = \mathbb{H}_V(\delta + i\rho)\phi_\rho - V\phi_\rho \in L^p(\Omega).
\]

(2.24)
We remark that $H_0(\delta + ip)\phi_\rho$ may be defined initially as an element in the dual space of $H^1_{\text{per}}(\Omega)$.

We claim that, for $|\rho|$ sufficiently large,

$$
\| V\{ H_0(\delta + ip)\}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \leq \frac{1}{2},
$$

(2.25)

Assume (2.25) for a moment. It is well known that (2.25) implies that $I + VH_0(\delta + ip)^{-1} : L^p(\Omega) \to L^p(\Omega)$ is invertible and that

$$
\| \{ I + V\{ H_0(\delta + ip)\}^{-1} \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \leq 2 \quad \text{for } |\rho| \text{ large.}
$$

(2.26)

Since

$$
\{ I + V\{ H_0(\delta + ip)\}^{-1} \} H_0(\delta + ip) \phi_\rho = (H_0(\delta + ip) + V) \phi_\rho = E \phi_\rho,
$$

in view of (2.24), we have

$$
H_0(\delta + ip) \phi_\rho = E\{ I + V\{ H_0(\delta + ip)\}^{-1} \}^{-1} \phi_\rho.
$$

It follows that, for $|\rho|$ sufficiently large,

$$
\phi_\rho = E H_0(\delta + ip)^{-1} \{ I + VH_0(\delta + ip)^{-1} \}^{-1} \phi_\rho.
$$

(2.27)

Thus, by (2.26),

$$
\| \phi_\rho \|_p \leq 2|E| \| H_0(\delta + ip)^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \| \phi_\rho \|_p
$$

or

$$
1 \leq 2|E| \| H_0(\delta + ip)^{-1} \|_{L^p(\Omega) \to L^p(\Omega)}.
$$

This is impossible since, by Theorem 2.1,

$$
\| H_0(\delta + ip)^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \leq 2\pi \| H_0(\delta + ip)^{-1} \|_{L^p(\Omega) \to L^2(\Omega)} \to 0
$$

as $|\rho| \to \infty$.

It remains to prove claim (2.25). For any $\varepsilon > 0$, we write $V = V_1 + V_2$ so that

$$
\| V_1 \|_{d/2} \leq \varepsilon \quad \text{and} \quad \| V_2 \|_{\infty} \leq C_\varepsilon.
$$
Then
\[
\| V \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \\
\leq \| V_1 \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} + \| V_2 \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \\
\leq \varepsilon \| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} + C \varepsilon \| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \\
\leq \varepsilon C + C \varepsilon \| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)}. 
\]

Choose \( \varepsilon \) small so that \( \varepsilon C \leq 1/4 \). Then we let \( |\rho| \) be so large that
\[
C \varepsilon \| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^p(\Omega) \to L^p(\Omega)} \leq \frac{1}{4}. 
\]

The claim is proved. \( \blacksquare \)

The next three sections are devoted to the proofs of Theorems 2.1 and 2.2. We point out that Theorem 2.1 alone would give the absolute continuity of \(-\Delta + V\) when \( d = 4, V \in L^r(\Omega) \) for some \( r > 2 \), and \( d \geq 5, V \in L^{d-2}(\Omega) \). In the case \( d \geq 5 \), this is the result obtained recently by Birman and Suslina [3].

3 The proof of Theorem 2.1

Let \( \psi \in L^2(\Omega) \); then
\[
\psi(x) = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n)e^{inx}, 
\]
where
\[
\hat{\psi}(n) = \frac{1}{(2\pi)^d} \int_{\Omega} e^{-iun} \psi(y) \, dy. 
\]

Note that
\[
\mathbb{H}_0(k)\psi(x) = \sum_{n \in \mathbb{Z}^d} (n + k)A(n + k)^T\hat{\psi}(n)e^{inx},
\]
where \( (n + k)^T \) denotes the transpose of the \( 1 \times d \) vector \( n + k \). Hence, at least formally,
\[
\{ \mathbb{H}_0(k) \}^{-1}\psi(x) = \sum_{n \in \mathbb{Z}^d} \frac{\hat{\psi}(n)e^{inx}}{(n + k)A(n + k)^T}. 
\]
Let
\[
\mathbf{k} = (\delta + i\rho)\mathbf{a} + \mathbf{b},
\] (3.4)
where \(\mathbf{a}\) is fixed in (2.17), \(\mathbf{b} \in \mathbb{L}\), and \(\delta\) is given by (2.22). Then
\[
(n + \mathbf{k})A(n + \mathbf{k})^T = (n + \mathbf{b})A(n + \mathbf{b})^T + 2\delta(n + \mathbf{b})A\mathbf{a}^T + (\delta^2 - \rho^2)A\mathbf{a}^T + 2i\rho(n + \mathbf{b} + \delta\mathbf{a})A\mathbf{a}^T
\]
where we have used \(A\mathbf{a} = (s_0, \ldots, 0)\) (see (2.17)) in the last equality.

Note that, by (2.22), \(b_1 + \delta a_1 = 1/2\). Thus
\[
(n + \mathbf{k})A(n + \mathbf{k})^T = (n + \mathbf{b})A(n + \mathbf{b})^T + 2\delta(n_1 + b_1)s_0 + (\delta^2 - \rho^2)a_1 s_0 + 2i\rho(n_1 + b_1 + \delta a_1)s_0,
\] (3.5)

Since \(n_1\) is an integer, \(|n_1 + 1/2| \geq 1/2\). It follows that
\[
|\text{Im}(n + \mathbf{k})A(n + \mathbf{k})^T| \geq |\rho|s_0.
\] (3.6)

Also, it is not hard to see that, if \(|\rho| \geq 1, n \in \mathbb{Z}^d\), then
\[
|\mathbf{k}| \approx |(n + \mathbf{b})A(n + \mathbf{b})^T - \rho^2 a_1 s_0| + \rho \left| n_1 + \frac{1}{2} \right|.
\] (3.7)

