A New Continued Fraction.

THE so-called half-regular continued fraction (C.F.) is defined by

$$b_0 + \frac{\epsilon_1}{b_1 +} \frac{\epsilon_2}{b_2 +} \cdots \frac{\epsilon_n}{+ b_n +} \frac{\epsilon_{n+1}}{b_{n+1}} + \cdots$$

when $|\epsilon_n| = 1$, b_0 is any integer, b_n a positive integer $(n \geqslant 1)$, $b_n + \epsilon_{n+1} \geqslant 1$ $(n \geqslant 1)$ with $b_n \geqslant 2$ if final; it includes the simple continued fraction as a particular case. Many types of such continued fractions have been studied in the last fifty years, notably by Hurwitz, Minkowski, Vahlen and Tietze. Among them is a special variety designated 'Canonical' by Perron, in which the aggregate of complete quotients is characterised by the property of being greater than $\frac{\sqrt{5}+1}{2}$.

Of the various half-regular C.F. developments of quadratic surds, special interest may be attached to those which are periodic and contain the least number of recurring elements. The first suggestion of such a development came from Minnegerode in 1873. We here propose a new C.F. development which is distinct from Minnegerode's and is applicable only to quadratic surds, resulting in a new periodic half-regular C.F. with the least number of recurring elements. It is remarkable that such a development should be implied in the method sketched by the famous mediæval Indian mathematician Bhaskara (born 1114 A.D.) for obtaining the integral solutions of the indeterminate equation $X^2 - Ny^2 = 1$, commonly and wrongly attributed to Pell (1610–1685 A.D.). It had been thought erroneously by various writers that Bhaskara's method also gave rise to the simple continued fraction. The present announcement is, therefore, intended to correct this view and lay bare for the first time the true nature of the new continued fraction implied in Bhaskara's method.

2. This new development will now be briefly described and a few properties noted without proof. Details are reserved for later publication.

Let $\frac{P + \sqrt{R}}{Q}$ be a surd in the standard form and a' the greatest integer in it, R being a non-square positive integer, and P, Q being any integers.

Then $\dfrac{P + \sqrt{R}}{Q}$ may be represented in one of two forms

(I) $a' + \dfrac{Q'}{P' + \sqrt{R}}$ or (II) $a' + 1 - \dfrac{Q''}{P'' + \sqrt{R}}$

(P', Q', P'', Q'' being all integers); and we choose a' or $a' + 1$ as the partial quotient for the continued fraction development according as (i) $| Q' | \lessgtr | Q'' |$ if $| Q' | \neq | Q'' |$ and (ii) $Q \lessgtr 0$ if $|Q'| = | Q'' |$.

After making the appropriate choice of I or II, we write it in the form $b_0 + \dfrac{\epsilon_1 Q_1}{P_1 + \sqrt{R}}$ where $| \epsilon_1 | = 1$ and $0 < \dfrac{Q_1}{P_1 + \sqrt{R}} < 1$; and proceed similarly with $\dfrac{P_1 + \sqrt{R}}{Q_1}$ for determining the next partial quotient, and so on.

Since it is easily seen that

$$\left| \frac{P'^2 - R}{P''^2 - R} \right| = \left| \frac{Q'}{Q''} \right| \text{ and therefore}$$

$$\left| P'^2 - R \right| \lessgtr \left| P''^2 - R \right| \text{ according as } \left| Q' \right| \lessgtr \left| Q'' \right|,$$

our condition implies a choice of the nearest square to R and so, the above development may be justifiably called '*The nearest square continued fraction*' or *the Bhaskara continued fraction* on account of its being implicit in his method.

3. *Properties :*

(i) The nearest square C.F. development of a quadratic surd gives a periodic half-regular continued fraction in which the cyclic part is 'canonical' and the number of recurring elements is never greater than that in any other periodic half-regular continued fraction development. The number of terms in the acyclic part also cannot exceed the corresponding number in the simple continued fraction development by more than 3.

