$W^{1,p}$ ESTIMATES FOR ELLIPTIC HOMOGENIZATION PROBLEMS IN NONSMOOTH DOMAINS

Zhongwei Shen

Abstract. Let $\mathcal{L}_\varepsilon = -\text{div}\big(A\big(\frac{x}{\varepsilon}\big)\nabla\big)$, $\varepsilon > 0$ be a family of second order elliptic operators with real, symmetric coefficients on a bounded Lipschitz domain Ω in $\mathbb{R}^n$, subject to the Dirichlet boundary condition. Assuming that $A(x)$ is periodic and belongs to VMO, we show that there exists $\delta > 0$ independent of ε such that Riesz transforms $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ are uniformly bounded on $L^p(\Omega)$, where $1 < p < 3+\delta$ if $n \geq 3$, and $1 < p < 4+\delta$ if $n = 2$. The ranges of p's are sharp. In the case of C^1 domains, we establish the uniform L^p boundedness of $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ for $1 < p < \infty$ and $n \geq 2$. As a consequence, we obtain the uniform $W^{1,p}$ estimates for the elliptic homogenization problem $\mathcal{L}_\varepsilon u_\varepsilon = \text{div} f$ in Ω, $u_\varepsilon = 0$ on $\partial\Omega$.

1. Introduction

This paper continues the study in [S] of the L^p boundedness of Riesz transforms associated with second order elliptic operators. Here we consider a family of second order elliptic operators of divergence form in a nonsmooth domain Ω,

$$\mathcal{L}_\varepsilon = -\text{div}\big(A\big(\frac{x}{\varepsilon}\big)\nabla\big), \quad \varepsilon > 0, \tag{1.1}$$

arising in the theory of homogenization, subject to the Dirichlet boundary condition. We assume that $A(x) = \big(a_{ij}(x)\big)$ is a $n \times n$ real symmetric matrix and satisfies the following elliptic and periodic conditions:

$$\mu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \frac{1}{\mu}|\xi|^2 \quad \text{for any } x,\ \xi \in \mathbb{R}^n, \tag{1.2}$$

$$A(x+y) = A(x) \quad \text{for any } x \in \mathbb{R}^n,\ y \in \mathbb{Z}^n, \tag{1.3}$$

where $\mu > 0$. We will also assume that the coefficient matrix $A \in VMO(\mathbb{R}^n)$; i.e.,

$$\lim_{t\to 0} \omega(t) = 0, \tag{1.4}$$

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where

$$\omega(t) = \sup_{\substack{x\in\mathbb{R}^n\\ 0<r<t}} \frac{1}{|B(x,r)|}\int_{B(x,r)} \left| A(y) - \frac{1}{|B(x,r)|}\int_{B(x,r)} A(z)dz \right| dy. \tag{1.5}$$

Under these assumptions we establish the uniform L^p boundedness of Riesz transforms $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ on Lipschitz or C^1 domains. Theorem 1.1 below is the main result of the paper. We point out that the ranges of p's in Theorem 1.1 as well as in Corollary 1.2 are sharp even for operators with constant coefficients [JK2].

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$. Suppose that the real symmetric matrix $A(x)$ satisfies conditions (1.2), (1.3) and (1.4). Then there exists a constant $\delta > 0$ depending only on μ, n, the Lipschitz character of Ω, and function $\omega(t)$ such that*

$$\|\nabla(\mathcal{L}_\varepsilon)^{-1/2} f\|_p \le C\,\|f\|_p \quad \textit{for any } f \in L^p(\Omega), \tag{1.6}$$

for $1 < p < 3+\delta$ if $n \ge 3$, and $1 < p < 4+\delta$ if $n = 2$, where C depends only on μ, n, p, the Lipschitz character of Ω, and $\omega(t)$. If Ω is a C^1 domain, estimate (1.6) holds for any $1 < p < \infty$.

As a direct consequence of Theorem 1.1, we obtain the uniform $W^{1,p}$ estimate for the elliptic homogenization problem

$$\mathcal{L}_\varepsilon u_\varepsilon = \operatorname{div} f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, there exists $\delta > 0$, depending only on n, μ, the Lipschitz character of Ω, and $\omega(t)$, such that if $f \in L^p(\Omega) \cap L^2(\Omega)$ with $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{6} + \delta$ for $n \ge 3$, and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{4} + \delta$ for $n = 2$, the unique solution to the Dirichlet problem (1.7) in $W_0^{1,2}(\Omega)$ satisfies*

$$\|\nabla u_\varepsilon\|_p \le C\,\|f\|_p, \tag{1.8}$$

where C depends only on n, μ, p, the Lipschitz character of Ω, and $\omega(t)$. If Ω is a C^1 domain, estimate (1.8) holds for any $1 < p < \infty$.

If the coefficients $a_{jk}(x)$ are periodic and Hölder continuous, it was shown by Avellaneda and Lin [AL4] that $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$. A similar theorem, based on the results in [AL1,AL3], was obtained by Alexopoulos [A]. We should point out that the results in [AL4] were established for systems of elliptic operators with periodic coefficients, and extended to bounded domains with $C^{1,\alpha}$ boundaries. In [CP] Caffarelli and Peral obtained the uniform interior $W^{1,p}$ $(2 < p < \infty)$ estimates under the assumption that $A(x)$ are continuous and periodic. By Theorem A in [S] this implies the boundedness of $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ on $L^p(\mathbb{R}^n)$ for operators with continuous and periodic coefficients. We mention that the boundedness of $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ on $L^p(\mathbb{R}^n)$ has been established in [ERS] for second-order periodic elliptic operators in divergence form with complex continuous coefficients. For related results on $W^{1,p}$ estimates and Riesz transforms for second order elliptic operators *without* the periodicity assumption, we refer the reader to [Au,AC,ACDH,AT1,AT2,AQ, B, BW,CD] and their references.