Now, for \(\xi \in \mathbb{C}\), we consider the family of operators,
\[
T_\xi \psi(x) = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n)e^{inx} \left\{ (n + \mathbf{k})A(n + \mathbf{k})^T \right\}^\xi.
\] (3.8)

**Lemma 3.1.** Suppose \(\text{Re } \xi \geq 0\); then
\[
\left\| T_\xi \psi \right\|_2 \leq C e^{|\text{Im } \xi|} |s_0|^{|\text{Re } \xi|} \left\| \psi \right\|_2.
\]
□

**Proof.** The lemma follows directly from the estimate
\[
\left\| (n + \mathbf{k})A(n + \mathbf{k})^T \right\|_2 \geq |s_0| e^{-\pi |\text{Im } \xi|} \text{ for } \text{Re } \xi \geq 0.
\] (3.9)
Lemma 3.2. Suppose \( \text{Re} \xi = \gamma \), where \( \gamma = 1 \) if \( d = 3 \), \( \gamma > 1 \) if \( d = 4 \), and \( \gamma \geq (d/2) - 1 \) if \( d \geq 5 \). Then

\[
\| T_\xi \psi \|_2 \leq C(\rho, \gamma) e^{\pi |\text{Im} \xi|} \| \psi \|_1
\]

with \( C(\rho, \gamma) \) bounded for \( |\rho| \geq 1 \). Furthermore, \( C(\rho, \gamma) \to 0 \) as \( |\rho| \to \infty \) for \( d = 3, \gamma = 1 \), and for \( d \geq 4, \gamma > (d/2) - 1 \).

Assuming Lemma 3.2, we now give the following proof.

Proof of Theorem 2.1. Note that \( \{ \mathbb{H}_0(\delta + i \rho) \}^{-1} = T_1 \). In the case \( d = 3 \), Theorem 2.1 follows directly from Lemma 3.2.

For \( d \geq 4 \), we apply the Stein interpolation theorem (see [18, Theorem 4.1, p. 205]) with estimates in Lemmas 3.1 (\( \text{Re} \xi = 0 \)) and 3.2. We obtain

\[
\| \{ \mathbb{H}_0(\delta + i \rho) \}^{-1} \psi \|_2 \leq \tilde{C}(\rho) \| \psi \|_p,
\]

where \( \tilde{C}(\rho) \leq C[C(\rho, \gamma)]^M \) for some \( M > 0 \) and

\[
\frac{1}{p} = \frac{t}{2} + \frac{1-t}{1}, \quad 1 = t \cdot 0 + (1-t) \cdot \gamma.
\]

(3.10)

From (3.10), we have \( p = 2\gamma/(\gamma + 1) \). Hence \( p = 2(d-2)/d \) if \( \gamma = (d/2) - 1 \). An inspection of Lemma 3.2 now yields Theorem 2.1.

Finally in this section we give the proof of Lemma 3.2, which completes the proof of Theorem 2.1.

Proof of Lemma 3.2. Note that, for \( \text{Re} \xi = \gamma > 0 \),

\[
\| T_\xi \psi \|_2 = (2\pi)^d \sum_{n \in \mathbb{Z}^d} \frac{|\hat{\psi}(n)|^2}{\left| (n + k)A(n + k)^T \xi \right|^2}
\]

\[
\leq (2\pi)^d e^{\pi |\text{Im} \xi|} \sum_{n \in \mathbb{Z}^d} \frac{|\hat{\psi}(n)|^2}{\left| (n + k)A(n + k)^T \right|^2} \gamma
\]

\[
\leq (2\pi)^d e^{\pi |\text{Im} \xi|} \| \psi \|_1 \sum_{n \in \mathbb{Z}^d} \frac{1}{\left| (n + k)A(n + k)^T \right|^2} \gamma
\]

where we use \( |\hat{\psi}(n)| \leq (2\pi)^{-d} \| \psi \|_1 \). Thus, by (3.7), we need to estimate

\[
\sum_{n \in \mathbb{Z}^d} \frac{1}{\left| (n + b)A(n + b)^T - \rho^2 a_1 s_0 \right| + \left| \rho \left( n_1 + \frac{1}{2} \right) \right|^2} \gamma.
\]

(3.11)
We may assume \( \rho \geq 1 \).

Let \( B \) be a \( d \times d \) symmetric, positive definite matrix such that \( A = B^2 \); that is, \( B = \sqrt{A} \). Then, for \( \rho \geq 1 \),

\[
| (n + b) A (n + b)^T - \rho^2 a_1 s_0 | = | (n + b) B |^2 - \rho^2 a_1 s_0 |
= \{ | (n + b) B | + \rho \sqrt{a_1 s_0} \} | (n + b) B | - \rho \sqrt{a_1 s_0}
\approx (\rho + |n|) | (n + b) B | - \rho \sqrt{a_1 s_0}.
\]

Thus, (3.11) is bounded by \( C(I_1 + I_2) \), where

\[
I_1 = \sum_{n \in \mathbb{Z}^d \atop |n| \geq C \rho} \frac{1}{|n|^{4\gamma}},
\]

and

\[
I_2 = \sum_{n \in \mathbb{Z}^d \atop |n| < C \rho} \frac{1}{\rho^{2\gamma}} \left\{ | (n + b) B | - \rho \sqrt{a_1 s_0} \right\}^{2\gamma}.
\]

Clearly,

\[
I_1 \leq C \int_{C \rho}^{\infty} \frac{r^{d-1}}{r^{4\gamma}} \, dr \leq \frac{C \gamma}{\rho^{4\gamma - d}}, \tag{3.12}
\]

where we assume that \( \gamma > d/4 \).

