

Understanding monetary transactions.

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The Simple Interest.

- Interest is **rent on borrowed money**.
We use the **notation P for principal**. This is the amount of money borrowed (or lent).
- We say **r is the interest rate**, if it is the agreed interest (rent) on one dollar per year. Of course, we change the name of the currency as appropriate!
The rate is often quoted as something like 7% which means $r = \frac{7}{100} = 0.07$. Thus the words “**per cent**” mathematically mean the fraction $\frac{1}{100}$.
- We shall typically let t denote the period of lending in years and thus the interest I accumulated in t years on a principal of P dollars at a rate r is given by the formula:

$$I = Prt.$$

Simple Interest continued.

- Thus after the t years, the total amount owed is the original principal P plus the interest I and thus has the formula;

$$\text{Accumulation } A = P + Prt = P(1 + rt).$$

- If we know three of the four quantities A, P, r, t then we can find the fourth. We should learn to recognize what is given and what is unknown.
- **Example 1.** If you invest \$770.91 at 8% simple interest, how much will your investment be worth in 15 months?
- We note that $r = 8\%$ or $r = 0.08$. Also $P = 770.91$. We are given $t = 15$ months which must be converted to years and thus $t = \frac{15}{12} = 1.25$. We are looking for A , the net accumulated value of the investment. Hence

$$A = 770.91(1 + 0.08 \cdot 1.25) = 848.001. \text{ or } \$848.$$

More examples.

- **Example 2.** If you invest \$1127.52 and after 18 months it is worth \$1229.00, what simple interest rate, expressed as a percentage and rounded to .01, did you receive?
- We are given $P = 1127.52$, $t = \frac{18}{12} = 1.5$ and $A = 1229$. We want r .

We recommend solving the formula $A = P(1 + rt)$ for r and then evaluating it. Thus:

$$r = \frac{\frac{A}{P} - 1}{t} = \frac{\frac{1229}{1127.52} - 1}{1.5}.$$

The answer comes out $r = 0.06000189206$. We multiply by 100 to make a per cent rate and report $r = 6\%$ after rounding.

Examples Continued.

- **Example 3.** If you invest \$1520.88 at 6% simple interest, after how many months, rounded to 0.01, will your investment be worth \$1657.00?
- We are given $P = 1520.88$, $r = 0.06$, $A = 1657$ and asked to find t .

As before, we solve our formula for t and evaluate:

$$t = \frac{\frac{A}{P} - 1}{r} = \frac{\frac{1657}{1520.88} - 1}{0.06}.$$

This gives $t = 1.491680255$. Be sure to multiply by 12 to make months. So the answer is 17.90016306 or 17.9 months after rounding.

A More Complicated Example.

- **Example 4.** Homer won a prize in the lottery of \$3000, payable \$1500 immediately and \$1500 plus 4% simple interest payable in 260 days. Getting impatient, Homer sells the promissory note to Moe for \$1440 cash after 170 days. Using a nominal 360 day year, find the simple interest rate, rounded to 0.01, earned by Moe.

- This is a simple interest problem of finding r , but needs careful set up. If Homer were to patiently wait the 260 days, he would earn

$$A = P(1 + rt) = 1500(1 + 0.04 \cdot \frac{260}{360}) = 1543.33 \text{ dollars} .$$

- **From Moe's perspective**, this is his A after a lending of \$1440 for a period of $260 - 170 = 90$ days. Thus for Moe, the calculated interest rate as in Example 2 is

$$r = \frac{\frac{A}{P} - 1}{t} = \frac{\frac{1543.33}{1440} - 1}{\frac{90}{360}} = 0.2870370361 = 28.7\% .$$

Such high rates are not uncommon for short term lenders!

The Greedy Lender.

- Suppose you lend somebody \$100 for a period of one year at 10% interest rate. You will receive an accumulated payback of \$110 at the end of the year.

But if you demand a repayment in six months, you will be entitled to receive \$105. Now, suppose you lend this total amount back to the borrower, then using the usual formula with $P = 105$, $r = 0.10$, $t = 0.5$ we get

$$105(1 + 0.10 \cdot 0.5) = 110.25 \text{ dollars!}$$

It is easy to see that the net formula is $100(1 + \frac{0.10}{2})^2$.

- Of course, you don't really want to carry out the transaction, just demand the money. This is called the accumulation by **compounding every six months or twice a year!**

Thus a greedy lender can claim more money by simply “imagining” a transaction!

The Compound Interest.