Our starting point for the proof of Theorem 1.1 is the following.

Theorem 1.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $A(x)$ be a real symmetric $n \times n$ matrix with bounded measurable entries satisfying (1.2). Let $\bar{p} > 2$. Suppose that there exist constants $C_0 > 1$, $\alpha_2 > \alpha_1 > 1$ and r_0 independent of $\varepsilon > 0$ such that for any ball $B(x_0, r)$ with the property that $0 < r < r_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, \alpha_2 r) \subset \Omega$, and for any weak solution of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in $\Omega \cap B(x_0, \alpha_2 r)$ and $u_\varepsilon = 0$ on $B(x_0, \alpha_2 r) \cap \partial\Omega$ (if $x_0 \in \partial\Omega$), one has $|\nabla u_\varepsilon| \in L^{\bar{p}}(\Omega \cap B(x_0, r))$ and*

$$\left(\frac{1}{r^n}\int_{\Omega\cap B(x_0,r)} |\nabla u_\varepsilon|^{\bar{p}} dx\right)^{1/\bar{p}} \leq C_0 \left(\frac{1}{r^n}\int_{B(x_0,\alpha_1 r)} |\nabla u_\varepsilon|^2 dx\right)^{1/2}. \tag{1.9}$$

Then there exists $\delta > 0$ depending only on μ, n, α_1, α_2, C_0 and the Lipschitz character of Ω such that (1.6) holds for $1 < p < \bar{p} + \delta$, where C depends only on μ, n, p, α_1, α_2, C_0 and the Lipschitz character of Ω.

Theorem 1.3 follows directly from Theorem B in [S], which states that given any second order elliptic operator $\mathcal{L}$ of divergence form with real, symmetric, bounded, measurable coefficients and any $p > 2$, the boundedness of the Riesz transforms $\nabla(\mathcal{L})^{-1/2}$ on $L^p(\Omega)$ is equivalent to the scale-invariant interior and boundary $W^{1,p}$ estimates (1.9) for weak solutions of $\mathcal{L}u = 0$. For the interior estimates, one may extend either the approximation method in [CP] or the compactness method developed in [AL1,AL2] to the case of VMO coefficients. As expected for nonsmooth domains, the main difficulty lies in the uniform boundary $W^{1,p}$ estimates.

For the second order systems of elliptic operators with periodic and Hölder continuous coefficients, Avellaneda and Lin [AL1] established a uniform boundary L^∞ estimate for the gradients of weak solutions of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ on $C^{1,\alpha}$ domains. Such L^∞ gradient estimate, however, fails in general for C^1 domains, even in the case of constant coefficients. Suppose $0 \in \partial\Omega$ and

$$\Omega \cap B(0, r_0) = \big\{(x', x_n) \in \mathbb{R}^n : x_n > \psi(x')\big\} \cap B(0, r_0). \tag{1.10}$$

Let $\Delta_r = \{(x', \psi(x')) : |x'| < r\}$. Our main novelty in the proof of Theorem 1.1 is to reduce the weak reverse Hölder inequality (1.9) to the following decay estimate,

$$\begin{aligned}\int_0^{tr}\int_{\Delta_r} &|u_\varepsilon(x', \psi(x') + s)|^{\bar{p}} dx'\, ds \\ &\leq C\, t^{\bar{p}+\alpha_0} \int_0^{2r}\int_{\Delta_{2r}} |u_\varepsilon(x', \psi(x') + s)|^{\bar{p}}\, dx' ds,\end{aligned} \tag{1.11}$$

for any $0 < t < 1$ and $0 < r < cr_0$, where $\alpha_0 > 0$, $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in $\Omega \cap B(0, 2r)$ and $u_\varepsilon = 0$ on Δ_{2r}. Note that estimate (1.11) only involves u, not ∇u. This, together with the observation that (1.11) holds for solutions of elliptic equations with constant coefficients, makes it accessible via a variant of the three-step compactness argument of Avellaneda and Lin in [AL1,AL2].

The paper is organized as follows. In section 2 we establish the uniform interior $W^{1,p}$ estimate for operators $\{\mathcal{L}_\varepsilon\}$ with VMO coefficients for any $p > 2$. The reduction from (1.9) to (1.11) as well as the proof of (1.11) for $\bar{p} = 3$ if $n \geq 3$, and for $\bar{p} = 4$ if $n = 2$, is given in section 3. A similar approach gives (1.9) on C^1 domains for any $2 < \bar{p} < \infty$. This is outlined in section 4. Finally in section 5 we give the proof of Theorem 1.1 and Corollary 1.2.

Throughout the rest of this paper, we assume that $A(x)$ is a real symmetric matrix satisfying conditions (1.2)-(1.4). We will denote $\partial/\partial x_j$ by ∂_j. The standard summation convention will also be used.

2. Interior $W^{1,p}$ estimates

Let $B(x_0, r)$ denote the ball centered at x_0 with radius r. For $\alpha > 0$, we use the notation $\alpha B = B(x_0, \alpha r)$. In this section we establish the following theorem.