To estimate \( I_2 \), note that, if \( x \in \mathbb{R}^d \) and \( |x - n| \leq 1/4 \), then

\[
| |x| - \rho \sqrt{a_1 s_0} | + |x_1| + 1 \leq C \left\{ | (n + b) B | - \rho \sqrt{a_1 s_0} \right\},
\]

where we use the fact that \( |b| \leq \sqrt{d} \) and \( |n_1 + 1/2| \geq 1/2 \). It follows that

\[
I_2 \leq \frac{C}{\rho^{2\gamma}} \int_{|x| \leq C \rho} \left\{ |x| - \rho \sqrt{a_1 s_0} + |x_1| + 1 \right\}^{2\gamma} dx
\leq \frac{C}{\rho^{2\gamma}} \int_{|y| \leq C \rho} \left\{ |y| - \rho \sqrt{a_1 s_0} + |(B^{-1} y)_1| + 1 \right\}^{2\gamma} dy
\leq \frac{C}{\rho^{2\gamma}} \int_{|x| \leq C \rho} \left\{ |x| - \rho \sqrt{a_1 s_0} + |x_1| + 1 \right\}^{2\gamma} dx,
\]

where the last inequality follows by a rotation. A dilation now gives

\[
I_2 \leq \frac{C}{\rho^{4\gamma - d}} \int_{|x| \leq C \rho} \left\{ |x| - \sqrt{a_1 s_0} + |x_1| + \frac{1}{\rho} \right\}^{2\gamma} dx.
\]
To take advantage of the term $|x_1|$ in the integrand, we use polar coordinates in $\mathbb{R}^d$ with $x_1 = r \cos \theta$. We obtain

$$I_2 \leq \frac{C}{\rho^{d/2}} \int_0^C \int_0^{\pi/2} r^{d-1} \, \frac{(\sin \theta)^{d-2} \, d\theta}{\left\{|r - \sqrt{a_1\rho^2} + r \cos \theta + \frac{1}{\rho}\right\}^{2\gamma}}$$

$$\leq \frac{C_\gamma}{\rho^{d/2}} \int_0^C \int_0^{\pi/2} r^{d-2} \, \frac{dr}{\left\{|r - \sqrt{a_1\rho^2} + \frac{1}{\rho}\right\}^{2\gamma-1}}$$

$$\leq \frac{C_\gamma}{\rho^{d/2}} \int_0^C \int_0^{\pi/2} \frac{dr}{\left\{|r + \frac{1}{\rho}\right\}^{2\gamma-1}}$$

$$\leq \frac{C_\gamma}{\rho^{d/2}} + \frac{C_\gamma}{\rho^{d/2}} \int_{1/\rho}^C \frac{dr}{\rho^{d/2-1}}.$$

This, together with (3.12), shows that, if $\gamma > d/4$, then the sum (3.11) is bounded by

$$\frac{C_\gamma}{\rho^{d/2}} + \frac{C_\gamma}{\rho^{d/2}} \int_{1/\rho}^C \frac{dr}{\rho^{d/2-1}}.$$

Thus, if $d = 3$ and $\gamma = 1$, then

$$\|T_\xi \psi\|_2 \leq C e^{\pi |\text{Im} \xi| |\rho|^{1/2}} \left(1 + \log |\rho|\right)^{1/2} \|\psi\|_1,$$

and

$$\|T_\xi \psi\|_2 \leq C_\gamma e^{\pi |\text{Im} \xi| |\rho|^{\gamma-1}} \|\psi\|_1 \text{ if } d = 4, \text{ Re } \xi = \gamma > 1,$$

$$\|T_\xi \psi\|_2 \leq C_\gamma e^{\pi |\text{Im} \xi| |\rho|^{\gamma-d/2+1}} \|\psi\|_1 \text{ if } d \geq 5, \text{ Re } \xi = \gamma \geq \frac{d}{2} - 1.$$

The lemma is proved. \(\square\)

### 4 The uniform Sobolev inequalities

The goal of this section is to establish the following uniform Sobolev inequalities.

**Theorem 4.1.** Let $p = 2d/(d+2)$, $q = 2d/(d-2)$, and $b \in \mathbb{R}^d$. Let $z \in \mathbb{C}$ such that $\text{Im} \, z \neq 0$ and $\text{Re} \, \sqrt{z} \geq c_0 > 0$; then

$$\|\{-(D + b)A(D + b)^T + z\}^{-1} \psi\|_q \leq C \|\psi\|_p,$$

with a constant $C$ independent of $b$ and $z$. \(\square\)
In the next section we deduce Theorem 2.2 from Theorem 4.1 by a localization argument. We remark that our proofs of Theorems 2.2 and 4.1 are based on an approach developed in [10] for similar estimates in $\mathbb{R}^d$.

For $\xi \in \mathbb{C}$, we consider the family of operators

$$S_\xi \psi(x) = \left(\xi - \frac{d}{2}\right) \sum_{n \in \mathbb{Z}^d} \frac{\hat{\psi}(n)e^{inx}}{|(n + b)B| + z} \xi,$$

where $B = \sqrt{A}$, $b \in \mathbb{R}^d$, and $z \in \mathbb{C}$ satisfies the conditions in Theorem 4.1. Clearly,

$$S_1 = \left(1 - \frac{d}{2}\right)\{(D + b)A(D + b)^T + z\}^{-1}.$$  \hspace{1cm} (4.2)

We begin with an $L^2$-$L^2$ estimate.

**Lemma 4.1.** Suppose $\text{Re} \xi \geq 0$; then

$$\|S_\xi \psi\|_{L^2(\Omega)} \leq C(|\xi| + 1)e^{\pi|\text{Im} \xi|} \text{Re} \xi \|\psi\|_{L^2(\Omega)}.$$ \hspace{1cm} $\square$

As for Lemma 3.1, the proof of Lemma 4.1 is easy. To apply the Stein interpolation theorem, we need to establish the $L^1$-$L^\infty$ estimate for $\text{Re} \xi = d/2$. To this end, we write

$$S_\xi \psi(x) = \int_{\Omega} G_\xi(x - y)\psi(y) \, dy,$$

where

$$G_\xi(x) = \frac{\xi - \frac{d}{2}}{(2\pi)^d} \sum_{n \in \mathbb{Z}^d} e^{inx} \frac{e^{inx}}{|(n + b)B| + z} \xi.$$ \hspace{1cm} (4.4)

Let $\mathcal{F}_\xi(\cdot)$ denote the Fourier transform of $|\cdot|^2 + z^{-1}$ in $\mathbb{R}^d$.

