Bhaskara and Samçlishta Kuttaka.

In a paper published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XVII, (1926), pp. 89—98, Mr. Saradakanta Ganguly draws attention to a stray rule regarding the solution of the general case of Simultaneous Indeterminate Equations of the first degree, found in two palm-leaf manuscript copies of *Lilavati*, with commentary in Oriya characters and also in two others of the same text in Andhra characters. In Mr. Ganguly's opinion, Bhaskara was really the author of these rules. This is however a matter of doubt. In this note we propose to re-open the question of genuineness of Bhaskara's authorship by examining the rule in the light of other Indian solutions of the same problem and also in the light of Indian mathematical tradition.

We adopt the modern notation to describe the rules and problems in question and refer the reader to Mr. Ganguly's essay for the literal rendering of the textual matter.

The rule for the solution of the general case of simultaneous indeterminate equations mentioned in the manuscripts aforesaid, is known as संश्लिष्ट बहुसामान्य कुट्टकसूत्रम् , which, for brevity, we may call Samçlishta Sutra.

It is interesting to note that the earliest Indian attempt of indeterminate analysis takes the form of solving simultaneous congruences of the type

$$x \equiv a_1 \ (\text{mod. } m_1)$$

$$x \equiv a_2 \ (\text{mod. } m_2)$$

$$x \equiv a_3 \ (\text{mod. } m_3),$$

$$\cdots \qquad \cdots \qquad \cdots$$

For, though Aryabhata mentions only the case of two congruences, the rules given by him can be made to serve for more than two cases also by repeating the process. This is actually done by Mahaviracharya, (vide *Ganitasara Sangraha*, verses $127\frac{1}{2}$ —$129\frac{1}{2}$, pp. 81-82, Text). Even if the intention of the original authors be doubtful, the commentators make it perfectly plain that in case the congruences are more than two, the solution of the first two congruences may be combined with the third just in the same way as the first two congruences, to obtain a solution satisfying all the three congruences, this process being capable of extension to any number of them. Further, it is in accordance with Indian tradition, that authors of original scientific treatises are generally very brief, their aim being to indicate the general outline of procedure and leave the details to be worked out by the reader. To quote Bhaskara :—

उपदेशलवं शास्त्रं कुरुते धीमतो यतः ।
तत्तु प्राप्यैव विस्तारं स्वयमेवोपगच्छति ॥
जले तैलं खले गुह्यं पात्रे दानं मनागपि ।
प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ॥

"A little instruction and guidance in science is sufficient for the intelligent student, for this alone will help him to develop his knowledge of his own accord. Science instilled into the intelligent mind has sufficient vitality in it to grow and expand by its own force even as a drop of oil on a sheet of water, a piece of secret confided to a villain, or a little act of charity to the deserving person."

The Samçlishta Sutra* which Mr. Ganguly attributes to Bhaskara amounts in effect to this:

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* The following is the text of the Sutra found in the palm-leaf manuscript (No 1885), Oriental Library, Mysore.

संश्लिष्ट बहुसामान्य कुट्टकसूत्रम् ।
हारे विभिन्ने गुणके च भिन्ने स्यादाद्य राशे र्गुणकस्तु साध्यः ।
द्वितीय भाज्यघ्न गुणः स आद्यः क्षेपोभवेत्क्षेपयुतो द्वितीयः ॥
द्वितीय भाज्यघ्न तदाद्य हारो भाज्यो भवेत्तत्र हरो हरः स्यात् ।
एवं प्रकल्प्यापि च कुट्टके ऽथ जातो गुणश्चाद्यहरेण निघ्नः ॥
गुणो भवेदाद्य गुणेन युक्तः हरघ्नहारोत्नहर प्रादिष्टः ।
अथ तृतीयेपि तथैव कुर्यात् एवं बहूनामपि साधयीत ॥

To solve the congruences:

$$a_1 x \equiv b_1 \ (\text{mod. } m_1)$$
$$a_2 x \equiv b_2 \ (\text{mod. } m_2)$$
$$\ldots \quad \ldots \quad \ldots$$

Let the solution of the first congruence be $x = k_1$ (mod. m_1)

i.e., $$x = k_1 + tm_1 \ \text{(say)}.$$

Substituting this in the second, we get

$$a_2 k_1 + a_2 tm_1 \equiv b_2 \ (\text{mod. } m_2)$$

i.e., $$a_2 m_1 t \equiv b_2 - a_2 k_1 \ (\text{mod. } m_2)$$

where t is the unknown to be determined.

Let the solution of this congruence be $t \equiv t_1$ (mod. m_2).