Theorem 2.1. *Let $u_\varepsilon \in W^{1,2}(3B)$ be a weak solution of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in $3B$. Then $|\nabla u_\varepsilon| \in L^p(B)$ and*

$$\left\{ \frac{1}{|B|} \int_B |\nabla u_\varepsilon|^p \, dx \right\}^{1/p} \leq C \left\{ \frac{1}{|2B|} \int_{2B} |\nabla u_\varepsilon|^2 dx \right\}^{1/2} \tag{2.1}$$

for any $p > 2$, where C depends only on μ, n, p and function $\omega(t)$.

Theorem 2.1 was proved in [CP] for the case of continuous coefficients. The proof in [CP] extends easily to the case that $A \in VMO(\mathbb{R}^n)$. We remark that if the coefficients are Hölder continuous, estimate (2.1) follows from the L^∞ estimates on ∇u_ε established in [AL1, Lemma 16]. Here we present a proof of Theorem 2.1, using the three-step compactness method of Avellaneda and Lin.

Let χ_j be the unique function such that $\mathcal{L}_1(\chi_j) = \partial_i a_{ij}$ on $\mathbb{R}^n$, χ_j is periodic with respect to $\mathbb{Z}^n$, and

$$\int_{[0,1]^n} \chi_j \, dx = 0. \tag{2.2}$$

The function $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$ is called the corrector for $\mathcal{L}_\varepsilon$. Observe that for any $\rho > 0$,

$$\mathcal{L}_\rho \{ x + \rho \chi(x/\rho) \} = 0 \quad \text{ in } \mathbb{R}^n. \tag{2.3}$$

It is also not hard to see that

$$\|\chi\|_{L^\infty(\mathbb{R}^n)} \leq C(n, \mu). \tag{2.4}$$

Let $\mathcal{L}_0 = -\partial_i b_{ij} \partial_j$ be a second order elliptic operator with real *constant* coefficients. Assume that coefficients satisfy $b_{ij} = b_{ji}$ and the ellipticity condition (1.2). Let $B_\theta =$

$B(0,\theta)$. Suppose that $\mathcal{L}_0 u_0 = 0$ in B_1. Then there exists C_0 depending only on μ and n such that

$$\sup_{|x|<\theta} |u_0(x) - u_0(0) - < x, (\nabla u_0)_{B_\theta} > | \le C_0 \theta^2 \|u_0\|_{L^\infty(B_1)}, \tag{2.5}$$

for any $\theta \in (0, 1/2)$, where $(\nabla u_0)_{B_\theta}$ denotes the average of ∇u_0 over B_θ.

For any fixed $\eta \in (0,1)$, we choose $\theta \in (0,1/2)$ so that $C_0\theta \le \theta^\eta$, where C_0 is given by (2.5). The following was proved in [AL1, Lemma 14] by a compactness argument.

Lemma 2.2. *There exists ε_0 depending only on μ, n and η such that*

$$\sup_{|x|<\theta} |u_\varepsilon(x) - u_\varepsilon(0) - < x + \varepsilon\chi(x/\varepsilon), (\nabla u_\varepsilon)_{B_\theta} > | \le \theta^{1+\eta} \|u_\varepsilon\|_{L^\infty(B_1)}, \tag{2.6}$$

for any $0 < \varepsilon < \varepsilon_0$ and any weak solution u_ε of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in B_1.

By an iteration argument, Lemma 2.2, together with (2.3) and (2.4), leads to the following.

Lemma 2.3. *Let η, θ and ε_0 be the same constants as in Lemma 2.2. There exists a constant C_1, depending only on μ, n and η, such that for any weak solution u_ε of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in B_1 with $\varepsilon_0\theta^{\ell+1} < \varepsilon \le \varepsilon_0\theta^\ell$,*

$$\sup_{|x|<\theta^\ell} |u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon d_\ell^\varepsilon - < x + \varepsilon\chi(x/\varepsilon), G_\ell^\varepsilon > | \le \theta^{\ell(1+\eta)} \|u_\varepsilon\|_{L^\infty(B_1)}, \tag{2.7}$$

where $d_\ell^\varepsilon \in \mathbb{R}$ and $G_\ell^\varepsilon \in \mathbb{R}^n$ are constants with the property

$$|d_\ell^\varepsilon| + |G_\ell^\varepsilon| \le C_1 \|u_\varepsilon\|_{L^\infty(B_1)}. \tag{2.8}$$

Proof. See [AL1, Lemma 16]. □

We are in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Fix $B = B(x_0, r_0)$. By translation and dilation, one may assume that $x_0 = 0$ and $r_0 = 1$. Also, by the local $W^{1,p}$ estimates for solutions of second order elliptic equations with VMO coefficients, one may further assume that $0 < \varepsilon < \theta\varepsilon_0$, where θ and ε_0 are given by Lemma 2.2.

Let $v(x) = u_\varepsilon(\varepsilon x)$. Then $\mathcal{L}_1 v = 0$ in $B(0, 2/\varepsilon)$. It follows that for any $2 < p < \infty$,

$$\|\nabla v\|_{L^p(B(0,\frac{1}{4\varepsilon_0}))} \le C \, \|\nabla v\|_{L^2(B(0,\frac{1}{2\varepsilon_0}))}. \tag{2.9}$$

Thus,

$$\left\{ \frac{1}{\varepsilon^n} \int_{B(0,\frac{\varepsilon}{4\varepsilon_0})} |\nabla u_\varepsilon|^p \, dx \right\}^{1/p} \le C \left\{ \frac{1}{\varepsilon^n} \int_{B(0,\frac{\varepsilon}{2\varepsilon_0})} |\nabla u_\varepsilon|^2 \, dx \right\}^{1/2}. \tag{2.10}$$