**Lemma 4.2.** The formula

$$G_\xi(x) = \frac{\xi - \frac{d}{2}}{\det(B)} \sum_{n \in \mathbb{Z}^d} e^{-ib(x + 2\pi n)} \mathcal{F}_\xi((x + 2\pi n)B^{-1})$$

holds for any $\xi$, with which the right-hand side converges absolutely. \hspace{1cm} $\square$

Proof. We may assume $\text{Re} \xi$ is large in order to justify the following computation. The formula can be extended then to any $\xi$, for which the right-hand side converges absolutely, by analytic continuation.
Note that
\[ F_\xi(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixy} \, dx. \]  
(4.5)

By a change of variables,
\[ F_\xi(yB^{-1}) = \frac{\det(B)}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ixy} \, dx, \]
\[ \quad \text{for} \quad \xi \in \mathbb{R}^d. \]

The inverse Fourier transform then gives
\[
\frac{1}{\|xB^2 + z\|^\xi} = \frac{1}{\det(B)} \int_{\mathbb{R}^d} e^{ixy} F_\xi(yB^{-1}) \, dy
= \frac{1}{\det(B)} \sum_{m \in \mathbb{Z}^d} \int_{\Omega} e^{i(x+y+2\pi m)B(y + 2\pi m)^{-1}} \, dy
= \frac{1}{\det(B)} \int_{\Omega} e^{i(x+y)B} \sum_{m \in \mathbb{Z}^d} e^{2\pi i x m} F_\xi((y + 2\pi m)B^{-1}) \, dy.
\]

In particular, if we let \( x = -(n + b) \), then
\[
\frac{1}{\|(n + b)B^2 + z\|^\xi} = \frac{1}{\det(B)} \int_{\Omega} e^{-i(x+y)b} e^{-i\xi y} \sum_{m \in \mathbb{Z}^d} e^{-2\pi i x b m} F_\xi((y + 2\pi m)B^{-1}) \, dy.
\]

This implies that
\[
\sum_{n \in \mathbb{Z}^d} e^{iux} \left[ \left\| (n + b)B^2 + z \right\|^\xi \right] = \frac{(2\pi)^d}{\det(B)} \sum_{n \in \mathbb{Z}^d} e^{-i(x+2\pi m)b} F_\xi((x + 2\pi m)B^{-1}).
\]

The lemma follows.

By Lemma 4.2, we have
\[
|G_\xi(x)| \leq C \left| \xi - \frac{d}{2} \right| \sum_{n \in \mathbb{Z}^d} \left| F_\xi((x + 2\pi m)B^{-1}) \right|. \tag{4.6}
\]

To estimate the sum, we use the following formula (see [5, pp. 288–289]):
\[
F_\xi(x) = \frac{2^{1-\xi}}{(2\pi)^{d/2} \Gamma(\xi)} \left( \frac{z}{|x|^2} \right)^{1/2(d/2-\xi)} K_{d/2-\xi} \left( \sqrt{2|x|^2} \right), \tag{4.7}
\]
where \( \Gamma(\xi) \) is the gamma function and \( K_\eta \) is the modified Bessel function of the third kind of order \( \eta \) (see [12, p. 108]).
We need two estimates on $K_\eta$ for $\Re \eta = 0$:

$$|K_\eta(\omega)| \leq Ce^{C|\Im \eta|} \frac{e^{-\Re \omega}}{|\omega|^{1/2}}$$
if $|\omega| \geq 1$, $\Re \omega > 0$, \hfill (4.8)

$$|\sin(\pi \eta)K_\eta(\omega)| \leq Ce^{C|\Im \eta|}$$
if $|\omega| \leq 1$, $\Re \omega > 0$. \hfill (4.9)

We remark that (4.8) follows from the formula

$$K_\eta(\omega) = \left(\frac{\pi}{2\omega}\right)^{1/2} \frac{e^{-\omega}}{\Gamma\left(\eta + \frac{1}{2}\right)} \int_0^\infty e^{-s\eta - 1/2} \left(1 + \frac{s}{2\omega}\right)^{-1/2} ds,$$

which is valid for $\Re \eta > -1/2$ and $|\arg \omega| < \pi$ [12, p. 140], while (4.9) is a consequence of

$$2\sin(\pi \eta)K_\eta(\omega) = \pi \left\{ \left(\frac{\omega}{2}\right)^\eta \sum_{j=0}^\infty \frac{\binom{\omega}{2j}}{\Gamma(j+1)\Gamma(j+\eta+1)} \right. \right.$$

$$\left. - \left(\frac{\omega}{2}\right)^{-\eta} \sum_{j=0}^\infty \frac{\binom{\omega}{2j}}{\Gamma(j+1)\Gamma(j-\eta+1)} \right\}$$

(see [12, p. 108]).

**Lemma 4.3.** Suppose $\Re \xi = d/2$; then

$$\|S_{\xi} \psi\|_\infty \leq Ce^{C|\Im \xi|}\|\psi\|_1.$$

\hfill $\Box$

**Proof.** In view of (4.3) and (4.6), it suffices to show that, for $\Re \xi = d/2,$

$$|\Im \xi| \sum_{n \in \mathbb{Z}^d} \left| F_{\xi}(\sqrt{z}(x + 2\pi n)B^{-1}) \right| \leq Ce^{C|\Im \xi|}, \quad x \in \Omega. \hfill (4.10)$$

To this end, we first note that, by (4.7), the left-hand side of (4.10) is bounded by

$$Ce^{C|\Im \xi|} \sum_{n \in \mathbb{Z}^d} |K_{\Im \xi}(\sqrt{z}(x + 2\pi n)B^{-1})|.$$

(4.11)