(ii) The square-root of a positive rational number, greater than 1 and not a perfect square, when developed as the nearest square C.F., has a period which begins immediately after the first term and ends with a partial quotient which is twice the first partial quotient, except when the rational number is of the form $n^2 + n + \frac{1}{2}$ (n being a positive integer). In the exceptional case, we have

$$\sqrt{n^2 + n + \tfrac{1}{2}} = n + 1 - \underset{\times}{\frac{1}{2}} + \underset{\times}{\frac{1}{(2n + 1)}}$$

[*N.B.*—This exceptional case is excluded in the properties (iii), (iv), (v) below.]

(iii) When the recurring cycle of the standard surd $\dfrac{\sqrt{R}}{Q}$ (> 1) in the new development does not contain a complete quotient of the form $\dfrac{p + q + \sqrt{R}}{p}$, where $R = p^2 + q^2$ and p, q integers such that $p \geqslant 2q > 0$, the properties of the cycle simulate those of the simple continued fraction.

(iv) When $R = p^2 + q^2$, and p, q integers such that $p \geqslant 2q > 0$, there cannot occur in the cyclic part of $\dfrac{\sqrt{R}}{Q}$ (> 1), Q being a positive integer, more than one complete quotient of the form $\dfrac{a + b + \sqrt{a^2 + b^2}}{a}$ where $R = a^2 + b^2$, $a \geqslant 2b > 0$; while

$$\frac{a + b + \sqrt{a^2 + b^2}}{a}$$

itself, when developed as the new C.F. will not have in its acyclic and cyclic part more than one remote or immediate successor of the same form but distinct from it.

Thus, it is possible to have *two* but *not more than two* equivalent surds of the type $\dfrac{p + q + \sqrt{p^2 + q^2}}{p}$ ($p > 2q > 0$).

For example, $\dfrac{487 + \sqrt{101^2 + 386^2}}{386}$ and $\dfrac{513 + \sqrt{139^2 + 374^2}}{374}$ are equivalent ($101^2 + 386^2 = 139^2 + 374^2$).

(v) If $\dfrac{\sqrt{R}}{Q}$ (> 1) has in its recurring period in the new development a complete quotient of the form $\dfrac{p + q + \sqrt{p^2 + q^2}}{p}$ [defined as in (iv) above], the cyclic part may be said to be '*almost*' symmetric being in the form

$$b_0 + \frac{\epsilon_1}{\underset{\times}{b_1}} + \cdots + \frac{\epsilon_v}{b_v} + \cdots + \frac{\epsilon_{v+r}}{b_{v+r}} + \cdots$$

$$+ \frac{\epsilon_{2v}}{\underset{\times}{2b_0}}, \text{ where}$$

$$b_v = 2, \ \epsilon_v = -1, \ b_{v-1} = b_{v+1} + 1,$$
$$\epsilon_{v+1} = 1 \text{ and}$$
$$b_{v+r} = b_{v-r}, \ \epsilon_{v+r} = \epsilon_{v-r+1} \ (v \geqslant r > 1)$$

which give the symmetries for b's and ϵ's.

(vi) The period of the square-root of a non-square positive integer has an even number of terms in the case contemplated in (v); and the denominators of the complete quotients up to the end of the period form a symmetric sequence of an odd number of terms with the middle term greater than 4.

Example: $\sqrt{58} = 8 - \frac{1}{\underset{\times}{3}} - \frac{1}{2} + \frac{1}{2} - \frac{1}{\underset{\times}{16}}$ and the denominators of 2nd, 3rd and 4th complete quotients can be verified to be 6, 7, 6 respectively.

4. Associated with this new continued fraction there is a theory of reduced quadratic forms, which remains to be fully worked out. The complete system of indefinite, primitive reduced forms for any given non-square positive integral determinant R consists of forms of the type (A, B, C) satisfying one of the following three sets of conditions:

(i) $B > 0, \ A^2 + \frac{1}{4} C^2 < R, \ C^2 + \frac{1}{4} A^2 < R$

(ii) $B = |C| + \frac{1}{2}|A| > \sqrt{R}, \ |A| < |C|,$
$C^2 + \frac{1}{4} A^2 = R$

(iii) $B = |A| - \frac{1}{2}|C| < \sqrt{R}, \ |A| > |C|,$
$A^2 + \frac{1}{4} C^2 = R$

If $A^2 + \frac{1}{4} C^2 = R = C^2 + \frac{1}{4} A^2$, the form (A, B, C) is not primitive except when $R = 5$.

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