By Caccioppoli's inequality, this implies that

$$\left\{ \frac{1}{\varepsilon^n} \int_{B(0,\frac{\varepsilon}{4\varepsilon_0})} |\nabla u_\varepsilon|^p \, dx \right\}^{1/p} \le C \sup_{|x|<\frac{\varepsilon}{\varepsilon_0}} \frac{|u_\varepsilon(x) - u_\varepsilon(0)|}{\varepsilon}. \tag{2.11}$$

Note that by Lemma 2.3, we have

$$\sup_{|x|<\frac{\varepsilon}{\varepsilon_0}} \frac{|u_\varepsilon(x) - u_\varepsilon(0)|}{\varepsilon} \le C \, \|u_\varepsilon\|_{L^\infty(B(0,1))}. \tag{2.12}$$

In view of (2.11) and (2.12), we have proved that

$$\int_{B(0,\frac{\varepsilon}{4\varepsilon_0})} |\nabla u_\varepsilon|^p dx \le C \varepsilon^n \|u_\varepsilon\|^p_{L^\infty(B(0,1))}. \tag{2.13}$$

It follows by translation that for any $y \in B(0, 1/2)$,

$$\int_{B(y,\frac{\varepsilon}{4\varepsilon_0})} |\nabla u_\varepsilon|^p dx \le C \varepsilon^n \|u_\varepsilon\|^p_{L^\infty(B(0,\frac{3}{2}))}. \tag{2.14}$$

By covering $B(0, 1/2)$ with a finite number of balls of radius $\varepsilon/4\varepsilon_0$, we may deduce from (2.14) that

$$\int_{\frac{1}{2}B} |\nabla u_\varepsilon|^p dx \le C \, \|u_\varepsilon\|^p_{L^\infty(\frac{3}{2}B)} \le C \, \|u_\varepsilon\|^p_{L^2(2B)}. \tag{2.15}$$

Since $u_\varepsilon - \beta$ is also a solution for any $\beta \in \mathbb{R}$, we may replace u_ε in the right side of (2.15) by $u_\varepsilon - \beta$. This, together with the Poincaré inequality, gives

$$\left\{ \frac{1}{|\frac{1}{2}B|} \int_{\frac{1}{2}B} |\nabla u_\varepsilon|^p dx \right\}^{1/p} \le C \left\{ \frac{1}{|2B|} \int_{2B} |\nabla u_\varepsilon|^2 dx \right\}^{1/2}. \tag{2.16}$$

By a simple covering argument it is not hard to see that estimate (2.16) is equivalent to (2.1). This completes the proof of Theorem 2.1. □

3. Boundary $W^{1,p}$ Estimates

Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\psi(0) = 0$. For $r > 0$, let

$$\Delta_r = \big\{ (x', \psi(x')) \in \mathbb{R}^n : \ |x'| < r \big\}, \tag{3.1}$$

$$D_r = \big\{ (x', x_n) \in \mathbb{R}^n : \ |x'| < r \ \text{ and } \ \psi(x') < x_n < \psi(x') + r \big\}. \tag{3.2}$$

Let $\bar{p} = 3$ for $n \ge 3$, and $\bar{p} = 4$ for $n = 2$. This section is devoted to the proof of the following theorem.

Theorem 3.1. *There exists a constant $C > 0$, depending only on μ, n, $\|\nabla\psi\|_\infty$ and function $\omega(t)$, such that if $u_\varepsilon \in W^{1,2}(D_{16r})$ is a weak solution of $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in D_{16r} and $u_\varepsilon = 0$ on Δ_{16r}, then $|\nabla u_\varepsilon| \in L^{\bar{p}}(D_r)$ and*

$$\left\{\frac{1}{r^n}\int_{D_r} |\nabla u_\varepsilon|^{\bar{p}}\, dx\right\}^{1/\bar{p}} \le C\left\{\frac{1}{r^n}\int_{D_{16r}} |\nabla u_\varepsilon|^2\, dx\right\}^{1/2}. \tag{3.3}$$

Let $d(x) = d(x', x_n) = |x_n - \psi(x')|$. To prove (3.3), we first show that it suffices to estimate the L^p norm of u_ε/d on D_{2r}.

Lemma 3.2. *Suppose that $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in D_{2r} and $u_\varepsilon = 0$ on Δ_{2r}. Then for any $2 < p < \infty$,*

$$\int_{D_r} |\nabla u_\varepsilon|^p dx \le C\int_{D_{2r}} \left|\frac{u_\varepsilon(x)}{d(x)}\right|^p dx, \tag{3.4}$$

where $C > 0$ depends only on μ, n, p, $\|\nabla\psi\|_\infty$ and function $\omega(t)$.

Proof. Let $\rho(x) = \text{dist}(x, \partial D_{4r})$. Then $\rho(x) \approx d(x)$ for any $x \in D_{3r}$. Choose $c = c(n, \|\nabla\psi\|_\infty) \in (0, 1/4)$ so small that $B(x, 2c\rho(x)) \subset D_{2r}$ for any $x \in D_r$. By interior estimate (2.1), we have

$$\int_{B(x,c\rho(x))} |\nabla u_\varepsilon(y)|^p dy \le C\int_{B(x,2c\rho(x))} \left|\frac{u_\varepsilon(y)}{\rho(y)}\right|^p dy \tag{3.5}$$

for any $x \in D_r$. Next we multiply both sides of (3.5) by $\rho(x)^{-n}$ and integrate the resulting inequality over D_r. This gives

$$\begin{aligned}\int_{D_r} |\nabla u_\varepsilon(y)|^p &\left\{\int_{\substack{x\in D_r\\ |x-y|<c\rho(x)}} \frac{dx}{\rho(x)^n}\right\} dy \\ &\le C\int_{D_{2r}} \left|\frac{u_\varepsilon(y)}{\rho(y)}\right|^p \left\{\int_{\substack{x\in D_r\\ |x-y|<2c\rho(x)}} \frac{dx}{\rho(x)^n}\right\} dy.\end{aligned} \tag{3.6}$$