Next we write the sum in (4.11) as $I_1 + I_2$, where $I_1$ is the sum over all $n$ such that $\Re \sqrt{z}(x + 2\pi n)B^{-1} \leq C$. It is easy to see that, by (4.8) and (4.9) and the estimate

$$|\Im \xi| \sin(\pi \Im \xi)| \leq Ce^{C|\Im \xi|},$$

$$I_1 \leq Ce^{C|\Im \xi|} \sum_{n \in \mathbb{Z}^d} |x + 2\pi n| \leq C,$$

(4.12)
To estimate $I_2$, we use (4.9) to obtain

$$I_2 \leq C e^{C \|\text{Im} \xi\|} \sum_{n \in \mathbb{Z}^d} e^{-c \text{Re} \sqrt{z} |x + 2\pi n|} \frac{1}{|z|^{1/4} |x + 2\pi n|^{1/2}}.$$

Note that, if $|y - x| \leq c_0 / \text{Re} \sqrt{z} \leq 1$, then

$$e^{-c \text{Re} \sqrt{z} |x + 2\pi n|} \frac{1}{|z|^{1/4} |x + 2\pi n|^{1/2}} \leq C e^{c \text{Re} \sqrt{z}} \frac{1}{|\text{Re} \sqrt{z} |y + 2\pi n|^{1/2}}.$$

Thus,

$$e^{-c \text{Re} \sqrt{z} |x + 2\pi n|} \frac{1}{|z|^{1/4} |x + 2\pi n|^{1/2}} \leq C \left( \frac{\text{Re} \sqrt{z}}{c_0} \right)^d \int_{|y - x| \leq (c_0 / \text{Re} \sqrt{z})} e^{-c \text{Re} \sqrt{z} |y + 2\pi n|} \frac{1}{|\text{Re} \sqrt{z} |y + 2\pi n|^{1/2}} \, dy.$$

This yields, by summation, that

$$I_2 \leq C e^{C \|\text{Im} \xi\|} \left( \frac{\text{Re} \sqrt{z}}{c_0} \right)^d \int_{|y| \geq (c / \text{Re} \sqrt{z})} e^{-c \text{Re} \sqrt{z} |y|} \frac{1}{|\text{Re} \sqrt{z} |y|^{1/2}} \, dy.$$

which, together with (4.12), gives (4.10). The proof is finished.

We are now in a position to give the following proof.

Proof of Theorem 4.1. With the estimates in Lemmas 4.1 ($\text{Re} \xi = 0$) and 4.3, we apply the Stein interpolation theorem (see [19, Theorem 4.1, p. 205]) to obtain

$$\|\{ (D + b)A(D + b)^T + z \}^{-1} \psi \|_q \leq C \|\psi\|_p,$$

where $q = p'$ and

$$\frac{1}{p} = \frac{t}{2} + \frac{1 - t}{1}, \quad 1 = t \cdot 0 + (1 - t) \cdot \frac{d}{2} \quad (4.13)$$

The theorem follows since, by (4.13),

$$t = 1 - \frac{2}{d} \quad \text{and} \quad p = \frac{2}{2 - t} = \frac{2d}{d + 2}.$$
5 Proof of Theorem 2.2

In this section we give the proof of Theorem 2.2 and thus complete the final step in the proof of Theorem 1.1.

Recall that

\[
\{\mathbb{H}_0(\delta + i\rho)\}^{-1}\psi = \sum_{n \in \mathbb{Z}^d} \frac{\hat{\psi}(n)e^{inx}}{(n + k)A(n + k)^T},
\]

where \(k = (\delta + i\rho)a + b\) (see (3.3)–(3.5)), and we need to show that

\[
\|\{\mathbb{H}_0(\delta + i\rho)\}^{-1}\psi\|_q \leq C\|\psi\|_p
\]

for \(p = 2d/(d + 2), q = p'\) with a constant \(C\) independent of \(\rho\) when \(|\rho| \geq 1\). We may assume that \(\psi \in C^\infty(\Omega)\).

First we write \(\psi = \sum_{j=-\infty}^{\infty} \psi_j\), where

\[
\psi_j = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n)e^{inx} \quad \text{for } j \geq 1,
\]

\[
\psi_j = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n)e^{inx} \quad \text{for } j \leq -1,
\]

and

\[
\psi_0 = \sum_{n \in \mathbb{Z}^d} \hat{\psi}(n)e^{inx}.
\]

Lemma 5.1. If estimate (5.2) holds for \(\psi_j\) with a constant \(C\) independent of \(j\), then it holds for \(\psi\). \(\square\)

Proof. Since \(\{\mathbb{H}_0(\delta + i\rho)\}^{-1}\) is a multiplier operator, by the 1-dimensional Littlewood-Paley theory on \([0, 2\pi]\) (see [21, Chapter XV]), we have
\[ \| \{ \mathbb{H}_0(\delta + i\rho) \}^{-1}\psi \|_q \leq C \left\| \left( \sum_{j=-\infty}^{\infty} |\{ \mathbb{H}_0(\delta + i\rho) \}^{-1}\psi_j|^2 \right)^{1/2} \right\|_q \]
\[ \leq C \left( \sum_{j=-\infty}^{\infty} \|\{ \mathbb{H}_0(\delta + i\rho) \}^{-1}\psi_j\|_q^2 \right)^{1/2} \]
\[ \leq C \left( \sum_{j=-\infty}^{\infty} \|\psi_j\|_p^2 \right)^{1/2} \]
\[ \leq C \left\| \left( \sum_{j=-\infty}^{\infty} |\psi_j|^2 \right)^{1/2} \right\|_p \]
\[ \leq C \|\psi\|_p, \]

where we use Minkowski’s inequality (see [18, p. 271]) and \( q > 2 > p \) in the second and fourth inequalities.

We give the estimate of \( \{ \mathbb{H}_0(\delta + i\rho) \}^{-1}\psi_j \) for \( j \geq 1 \) in detail. The case \( j \leq 0 \) may be handled in the same manner.

