DEPARTMENT OF MATHEMATICS

Ma162 SECOND EXAM Spring 2004
March 8, 2004

DO NOT TURN THIS PAGE UNTIL YOU ARE INSTRUCTED TO DO SO.

Be sure to show all work and justify your answers.
There are 5 problems and a total of 6 pages including this one. You are allowed the use of calculators.

Problem	Maximum Score	Actual Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

NAME: ____________________________
SECTION NO: ____________________________
STUDENT #: ____________________________

1. Consider a triangular region with vertices $A(3,1)$, $B(7,5)$, $C(0,5)$ in the plane.

   It is given that it is the region of feasible points of some set of inequalities. Answer the following questions.

   (a) Determine the inequality corresponding the to line AC satisfied by the region.

   $$y + 4x/3 \geq 5.$$

   (b) Determine the inequality corresponding the to line BC satisfied by the region.

   $$y \leq 5.$$

   (c) Is the inequality $y \geq 1$ satisfied by all the points of the region? Explain why or why not.
   Ans: Evaluate the function y at all the corners. The smallest value is 1. Since the region is bounded, there is no infinite limit to worry about. So, the answer is YES!

(d) Find a non constant linear function ($f(x, y) = ax + by$) which has a maximum value at A and C. We need something parallel to the line AC. Try $k(y + 4x/3)$. This has values $5k, 43k/3, 5k$ at the three points. If we try $k = 3$, then the values become $15, 43, 15$. This has a minimum along AC. If we want a maximum instead, we just take $k = -3$, i.e. the function is: $-(3y + 4x)$.The most general answer is $-p(3y + 4x)$ where $p > 0$.

2. You are given the inequalities:

$$y - x \leq 3 \ , \ x - y \leq 3 \ , \ 4y - x \leq 21 \ , \ 4x - y \leq 21.$$

Also, as usual, assume that

$$x \geq 0 \ , \ y \geq 0.$$

Determine the common point of $4y - x = 21, 4x - y = 21$. $(7, 7)$

Make a rough sketch of the region by plotting the corner points and bounding lines. Be sure to label all parts - points as well as lines. Be sure to plot your points with a reasonably accurate scale. **Ans:** It is a hexagon with corners $[[0, 0], [3, 0], [6, 3], [7, 7], [3, 6], [0, 3]]$.

Does the function $x + y + 1$ have a maximum on the region? If it does, be sure to identify the point where the maximum is achieved as well as the maximum value.

Does the function $x + y + 1$ have a minimum on the region? If it does, be sure to identify the point where the maximum is achieved as well as the maximum value.

The region being bounded has well defined max/min.

The values of the given function at the six points are: [1, 4, 10, 15, 10,4] and so the max and min are respectively $15, 1$ at the points $(7, 7), (0, 0)$.

3. Matt wishes to live on a diet of steak and potatoes but wishes to make sure he follows diet principles too. He has been told that a person should have at most 60 grams of fat and at least 40 grams of protein as well as at least 50 grams of carbohydrates. His chosen recipe of one steak serving supplies $5, 20, 15$ grams of carbohydrates, protein and fat respectively, while the potato serving supplies $15, 5, 2$ grams of the same nutrients respectively.

The cost per serving of steak is \$4, while that for potato is only \$2.

Set up a linear programming problem for Matt if he wishes to satisfy the dietary constraints and eat as cheaply as he can.

Be sure to identify the variables and record all the constraints.

Let x be number of steak servings and let y be the number of potato servings. Then the inequalities are:

Cost to minimize: $4x + 2y$.

Positivity: $x \geq 0, y \geq 0$.

Fat: $15x + 2y \leq 60$.

Protein: $20x + 5y \geq 40$.

Carbs: $5x + 15y \geq 50$.

4. You are given an intermediate simplex tableau associated with a linear programming problem.

$$M = \left[\begin{array}{ccccccc|c} x & y & z & s & t & u & P & \text{Constant} \\ \hline -2 & 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & -3 & -1 & 1 & 0 & 0 & 2 \\ 6 & 0 & 1 & 0 & 0 & 1 & 0 & 8 \\ \hline -4 & 0 & 7 & 3 & 0 & 0 & 1 & 4 \end{array}\right]$$

(a) Locate the pivot element and carry out the next iteration of the simplex method.
The pivot must be in column 1 due to -4 at the bottom and the ratios are $2/2, 8/6$ in rows $2, 3$ resp. So the pivot is in row 2 col. 1.
The algorithm produces a new matrix:

$$M = \left[\begin{array}{ccccccc|c} x & y & z & s & t & u & P & \text{Constant} \\ \hline 0 & 1 & -1 & 0 & 1 & 0 & 0 & 4 \\ 1 & 0 & -3/2 & -1/2 & 1/2 & 0 & 0 & 1 \\ 0 & 0 & 10 & 3 & -3 & 1 & 0 & 2 \\ \hline 0 & 0 & 1 & 1 & 2 & 0 & 1 & 8 \end{array}\right]$$

(b) Explain why your tableau after the above step is in final form.
No more negatives left in the bottom row, so we are done!

(c) Report the solution to the original primal problem using the final form of the tableau. You should be able to deduce the names of the original and the slack variables by followoing the usual convention.
Be sure to indicate the values of the variables and the maximum value.
Since there are clearly three inequalities, the last four columns on the left hand side (under s, t, u, P) must be the original slack variables (including the function variable P). So, the original variables are x, y, z.
x is a pivot variable in row 2 and has value 1 in the right hand side of row 2. y is a pivot variable in row 1 and hence has value 4 in the right hand side of row 1. z is non pivot and hence 0.
The maximum value of the function is 8 in the right hand side of the bottom row.

5. (a) You are given the following primal problem.

$$\text{Minimize } C = 5x + 8y + 8z.$$

Such that:

$$x + 2y + z \geq 2 \ , \ x + y + 2z \geq 3 \ , \ x \geq 0 \ , \ y \geq 0 \ , \ z \geq 0.$$

Construct the dual problem associated with the given primal problem. Be sure to identify the variables and list all the inequalities.
Make a tabular form while writing the coefficients vertically - since the given problem is a minimization. (We would write horizontally for a maximization.)

$$M = \left[\begin{array}{cc|c} 1 & 1 & 5 \\ \hline 2 & 1 & 8 \\ \hline 1 & 2 & 8 \\ \hline 2 & 3 & 0 \end{array}\right]$$

This says that there are two variables (two columns on the left) and three inequalities. Let the variables be x, y. Then we must have the dual problem:

$$x, y \geq 0, x + y \leq 5, 2x + y \leq 8, x + 2y \leq 8 \text{ Maximize } 2x + 3y.$$

This is not asked, but for your understanding, here is the set up of the initial tableau. Do note the negative signs in the bottom row!

$$\left[\begin{array}{cccccc|r} x & y & u & v & w & P & \text{Constant} \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 5 \\ 2 & 1 & 0 & 1 & 0 & 0 & 8 \\ 1 & 2 & 0 & 0 & 1 & 0 & 8 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

(b) You are given the following final tableau corresponding to a maximization problem, with the usual convention about the original and slack variables.

Identify the variables for the dual minimization problem and state its solution.

$$M = \left[\begin{array}{cccccc|r} u & v & x & y & z & P & \text{Constant} \\ \hline 1 & 0 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & -3 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 3 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 4 \end{array}\right]$$

By the usual convention, the dual variables must correspond to the columns of x, y, z with P related to the function.

The dual values are simply read in the bottom row $x = 1, y = 0, z = 1$ with the minimum value of the function equal to 4.