MA214
011003

Solving for y_p

1. **Basic formulas** Learn, memorize and understand the following derivative formulas:

$$f(D)e^{at}v = e^{at}f(D+a)y\ ,\ f(D)tv = tf(D)v + f'(D)v$$

and more generally

$$f(D)t^r v = t^r f(D)v + rt^{r-1}f'(D)v + r(r-1)/2t^{r-2}f''(D)v + \cdots$$

Here $f(D)$ denotes a polynomial in the operator D, f', f'' etc. denote its derivatives as polynomials and so on.

You should also memorize/observe the operators which kill a desired function. Thus we have:

$f(D)t^s$ is zero if D^{s+1} divides $f(D)$.

$f(D)e^{at} = 0$ if $(D-a)$ divides $f(D)$.

$f(D)\sin(at) = 0$ if $D^2 + a^2$ divides $f(D)$. Ditto for $\cos(at)$.

$f(D)e^{at}t = 0$ if $f(D)$ is divisible by $(D-a)^2$. Find similar other results.

2. **Superposition** If the right hand side has several terms, then you can take one term at a time, solve and then combine the final answers together.

3. **Step 1: removing the exponential.** To solve $f(D)y = e^{at}g(t)$ first make a substitution $y = e^{at}v$.Then you have to solve $f(D+a)v = g(t)$ for v and plug back this solution in the formula for y. Thus, the factor e^{at} can be removed!

4. **Step 2: solving for polynomial on the right hand side** We want to solve $f(D)y = g(t)$ where $g(t)$ is a polynomial of degree d. First factor out as much power of D from $f(D)$ as possible, i.e. write $f(D) = u(D)D^r$ so that $u(0) \neq 0$. Further by dividing by $u(0)$ on both sides, assume (arrange) that $u(0) = 1$, so that $u(D) = 1 - h(D)$ for some $h(D)$ such that $h(0) = 0$.

Now set $w = D^r y$. The equation to solve first is

$$(1 - h(D))w = g(t).$$

The solution is simply given by the formula:

$$w = (1 + h(D) + h(D)^2 + \cdots)g(D)$$

where we only take enough powers of $h(D)$ so that we get zero by acting on $g(t)$. In any case, at most d powers are needed!

Finally, we find our y by solving $D^r y = w$. This amounts to integrating (the polynomial) w r times!

5. **Step 3: solving for $t^r \sin(at)$ or $t^r \cos(at)$ on the r.h.s.**

Here, there are two methods. One is to replace the $\sin(at)$ or $\cos(at)$ terms by e^{ait} and working as above, except with complex numbers. Finally, the original problem is solved by taking appropriate real/imaginary parts of the answer. This method, though elegant, can be difficult for people not good with complex algebra. So, the method of undetermined coefficients as in the book may work better in practice. There are, still shortcuts to the calculations which will be explained in class.