Fix \( j \geq 1 \). Note that \( n_1 \approx 2^j \) if \( \hat{\psi}_j(n) \neq 0 \). In view of (3.5), we let

\[ z_j = -\rho^2 a_1 s_0 + 2i\rho \cdot 2^j \cdot s_0, \quad (5.4) \]

and we consider the operator

\[ \{ (D + b)A(D + b)^T + z_j \}^{-1}\psi = \sum_{n \in \mathbb{Z}^d} \frac{\hat{\psi}(n)e^{inx}}{(n + b)(B_j^2 + z_j)} \quad (5.5) \]

(see (4.1) and (4.2)).

**Lemma 5.2.** Let \( j \geq 1 \), and let \( \rho \in \mathbb{R}, |\rho| \geq 1 \). Then there exists a constant \( C \) independent of \( j \) and \( \rho \) such that

\[ \left\| \{ \mathbb{H}_0(\delta + i\rho) \}^{-1}\psi_j - \{ (D + b)A(D + b)^T + z_j \}^{-1}\psi_j \right\|_q \leq C \|\psi_j\|_p, \]
Proof. Note that, by (5.1) and (5.5),

\[
\{\mathbb{H}_0(\delta + ip)\}^{-1}_\psi - \{(D + b)A(D + b)^T + zI\}^{-1}_\psi
= \sum_{n \in \mathbb{Z}^d} \hat{\psi}_j(n)e^{inx}\frac{[(n + b)B^2 + zI - (n + k)A(n + k)^T]}{[(n + k)A(n + k)^T] \cdot [(n + b)B^2 + zI]} \tag{5.6}
\]

Now we consider the second-order elliptic operator \( P = DAD^T \) on the d-torus \([0, 2\pi)^d \approx \mathbb{R}^d/(2\pi\mathbb{Z})^d \). \( P \) has a complete set of eigenfunctions \( \{e^{inx}, n \in \mathbb{Z}^d\} \) with the corresponding eigenvalues \( \{nAn^T, n \in \mathbb{Z}^d\} \). Thus,

\[
\chi_M \psi = \sum_{nAn^T \in [(M-1)^2, M^2)} \hat{\phi}(n)e^{inx} = \sum_{nAn^T \in [(M-1)^2, M^2)} \hat{\phi}(n)e^{inx} \tag{5.7}
\]

is the projection of \( \phi \) to the subspace of \( L^2(\Omega) \), spanned by eigenfunctions with eigenvalues in \( [(M-1)^2, M^2) \). It then follows from a general result in [17, Theorem 2.2(i), p. 127] that, for any \( \phi \in L^2(\Omega) \),

\[
\|\chi_M \phi\|_2 \leq CM^{1/2} \|\phi\|_p \tag{5.8}
\]

since \( p = 2d/(d+2) < p_4 = 2(d+1)/(d+3) \) and \( \sigma(p, d) = d[1/p-1/p']-1 = 1 \). By duality, (5.8) implies that, if \( q = p' \), then

\[
\|\chi_M \phi\|_q \leq CM^{1/2} \|\phi\|_2 \tag{5.9}
\]

Hence, by (5.6), Minkowski’s inequality, (5.8), and (5.9), we have

\[
\left\|\{\mathbb{H}_0(\delta + ip)\}^{-1}_\psi - \{(D + b)A(D + b)^T + zI\}^{-1}_\psi\right\|_q
\leq \sum_{M=1}^{\infty} \left\| \sum_{n \in \mathbb{Z}^d \atop M-1 \leq nB < M} \hat{\psi}_j(n)e^{inx}\frac{[(n + b)B^2 + zI - (n + k)A(n + k)^T]}{[(n + k)A(n + k)^T] \cdot [(n + b)B^2 + zI]} \right\|_q
\leq CM^{1/2} \sum_{M=1}^{\infty} \left\| \sum_{n \in \mathbb{Z}^d \atop M-1 \leq nB < M} \hat{\psi}_j(n)e^{inx}\frac{[(n + b)B^2 + zI - (n + k)A(n + k)^T]}{[(n + k)A(n + k)^T] \cdot [(n + b)B^2 + zI]} \right\|_2
\]
We may assume \( \varepsilon \) if \( \varepsilon \) is small. Thus, the left-hand side of (5.10) is bounded by

\[
\left\| \sum_{n \in \mathbb{Z}^d} \hat{\psi}_j(n)e^{inx} \right\|_2^2 \leq C \sum_{M=1}^{\infty} M^{1/2} \sup_{n \in \mathbb{Z}^d, \ |nB| \in [M-1,M]} \left\| \frac{\{(n+b)B^2 + z_j - (n+k)A(n+k)\} \cdot ((n+b)B+ z_j)}{(n+k)A(n+k)^\top \cdot ((n+b)B+ z_j)} \right\|
\]

where we use (3.5), (3.7), and (5.4) in the last inequality.

It remains to show that, for \( \rho \in \mathbb{R} \) and \( |\rho| \geq 1 \),

\[
\sum_{M=1}^{\infty} M \sup_{n \in \mathbb{Z}^d, \ |nB| \in [M-1,M]} \frac{|\rho|2^j}{\left\| \{(n+b)B^2 - \rho^2a_1s_0| + 2^j|\rho\} \right\|^2} \leq C.
\]

(5.10)

We may assume \( \rho \geq 1 \).