Finally we observe that if $|x - y| < c\rho(x)$, then

$$\rho(y) \le \rho(x) + |x - y| \le (1 + c)\rho(x). \tag{3.7}$$

Similarly, $\rho(x) \le \rho(y) + |x - y| \le \rho(y) + c\rho(x)$ and thus $(1 - c)\rho(x) \le \rho(y)$. Hence $\rho(y) \approx \rho(x)$. It follows that for any $y \in D_r$,

$$\int_{\substack{x\in D_r\\ |x-y|<\rho(x)}} \frac{dx}{\rho(x)^n} \ge \frac{c}{\rho(y)^n}\int_{\substack{x\in D_r\\ |x-y|<c\rho(y)}} dx \ge c, \tag{3.8}$$

and for any $y \in D_{2r}$,

$$\int_{\substack{x\in D_r\\ |x-y|<2c\rho(x)}} \frac{dx}{\rho(x)^n} \le \frac{C}{\rho(y)^n}\int_{\substack{x\in D_r\\ |x-y|\le C\rho(y)}} dy \le C. \tag{3.9}$$

The desired estimate (3.4) now follows from (3.6), (3.8) and (3.9). □

By the well known De Giorgi-Nash regularity estimates and the Poincaré inequality,

$$\begin{aligned}\left\{\frac{1}{r^n}\int_{D_r}|u_\varepsilon|^p dx\right\}^{1/p} &\le C\left\{\frac{1}{r^n}\int_{D_{2r}}|u_\varepsilon|^2 dx\right\}^{1/2}\\ &\le Cr\left\{\frac{1}{r^n}\int_{D_{2r}}|\nabla u_\varepsilon|^2 dx\right\}^{1/2}\end{aligned}\tag{3.10}$$

for any $2<p<\infty$. In view of (3.4) and (3.10), we see that the estimate (3.3) would follow from

$$\int_0^r\int_{|x'|<r}\left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{\bar p}dx'ds\le\frac{C}{r^{\bar p}}\int_0^{4r}\int_{|x'|<4r}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds.\tag{3.11}$$

Lemma 3.3. *Suppose that there exist positive constants* ε_0, α_0 *and* C *depending only on* μ, n, $\|\nabla\psi\|_\infty$ *and function* $\omega(t)$ *such that for* $\frac{\varepsilon}{\varepsilon_0}<\frac{t}{r}<1$,

$$\begin{aligned}&\int_0^t\int_{|x'|<r}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds\\ &\qquad\le C\left(\frac{t}{r}\right)^{\bar p+\alpha_0}\int_0^{2r}\int_{|x'|<2r}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds\end{aligned}\tag{3.12}$$

whenever $\mathcal{L}_\varepsilon u_\varepsilon=0$ *in* D_{2r} *and* $u_\varepsilon=0$ *on* Δ_{2r}. *Then estimate (3.3) holds.*

Proof. By the boundary $W^{1,p}$ estimates on Lipschitz domains for operators with VMO coefficients (see Theorem C and its proof in [S]), estimate (3.3) holds for $\varepsilon\ge(\varepsilon_0/4)$. Assume that $0<\varepsilon<\varepsilon_0/4$. Since $w(x)=u_\varepsilon(\varepsilon x)$ is a weak solution of $\mathcal{L}_1 w=0$, the same $W^{1,p}$ estimates also give us

$$\begin{aligned}&\int_0^{\frac{\varepsilon r}{\varepsilon_0}}\int_{|x'|<\frac{\varepsilon r}{\varepsilon_0}}\left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{\bar p}dx'ds\\ &\qquad\le\frac{C}{(\varepsilon r)^{\bar p}}\int_0^{\frac{2\varepsilon r}{\varepsilon_0}}\int_{|x'|<\frac{2\varepsilon r}{\varepsilon_0}}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds.\end{aligned}\tag{3.13}$$

By covering Δ_r with surface balls of radius $\varepsilon r/\varepsilon_0$, we may deduce from (3.13) that

$$\begin{aligned}&\int_0^{\frac{\varepsilon r}{\varepsilon_0}}\int_{|x'|<r}\left|\frac{u_\varepsilon(x',\psi(x')+s)}{s}\right|^{\bar p}dx'ds\\ &\qquad\le\frac{C}{(\varepsilon r)^{\bar p}}\int_0^{\frac{2\varepsilon r}{\varepsilon_0}}\int_{|x'|<2r}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds\\ &\qquad\le\frac{C}{r^{\bar p}}\int_0^{4r}\int_{|x'|<4r}|u_\varepsilon(x',\psi(x')+s)|^{\bar p}dx'ds,\end{aligned}\tag{3.14}$$

where we have used the assumption (3.12) in the last step.