Then $$x = k_1 + m_1 t_1 \quad (\text{mod. } m_1 m_2)$$
$$= k_1 + m_1 t_1 + u\, m_1 m_2 \quad \text{(say)}$$

This again may be substituted in the third congruence and solved for u and so on.

There is nothing striking or original in this method and it will obviously suggest itself to any one who knows how to solve the first congruence. It is a mere repetition of a process with no new principle involved and does not justify an explicit enunciation by a mathematician of the attainments of Bhaskara. While Aryabhata, Brahmagupta and Mahaviracharya had thought it fit to suggest similar obvious extensions of their method as being implied in the particular cases discussed by them, it is at least strange that Bhaskara who is content with उपदेशलव should deem it proper to dwell on it at length. If he should have really done so, he has gone out of the way to play the role of the prolix commentator. It is, therefore, very likely that a commentator has interpolated it in Bhaskara's text.

Mr. Ganguly suggests that this saṃçlishta rule must have been felt difficult in those days, and as it had no application in astronomy, readers might not be inclined to waste their time in learning and copying the rule, etc. In my opinion, the omission is not due to the difficulty for there is none in it, but may be due to its being too obvious for explicit mention. In this conection it must be noticed that other Hindu mathematicians have not neglected the problem on account of its difficulty but have shown their full appreciation by suggesting alternative methods of solution. Aryabhata was the first to hint the general problem of finding a number having given residues with respect to given moduli. Mahaviracharya who lived more than 250 years before Bhaskara has given a rule different from the samçlishta rule. He solves the congruences $a_1 x \equiv b_1$ (mod, m_1), $a_2 x \equiv b_2$ (mod. m_2), etc. separately and obtains $x \equiv k_1$ (mod m_1), $x \equiv k_2$ (mod m_2)... etc. which he manipulates in Aryabhata's fashion to get the final result. This is certainly more original and natural than the samçlishta rule and is also the accepted modern method. Bhaskara indeed goes further and suggests a novel method of solving simultaneous equations in the apparently particular case of the same modulus for all congruences. There is no loss of generality in this uniform modulus, for we can replace the given set of congruences by the equivalent set

$$\frac{m}{m_1} a_1 x \equiv \frac{m}{m_1} b_1 \text{ (mod } m), \quad \frac{m}{m_2} a_2 x = \frac{m}{m_2} b_2 \text{ (mod } m) \ \ldots\ldots$$

where m is the L. C M. of m_1 m_2, ... We shall deal with this aspect of Bhaskara's Samçlishta Kuttaka* in some detail presently. Again, one of the commentators of Aryabhata in a work Kuttakara Çiromani† explicitly devoted to Kuttaka problems distinguishes three varieties:

(1) that which has a single modulus but different residues;

(2) that in which there is only one dividend co-efficient but different divisors as well as different residues;

(3) that in which the dividend co-efficients, the divisors and residues are all different.

The rule given for the last case is just the same as Mahaviracharya's.

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* Samçlishta kuttaka is different from the spurious samçlishta rule.

† This is a work of one Devaraja; it is meant to be a commentary and supplement to Aryabhata's Kuttaka Sutram. Four manuscript copies of this work are available in the Oriental Library, Mysore.

Brahmagupta also suggests the same method. Thus, nowhere does the spurious samçlishta rule seem to be recognised; at any rate, even if recognised, probably it is not considered more worthy of adoption than the other one.

It is wrong to argue that the omission of the samçlishta rule in many of the texts of *Lilavati* and *Bijaganita* as well as the absence of reference to it in other Hindu mathematical texts, is due to its difficulty. For more difficult rules such as the Chakravala or the cyclic method of solving indeterminate equations of the second degree have not been omitted but found their due place among all the known manuscript copies of *Siddhanta Çiromani*. As regards the application to astronomy, it is not difficult to devise one, when applications have been found for even the Chakravala in Indian astronomy. In fact Brahmagupta gives an important astronomical application, *viz.*, 'Given the residual revolutions of different planets, to find the elapsed number of days.'

So much for the spurious rule attributed to Bhaskara. Apart from this, there is a well-known rule for Samçlishta Kuttaka, the importance and significance of which have not been sufficiently recognised. According to Mr. Ganguly, the wording of this well-known rule strongly suggests that Bhaskara must have considered the case of the spurious rule, without which the chapter would remain incomplete. The invalidity of this argument can be felt forcibly when applied to a similar rule * given by the younger Aryabhata without any discussion of the case of different divisors, corresponding to the spurious rule Unlike the spurious rule, this well-known rule is very popular and copied by Bhaskara's successors. There is something deeper in the rule than what appears at first sight. It furnishes a novel method of solving simultaneous congruence equations, with a common modulus. As we have already pointed out, the common modulus does not in any way make the method lose its generality. But a note of warning is necessary. Bhaskara has not done him-

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* गुणकैक्यं संश्लिष्टे भाज्यः शेषैक्यकं भवेत् क्षेपः ।
तुल्यच्छेदे कर्म मन्दार्थं कथ्यते विततः ॥

Maha Siddhanta of Aryabhata.