To prove (5.10), note that, if \( |nB| \in [M-1,M] \) and \( M \geq C\rho \), then

\[
\left\| (n+b)B^2 - \rho^2a_1s_0 | + 2^j\rho \geq c(M^2 + 2^j\rho).
\]

On the other hand, if \( |nB| \in [M-1,M] \) and \( M \leq C\rho \), then

\[
\left\| (n+b)B^2 - \rho^2a_1s_0 | + 2^j\rho \right.
\]

if \( \varepsilon \) is small. Thus, the left-hand side of (5.10) is bounded by

\[
C \sum_{1 \leq M \leq C\rho} \frac{M2^j}{\rho\left\| |M - \rho \sqrt{a_1s_0}| + 2^j\right\|^2} + C \sum_{M \geq C\rho} \frac{M\rho2^j}{(M^2 + 2^j\rho)^2} = I_1 + I_2.
\]
It is not hard to see that, if $|r - M| \leq 1/2$ and $M \geq 1$, then
\[
\frac{M^2}{\rho \{|M - \rho \sqrt{a_1 s_0} + 2i| \}^2} \leq \frac{Cr^2}{\rho \{|r - \rho \sqrt{a_1 s_0} + 2i| \}^2}.
\]
It follows that
\[
I_1 \leq C \int_0^C \frac{r^2 dr}{\rho \{|r - \rho \sqrt{a_1 s_0} + 2i| \}^2} \leq C \int_0^C \frac{2^i \rho^{-1} dr}{\{|r - \sqrt{a_1 s_0} + 2i \rho^{-1}| \}^2} \leq C \int_{|r - \sqrt{a_1 s_0}| \leq 2^i \rho^{-1}} \frac{dr}{2^i \rho^{-1}} + C \int_{|r - \sqrt{a_1 s_0}| \geq 2^i \rho^{-1}} \frac{2^i \rho^{-1} dr}{|r - \sqrt{a_1 s_0}|^2} \leq C.
\]
Similarly,
\[
I_2 \leq C \int_{C \rho}^\infty \frac{\rho^2 r dr}{(r^2 + 2^i \rho)^2} \leq C \int_0^\infty \frac{r dr}{(r^2 + 1)^2} \leq C.
\]
Now (5.10) is proved, and thus the proof of Lemma 5.2 is complete.

Finally, we give the following proof.

Proof of Theorem 2.2. In view of Lemmas 5.1 and 5.2, it suffices to show that
\[
\|((D + b)A(D + b)^T + z_i)^{-1} \psi \|_q \leq C \|\psi\|_p.
\]
But this follows immediately from Theorem 4.1, since, for $z_i = -\rho^2 a_1 s_0 + 2i \rho \cdot 2^i \cdot s_0$, we have $\text{Im} z_i = 2 \rho \cdot 2^i \cdot s_0 \neq 0$ and
\[
\text{Re} \sqrt{z_i} = |z_i|^{1/2} \cos \left(\frac{1}{2} \text{arg}(z_i)\right) \geq \frac{1}{2} \frac{|\text{Im} z_i|}{|z_i|^{1/2}} \geq c \frac{\rho |2^i|}{|\rho| + \sqrt{|\rho| |2^i|}} \geq c \min \left(2^i, \sqrt{|\rho| |2^i|}\right) \geq c_0 > 0.
\]
6 Potentials in weak-$L^{d/2}(\Omega)$ space

In this section we give the proof of Theorem 1.2. We use $\| \cdot \|_{p,q}$ to denote the norm in the Lorentz space $L^{p,q}(\Omega)$, where $1 < p \leq \infty$, $1 \leq q \leq \infty$ (see [19, Chapter V]). We need the following inequalities:

$$\| f \|_{q_0,2} \leq C \{ \| \nabla f \|_{2} + \| f \|_{2} \}$$

(6.1)

and

$$\| fg \|_{p_0,2} \leq C \| g \|_{d/2,\infty} \| f \|_{q_0,2},$$

(6.2)

where $p_0 = 2d/(d + 2)$ and $q_0 = 2d/(d - 2)$. Inequalities (6.2) and (6.3), which may be proved by the real interpolation, are substitutes, in the case of Lorentz spaces, for the usual Sobolev inequality and the Hölder inequality.

Let $V \in L^{d/2,\infty}(\Omega)$ be a real function satisfying condition (1.9). We may write $V = V_1 + V_2$ such that

$$\| V_1 \|_{d/2,\infty} \leq \varepsilon_0 \quad \text{and} \quad \| V_2 \|_{\infty} \leq C.$$  

(6.3)

It follows from (6.1) and (6.2) that

$$\int_{\Omega} |V| |f|^2 \, dx \leq C \varepsilon_0 \int_{\Omega} |\nabla f|^2 \, dx + C \int_{\Omega} |f|^2 \, dx.$$

(6.4)

Using the same arguments as in Section 2, we see that, if $\varepsilon_0$ is sufficiently small, we may define the operators $D A D^\top + V$ on $\mathbb{R}^d$ and $(D + k) A (D + k)^\top + V$ on $\Omega$ through the corresponding quadratic forms.

Next note that, by (6.1) and (6.2), $V \phi \in L^{p_0,2}(\Omega)$ if $V \in L^{d/2,\infty}(\Omega)$ and $\phi \in H^1_{\text{per}}(\Omega)$. With this observation, a careful inspection of the arguments in the proof of Theorem 1.1 in Section 2 shows that Theorem 1.2 follows from the following estimates:

$$\| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^{p_0,2}(\Omega) \to L^{p_0,2}(\Omega)} \rightarrow 0 \quad \text{as} \ |\rho| \rightarrow \infty,$$

(6.5)

$$\| \{ \mathbb{H}_0 (\delta + i \rho) \}^{-1} \|_{L^{p_0,2}(\Omega) \to L^{q_0,2}(\Omega)} \leq C \quad \text{for} \ |\rho| \geq 1.$$  

(6.6)

Also it is not hard to see that (6.5) is a direct consequence of Theorem 2.1 and the real interpolation (see [19, Theorem 3.15, p. 197]). We omit the details.

To show (6.6), we need the following theorem.
Theorem 6.1. Let $p_1 \leq p \leq p_2$ and $(1/q) = (1/p) - (2/d)$, where $p_1 = 2d/(d+1)/(d^2+3d+4)$ and $p_2 = 2(d+1)/(d+3)$. Then there exists $C > 0$ depending on $A$ and $d$ such that, for $|p| \geq 1$,

$$
\left\| \{\mathbb{I}_0(\delta + ip)\}^{-1} \right\|_{L^p(\Omega) \to L^q(\Omega)} \leq C.
$$
(6.7)

Note that $p_0 = 2d/(d+2) \in (p_1, p_2)$. Estimate (6.6) follows from Theorem 6.1, again by the real interpolation. Finally, we give the following proof.