Finally we let $f(x', s) = s^{-1} u_\varepsilon(x', \psi(x') + s)$ and write

$$\begin{aligned} &\int_0^r \int_{|x'|<r} |f(x',s)|^{\bar{p}} dx' ds \\ &= \left\{ \int_0^{\frac{\varepsilon r}{\varepsilon_0}} \int_{|x'|<r} + \sum_{j=1}^{j_0} \int_{\frac{2^{j-1}\varepsilon r}{\varepsilon_0}}^{\frac{2^j \varepsilon r}{\varepsilon_0}} \int_{|x'|<r} + \int_{\frac{2^{j_0}\varepsilon r}{\varepsilon_0}}^{r} \int_{|x'|<r} \right\} |f(x',s)|^{\bar{p}} dx' ds, \end{aligned} \tag{3.15}$$

where $2^{-j_0-1} \le \frac{\varepsilon}{\varepsilon_0} \le 2^{-j_0}$. The first term in the right side of (3.15) is handled by (3.14), while the estimate of the last term is trivial. To control the term involving the summation over j, we use the assumption (3.12). This gives estimate (3.11), from which inequality (3.3) follows. □

It remains to establish estimate (3.12). By dilation we may assume that $r = 1$. We first observe that (3.12) holds for solutions of the elliptic equations with constant coefficients. Indeed suppose that $\mathcal{L}_0 u_0 = -\partial_i b_{ij} \partial_j u_0 = 0$ in $D_{3/2}$ and $u_0 = 0$ on $\Delta_{3/2}$, where b_{ij} are real constants satisfying $b_{ij} = b_{ji}$ and (1.2). By the L^2 regularity estimates on Lipschitz domains [JK1], we have

$$\int_{|x'|<1} |(\nabla u_0)^*(x', \psi(x'))|^2 dx' \le C \int_{D_{3/2}} |u_0|^2 dx, \tag{3.16}$$

where $(\nabla u_0)^*(x', \psi(x')) = \sup\{|\nabla u_0(x', x_n)| : \psi(x') < x_n < \psi(x_n) + r\}$. This, together with the observation that $|u_0(x', s + \psi(x'))| \le s (\nabla u_0)^*(x', \psi(x'))$ and the boundary Hölder estimates, gives

$$\int_0^t \int_{|x'|<1} |u_0(x', \psi(x') + s)|^3 dx' ds \le C t^{3+\beta} \int_0^{\frac{3}{2}} \int_{|x'|<\frac{3}{2}} |u_0(x', \psi(x') + s)|^3 dx' ds, \tag{3.17}$$

for any $0 < t < 1$, where $\beta > 0$ and $C > 0$ depend only on μ, n and $\|\nabla \psi\|_\infty$. Note that if $n = 2$, one has a stronger boundary Hölder estimate,

$$|u_0(x', \psi(x') + s)| \le C s^{\frac{1+\beta}{2}} \int_{D_{3/2}} |u_0| \, dy \tag{3.18}$$

for some $\beta > 0$ depending only on μ and $\|\nabla \psi\|_\infty$. Together with (3.16), this shows that for $n = 2$,

$$\int_0^t \int_{|x'|<1} |u_0(x', \psi(x') + s)|^4 dx' ds \le C t^{4+\beta} \int_0^{\frac{3}{2}} \int_{|x'|<\frac{3}{2}} |u_0(x', \psi(x') + s)|^4 dx' ds \tag{3.19}$$

for any $0 < t < 1$.

We now fix $0 < \alpha < \beta$. Choose $t_0 \in (0, 1/4)$ so that $2C t_0^{\beta - \alpha} \le 1$, where β and C are given by (3.17) and (3.19).

Lemma 3.4. *There exists $\varepsilon_0 > 0$ depending only on μ, n and $\|\nabla\psi\|_\infty$ such that for any $0 < \varepsilon \le \varepsilon_0$,*

$$
\begin{aligned}
&\int_0^{t_0}\int_{|x'|<1} |u_\varepsilon(x',\psi(x')+s)|^{\bar{p}}dx'ds \\
&\qquad\qquad \le t_0^{\bar{p}+\alpha}\int_0^2\int_{|x'|<2} |u_\varepsilon(x',\psi(x')+s)|^{\bar{p}}dx'ds,
\end{aligned} \tag{3.20}
$$

whenever $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in D_2 and $u_\varepsilon = 0$ on Δ_2.

Proof. We prove estimate (3.20) by contradiction. Suppose that there exist $\{\mathcal{L}^k\}$, $\{\varepsilon_k\}$ and $\{u_{\varepsilon_k}\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$,

$$
\mathcal{L}^k_{\varepsilon_k} u_{\varepsilon_k} = -\partial_i\left(a^k_{ij}(\frac{x}{\varepsilon_k})\partial_j u_{\varepsilon_k}\right) = 0 \tag{3.21}
$$

in $\{(x',x_n) : |x'| < 2 \text{ and } \psi_k(x') < x_n < \psi_k(x')+2\}$, $u_{\varepsilon_k} = 0$ on $\{(x',\psi_k(x')) : |x'| < 2\}$, and

$$
\begin{aligned}
&\int_0^2\int_{|x'|<2} |u_{\varepsilon_k}(x',\psi_k(x')+s)|^{\bar{p}}dx'ds = 1, \\
&\int_0^{t_0}\int_{|x'|<1} |u_{\varepsilon_k}(x',\psi(x')+s)|^{\bar{p}}dx'ds > t_0^{\bar{p}+\alpha},
\end{aligned} \tag{3.22}
$$

where the coefficients $a^k_{ij}(x)$ of $\mathcal{L}^k$ are real symmetric and satisfy (1.2)-(1.3), $\|\nabla\psi_k\|_\infty \le M$ and $\psi_k(0) = 0$. Let

$$
b^k_{ij} = \int_{[0,1]^n} a^k_{i\ell}(y)\{\delta_{j\ell} - \partial_\ell \chi^k_j(y)\}dy, \tag{3.23}
$$

where $\chi^k = (\chi^k_1,\dots,\chi^k_n)$ are correctors for $\mathcal{L}^k$. Since b^k_{ij} are bounded, by passing to a subsequence, we may assume that

$$
b_{ij} = \lim_{k\to\infty} b^k_{ij} \tag{3.24}
$$

exists for $1 \le i,j \le n$. It is known that the constant matrix (b_{ij}) is symmetric and satisfies (1.2) [BLP].