This is just a paraphrase of Bhaskara's एको हरश्चेत्........

self justice in this rule, for he does not point out the cases of failure though it is his usual custom † to point out खिल cases.

The point is this: If we have a system of congruences

$$\left.\begin{array}{l} a_1 x \equiv b_1 \\ a_2 x \equiv b_2 \\ a_3 x \equiv b_3 \\ \dots \quad \dots \end{array}\right\} \text{(mod. } m) \quad \dots \qquad \dots \quad (\alpha)$$

any linear combination

$$\Sigma\, l_r\, a_r\, x \equiv \Sigma\, l_r\, b_r \text{ (mod. } m) \quad \dots \qquad \dots \quad (\beta)$$

gives the value of x which satisfies all the given congruences, provided $\Sigma\, l_r\, a_r$ is prime to m and the congruences are consistent. While Bhaskara has recognised the principle of the linear combination, he has failed to note the limitation about $\Sigma\, l_r a_r$ as well as the consistence of the congruences. Bhaskara's method consists in using unity for each of $l_1, l_2, \dots$ and solving the representative congruence $\Sigma\, a_r\, x \equiv \Sigma\, b_r$ (mod. m). Bhaskara's rule is particularly serviceable in the congruences

$$a_1 x \equiv b_1 \bmod (m_1),\ a_2\, x \equiv b_2 \bmod (m_2),\ \dots \dots$$

where $m_1, m_2, \dots$ are prime to each other and m_1 is prime to a_1, m_2 to a_2 and so on. Thus, taking the illustration for the spurious rule,

$$7x \equiv 3 \text{ (mod. } 62); \ 6x \equiv 5 \text{ (mod. } 101); \ 8x \equiv 9 \text{ (mod. } 17)$$

we may combine them into a single representative congruence

$$\begin{aligned} (7 \times 17 \times 101 + 6 \times 17 \times 62 + 8 \times 62 \times 101)\, x \\ \equiv 3 \times 17 \times 101 + 5 \times 17 \times 62 + 9 \times 62 \times 101 \\ \text{(mod. } 62 \times 101 \times 17) \end{aligned}$$

i.e. $\quad 68439x \equiv 66779 \text{ (mod. } 106454)$

which yields $\quad x \equiv 70583 \text{ (mod. } 106454).$

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† cf: येनच्छिन्नौ भाज्य हारौ न तेन क्षेपश्चैतद्दुष्टमुद्दिष्टमेव ।

or again

रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणोनचेत् ।

This shows again how this rule about Samclishta Kuttaka covers also the case of the spurious rule and the completeness of the chapter does not suffer by the omission of the latter.

We may conveniently suppose, without loss of generality, that the H. C. F. of $a_1, a_2, \ldots$ in the congruences (α) is prime to m, for if there be a common factor between the H, C. F. and m, the whole congruence may be divided out by this common factor. If the congruences are consistent, there is only one value of x less than m which satisfies all of them. If $\Sigma\, l_r a_r$ and m have the highest common factor, say d, then there are m/d values ($< m$) for the congruence (β), only one of which is bound to satisfy (α) and we have to pick out this value by trial.

It is important to note that the least value of (β) need not always satisfy (α) except when $\Sigma\, l_r a_r$ is prime to m. In Bhaskara's example, the least value of the representative congruence $15x \equiv 21$ (mod. 63) or $5x \equiv 7$ (mod. 21) seems to satisfy the given congruences

$$\left.\begin{aligned} 5x &\equiv 7 \\ 10x &\equiv 14 \end{aligned}\right\} \text{(mod. 63).}$$

which is a matter of chance and has perhaps led to Bhaskara's oversight in assuming probably that every solution of the new congruence is the required one.

If the given congruences be

$$\left.\begin{aligned} 5x &\equiv 7 \\ 30x &\equiv 42 \end{aligned}\right\} \text{(mod. 63).}$$

Bhaskara's method will give $35x \equiv 49$ (mod. 63)

or $5x \equiv 7$ (mod. 9) as the representative congruence with $x \equiv 5$ as the least value. This evidently does not satisfy the given congruences. We have to choose the next higher value 14, which suits them. Ganésa in his commentary *Buddhivilasini* of *Lilavati* has struck this note of warning: The quotient as it comes out in this operation is not to be taken; but it is to be separately sought with the several original multipliers applied to this quantity and divided by the divisor as given.

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