Proof of Theorem 6.1. By the same localization arguments as in the proofs of Lemmas 5.1 and 5.2, we may reduce estimate (6.6) to

$$
\left\| \{ (D + b)A(D + b)^T + z \}^{-1} \psi \right\|_q \leq C \| \psi \|_p,
$$
(6.8)

where $b \in \mathbb{R}^d$, $z \in \mathbb{C}$, $\text{Im} \, z \neq 0$, and $\text{Re} \, \sqrt{z} \geq c_0 > 0$. We should point out that in place of (5.8) and (5.9) in the proof of Lemma 5.2, one needs to use the assumption $p_1 \leq p \leq p_2$. This gives $q' \leq p_d = 2(d+1)/(d+3)$, $p \leq p_d$, and $\sigma(q', d) = d[1/q' - 1/q] - 1$, $\sigma(p, d) = d[1/p - 1/p'] - 1$. The result in [17, Theorem 2.2, p. 127] then implies that

$$
\|X_M\|_{L^2 \to L^q} \|X_M\|_{L^q \to L^2} = \|X_M\|_{L^q \to L^2} \|X_M\|_{L^p \to L^2} \leq CM^{\sigma(q', d)/2} M^{\sigma(p, d)/2}
$$
(6.9)

where we also use $(1/q) = (1/p) - (2/d)$ in the last equality.

To prove (6.8), we define $S_\xi \psi$ as in (4.1). We claim that, if $\text{Re} \, \xi = (d - 1)/2$, then

$$
|G_\xi(x)| \leq Ce^{C|\text{Im} \, \xi|} \left\{ 1 + \sum_{|x + 2\pi n| \leq C} \frac{1}{|x + 2\pi n|} \right\},
$$
(6.10)

where $G_\xi(\cdot)$ is the integral kernel of the operator $S_\xi$ (see (4.4)). Indeed, by (4.6) and (4.7), we have

$$
|G_\xi(x)| \leq Ce^{C|\text{Im} \, \xi|} \sum_{n \in \mathbb{Z}^d} \frac{|z|^{1/4}}{|x + 2\pi n|^{1/2}} \cdot |K_{3/2 - i \text{Im} \, \xi} (\sqrt{z}|x + 2\pi n|B^{-1})|.
$$
(6.11)

We need the following estimates on the Bessel function $K_\eta(\omega)$ for $\text{Re} \, \eta = 1/2$:

$$
|K_\eta(\omega)| \leq Ce^{C|\text{Im} \, \eta|/|\omega|^{1/2}} \quad \text{if } |\omega| \leq 1,
$$
(6.12)

$$
|K_\eta(\omega)| \leq Ce^{C|\text{Im} \, \eta|e^{-\text{Re} \, \omega}}/|\omega|^{1/2} \quad \text{if } |\omega| \geq 1 \text{ and } \text{Re} \, \omega > 0.
$$
(6.13)
Using (6.12) and (6.13), we obtain from (6.11) that
\[
|G_\xi(x)| \leq C e^{C|\text{Im }\xi|} \sum_{|z|^{1/2}/(x+2\pi n)B^{-1}_{x}} \frac{|z|^{1/4}}{|x+2\pi n|^{1/2}} \cdot \frac{1}{|z|^{1/4}|x+2\pi n|^{1/2}} \\
+ C e^{C|\text{Im }\xi|} \sum_{|z|^{1/2}/(x+2\pi n)B^{-1}_{x}>1} \frac{|z|^{1/4}}{|x+2\pi n|^{1/2}} \cdot \frac{e^{-Re\sqrt{z}(x+2\pi n)B^{-1}_{x}}}{|z|^{1/4}|x+2\pi n|^{1/2}} \\
\leq C e^{C|\text{Im }\xi|} \sum_{|x+2\pi n|\leq C} \frac{1}{|x+2\pi n|} \\
+ C e^{C|\text{Im }\xi|} \sum_{|x+2\pi n|\geq C} e^{-c|x+2\pi n|} \\
\leq C e^{C|\text{Im }\xi|} \left\{ 1 + \sum_{|x+2\pi n|\leq C} \frac{1}{|x+2\pi n|} \right\},
\]
where we also use the assumption Re\sqrt{z} \geq c_0. Estimate (6.10) is then proved.

It follows from (6.10) and the well-known fractional integral estimates (see [18]) that, for Re\xi = (d - 1)/2,
\[
\|S_\xi \psi\|_q \leq C \|\psi\|_p, \tag{6.14}
\]
where
\[
\frac{1}{q} = \frac{1}{p} - \frac{d - 1}{d} \quad \text{and} \quad 1 < \tilde{p} < \frac{d}{d-1}. \tag{6.15}
\]
With (6.14) and the L^2-estimate in Lemma 4.1 for Re\xi = 0 at our disposal, we now apply the Stein complex interpolation theorem. We conclude that
\[
\|S_1 \psi\|_q \leq C \|\psi\|_p, \tag{6.16}
\]
where
\[
1 = t \cdot 0 + (1 - t) \cdot \frac{d - 1}{2},
\]
\[
\frac{1}{q} = \frac{t}{2} + \frac{1-t}{q},
\]
\[
\frac{1}{p} = \frac{t}{2} + \frac{1-t}{\tilde{p}}. \tag{6.17}
\]
An inspection of (6.15) and (6.17) shows that (6.16), and hence (6.8), holds for
\[
\frac{1}{q} = \frac{1}{p} - \frac{2}{d} \quad \text{and} \quad \frac{1}{2} + \frac{2}{d} - \frac{1}{d-1} < \frac{1}{p} < \frac{1}{2} + \frac{1}{d-1}.
\]
In particular, (6.8) is true for any $p \in [p_1, p_2]$. The proof is then complete. ■

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