Note that the sequence $\{\psi_k\}$ is equi-continuous on $\{x' : |x'| \le 2\}$. Thus, without loss of generality, we may assume that ψ_k converges uniformly to ψ_0 on $\{x' : |x'| \le 2\}$. Clearly $\|\nabla\psi_0\|_\infty \le M$ and $\psi_0(0) = 0$. By the classical regularity estimates, $\{u_{\varepsilon_k}\}$ is uniformly Hölder continuous on $\{(x',x_n) : |x'| \le r \text{ and } \psi_k(x') \le x_n \le \psi_k(x')+r\}$ for any $1 < r < 2$. It follows that the sequence $\{u_{\varepsilon_k}(x',\psi_k(x')+s)\}$ is equi-continuous on $Q_r = \{(x',s) : |x'| \le r \text{ and } 0 \le s \le r\}$ for any $1 < r < 2$. Hence, by passing to a subsequence, we may assume that $u_{\varepsilon_k}(x', \psi_k(x') + s)$ converges uniformly to $u_0(x', \psi_0(x') + s)$ on $Q_{3/2}$. We may also assume that $u_{\varepsilon_k}(x', \psi_k(x') + s)$ converges weakly to $u_0(x', \psi_0(x') + s)$ in $W^{1,2}(Q_{3/2})$.

By (3.22) and the uniform convergence of u_{ε_k}, we have

$$\begin{aligned} &\int_0^{\frac{3}{2}} \int_{|x'|<\frac{3}{2}} |u_0(x', \psi_0(x') + s)|^{\bar{p}} dx' ds \le 1, \\ &\int_0^{t_0} \int_{|x'|<1} |u_0(x', \psi_0(x') + s)|^{\bar{p}} dx' ds \ge t_0^{\bar{p}+\alpha}. \end{aligned} \tag{3.25}$$

This contradicts with estimates (3.17) and (3.19), since u_0 is a solution of $b_{ij}\partial_i\partial_j u_0 = 0$, with b_{ij} given by (3.24) [BLP], in $\{(x', x_n) : |x'| < 3/2 \text{ and } \psi_0(x') < x_n < \psi_0(x') + 3/2\}$ and $u_0 = 0$ on $\{(x', \psi_0(x')) : |x'| < 3/2\}$. □

We are now in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let ε_0 be given by Lemma 3.4. It suffices to establish estimate (3.12) for $r = 1$.

Let $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in D_2 and $u_\varepsilon = 0$ on Δ_2 for some $\varepsilon < \varepsilon_0$. Let $v(x) = u_\varepsilon(\theta x)$ where $0 < \varepsilon/\theta \le \varepsilon_0$. Then $\mathcal{L}_{\frac{\varepsilon}{\theta}} v = 0$ in

$$\left\{(x', x_n) :\ |x'| < 2\theta^{-1} \text{ and } \ \theta^{-1}\psi(\theta x') < x_n < \theta^{-1}\psi(\theta x') + 2\theta^{-1}\right\}. \tag{3.26}$$

Let $\psi_\theta(x') = \theta^{-1}\psi(\theta x')$. Note that $\|\nabla\psi_\theta\|_\infty = \|\nabla\psi\|_\infty$ and $\psi_\theta(0) = 0$. It follows from Lemma 3.4 that

$$\begin{aligned} &\int_0^{t_0} \int_{|x'|<1} |v(x', \theta^{-1}\psi(\theta x') + s)|^{\bar{p}} dx' ds \\ &\qquad \le t_0^{\bar{p}+\alpha} \int_0^2 \int_{|x'|<2} |v(x', \theta^{-1}\psi(\theta x') + s)|^{\bar{p}} dx' ds. \end{aligned} \tag{3.27}$$

By a change of variables, this gives

$$\begin{aligned} &\int_0^{\theta t_0} \int_{|x'|<\theta} |u_\varepsilon(x', \psi(x') + s)|^{\bar{p}} dx' ds \\ &\qquad \le t_0^{\bar{p}+\alpha} \int_0^{2\theta} \int_{|x'|<2\theta} |u_\varepsilon(x', \psi(x') + s)|^{\bar{p}} dx' ds. \end{aligned} \tag{3.28}$$

By covering $\{x' : |x'| < 1 + \theta\}$ with balls of radius θ, we may deduce from (3.28) that

$$\begin{aligned} &\int_0^{\theta t_0} \int_{|x'|<1+\theta} |u_\varepsilon(x', \psi(x') + s)|^{\bar{p}} dx' ds \\ &\qquad \le C_n t_0^{\bar{p}+\alpha} \int_0^{2\theta} \int_{|x'|<1+2\theta} |u_\varepsilon(x', \psi(x') + s)|^{\bar{p}} dx' ds, \end{aligned} \tag{3.29}$$

where C_n depends only on n. Now suppose that $t_0^{k+1}\varepsilon_0 < \varepsilon \le t_0^k \varepsilon_0$ for some $k \ge 1$. Since $\varepsilon/t_0^j \le t_0^{k-j}\varepsilon_0$, it follows from (3.29) with $\theta = t_0^j$ that

$$\begin{aligned}\int_0^{t_0^{j+1}} \int_{|x'|<1+t_0^j} &|u_\varepsilon(x',\psi(x')+s)|^{\bar p} dx' ds \\ &\le C_n t_0^{\bar p+\alpha} \int_0^{t_0^{j-1}} \int_{|x'|<1+t_0^{j-2}} |u_\varepsilon(x',\psi(x')+s)|^{\bar p} dx' ds,\end{aligned} \tag{3.30}$$

for $j = 1, \dots, k$. By choosing t_0 small we may assume that $C_n t^{\alpha-\alpha_0} \le 1$ for some $0 < \alpha_0 < \alpha$. This implies that

$$\begin{aligned}\int_0^{t_0^j} \int_{|x'|<1} &|u_\varepsilon(x',\psi(x')+s)|^{\bar p} dx' ds \\ &\le C(C_n t_0^{\bar p+\alpha})^j \int_0^2 \int_{|x'|<2} |u_\varepsilon(x',\psi(x')+s)|^{\bar p} dx' ds \\ &\le C(t_0^j)^{\bar p+\alpha_0} \int_0^2 \int_{|x'|<2} |u_\varepsilon(x',\psi(x')+s)|^{\bar p} dx' ds,\end{aligned} \tag{3.31}$$

for $j = 1, \dots, k$, where $t_0^k \approx \varepsilon/\varepsilon_0$. It is not hard to see that this yields the desired estimate (3.12) for $r = 1$. □

4. The case of C^1 boundary

Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a C^1 function with compact support such that $\psi(0) = |\nabla\psi(0)| = 0$. Suppose that $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in D_{16r} and $u_\varepsilon = 0$ in Δ_{16r}. Then for any $2 < p < \infty$,

$$\left\{ \frac{1}{r^n} \int_{D_r} |\nabla u_\varepsilon|^p dx \right\}^{1/p} \le C \left\{ \frac{1}{r^n} \int_{D_{16r}} |\nabla u_\varepsilon|^2 dx \right\}^{1/2}, \tag{4.1}$$

where C depends on μ, n, p, $\omega(t)$ as well as on the modulus of continuity of $\nabla\psi$,

$$\eta(t) = \sup\big\{ |\nabla\psi(x') - \nabla\psi(y')| : \ |x'-y'| < t \big\}. \tag{4.2}$$

The proof of (4.1) follows the same line of argument as in the case of Lipschitz boundary. We remark that the $W^{1,p}$ boundary estimates for operators with VMO coefficients hold on C^1 domains for any $2 < p < \infty$ (see e.g. [AQ,S]). One also has a stronger Hölder estimate

$$|u_0(x',\psi(x')+s)| \le C\, s^\beta \int_{D_{3/2}} |u_0| dy \tag{4.3}$$

for any $0 < \beta < 1$, where u_0 is a solution of an elliptic equation with constant coefficients $\mathcal{L}_0 u_0 = 0$ in $D_{3/2}$ and $u_0 = 0$ on $\Delta_{3/2}$. This, together with (3.16), shows that for any $2 < p < \infty$, $0 < t < 1$ and $0 < \alpha < 1$,

$$\int_0^t \int_{|x'|<1} |u_0(x',\psi(x')+s)|^p dx' ds \le C t^{p+\alpha} \int_0^{\frac{3}{2}} \int_{|x'|<\frac{3}{2}} |u_0(x',\psi(x')+s)|^p dx' ds \tag{4.4}$$

where $C > 0$ depends only on μ, n, p, α and function $\eta(t)$. The rest of the proof is exactly the same as in the Lipschitz case. We omit the details.

It worths pointing out that by Sobolev imbedding, estimate (4.1) implies the uniform boundary C^α estimate for any $0 < \alpha < 1$,

$$|u_\varepsilon(x)| \le C \left(\frac{d(x)}{r}\right)^\alpha \left\{ \frac{1}{r^n} \int_{D_{2r}} |u_\varepsilon(y)|^2 dy \right\}^{1/2} \tag{4.5}$$

for $x \in D_r$, where $C > 0$ depends on μ, n, α as well as on functions $\omega(t)$ and $\eta(t)$.

5. Proof of Theorem 1.1 and Corollary 1.2

By uniform interior $W^{1,p}$ estimates in section 2 and uniform boundary estimates in section 3, the weak reverse Hölder inequality (1.9) holds for $\bar{p} = 3$ in the case $n \ge 3$, and for $\bar{p} = 4$ in the case $n = 2$. If Ω is a C^1 domain, the interior estimates in section 2 and boundary estimates in section 4 give (1.9) for any $2 < \bar{p} < \infty$. It then follows from Theorem 1.3 that the Riesz transforms $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ are uniformly bounded on $L^p(\Omega)$ for $1 < p < 3 + \delta$ in the case $n \ge 3$, and for $1 < p < 4 + \delta$ in the case $n = 2$. If Ω is a C^1 domain, the Riesz transforms are uniformly bounded on $L^p(\Omega)$ for any $1 < p < \infty$. This completes the proof of Theorem 1.1.

Let $q > 2$. Suppose that $\nabla(\mathcal{L}_\varepsilon)^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < q$. By duality it follows that $(\mathcal{L}_\varepsilon)^{-1/2}\nabla$ is bounded on $L^p(\Omega)$ for $q' < p < \infty$. Consequently, $\nabla(\mathcal{L}_\varepsilon)^{-1}\text{div}$ is bounded on $L^p(\Omega)$ for any $q' < p < q$. Corollary 1.2 follows from this and Theorem 1.1.

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Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA.

E-mail address: `shenz@ms.uky.edu`