- If one gets greedier and imagines compounding the interest m times a year then it is easy to see that in each of the m periods, we get the accumulation by multiplying the starting principal for that period by $(1 + \frac{r}{m})$ and thus the full formula for the interest after one year is:

$$A = P \left(1 + \frac{r}{m}\right)^m .$$

- It is useful to develop some **new notation**.

Assume that we are **compounding m times each year**. Thus in t years, we shall have mt periods of compounding and we define:

Periodic interest rate $i = \frac{r}{m}$ and Total term of loan in periods $n = tm$.

- This gives us the formula for accumulated amount when we compound m times a year as:

This Greed has a Limit!

- Continuing our example of lending \$100 for one year at a rate of 10%. If we compound it m times a year, then we have the formula $A_m = 100 \left(1 + \frac{0.10}{m}\right)^m$.
- We can calculate the accumulation for different values of m :

m	1	11	51	101
A_m	110.00	110.46717	110.50627	110.51162

- Thus, though increasing, it is not growing very fast. Indeed, using techniques of algebra it is possible to show that the limit of the quantity A_m as m goes to infinity is a famous function of mathematics, namely

$$\lim_{m \rightarrow \infty} P \left(1 + \frac{r}{m}\right)^m = P \exp(r).$$

Thus, even if we imagine infinite compounding, our accumulation for the above $P = 100, r = 10\%$ is only $100 \exp(0.10) = 110.5170918$.

The Compound Interest Formulas.

- To summarize, we have the formula that for principal P , annual rate r , period t years and compounded m times a year, we have

$$A_m = P(1 + i)^n \text{ where } i = \frac{r}{m}, n = tm.$$

- We describe the idea of infinite compounding as **continuous compounding**. The accumulation if we **compound continuously** is given by the formula:

$$A_C = P \exp(rt).$$

- Typically, we just write A for accumulation, but mention the value of m in words.

Examples of Compound Interest.

- **Example 5.** If you invest \$5000.00 at 9% compounded bi-weekly, how much will your investment be worth in 8 years?
- We have $P = 5000$, $r = 0.09$, $t = 8$. The meaning of the phrase bi-weekly is that it is compounded once every two weeks or $m = 26$ using a nominal year of 52 weeks. We have $i = \frac{0.09}{26} = 0.003461538462$ and $n = 8 \cdot 26 = 208$. Thus

$$A = 5000(1 + 0.003461538462)^{208} = 10259.40275.$$

- **Warning:** It is crucial to learn good calculator techniques here, since if you don't keep enough accuracy for i , then the power calculation introduces a lot of error and multiplication by a large P makes a very inaccurate amount. One should try not to copy down intermediate results, but store and reuse them for better accuracy!

More Examples.

- **Example 6.** How much did you invest at 8% compounded bi-weekly if 15 years later the investment is worth \$97000.00?
- If the investment is P then our formula gives:

$$97000 = P \left(1 + \frac{0.08}{26}\right)^{(15 \cdot 26)}$$

which can be solved for P as:

$$P = 97000 \cdot \left(\left(1 + \frac{0.08}{26}\right)^{(-15 \cdot 26)} \right).$$

- This evaluates to 29269.71472. It is an excellent idea to double check that this value of P does generate the 97000 i.e.

$$29269.71472 \left(1 + \frac{0.08}{26}\right)^{15 \cdot 26} = 97000$$

within reasonable accuracy! The computer answer is 96999.99712.

Effective Rate.

- Often, lending terms are described by different rates and different number of compoundings per year. It is necessary to be able to compare them to decide which is a better rate.
- One way to do this is to find an **effective rate** r_{eff} , which is defined as a simple interest rate which will give the same yield as the given scheme.
- Thus, if we invest one dollar at $r\%$ annual rate compounded m times a year, then our net yield is $(1 + \frac{r}{m})^m$ and if r_{eff} is to be the effective rate, then we have:

$$\left(1 + \frac{r}{m}\right)^m = 1 + r_{eff}$$

so we have the formula;

$$r_{eff} = \left(1 + \frac{r}{m}\right)^m - 1.$$

Example of Effective Rate.

- **Example 7.** Bank A is offering an interest rate of 6.60% compounded monthly, while bank B is offering an interest rate of 6.69% compounded quarterly.
What are the effective rates of the two banks expressed as percents and for the investor, which bank offers the better rate?
- We apply the formula for the effective rate to get:
The r_{eff} for bank A is: $\left(1 + \frac{0.066}{12}\right)^{12} - 1 = 0.06803356$
and the r_{eff} for bank B is: $\left(1 + \frac{0.0669}{4}\right)^4 - 1 = 0.06859714600$.
- The reported answers should be 6.80% and 6.86% respectively, with bank B declared as having a better rate.
- Note that if the problem was about borrowing from the bank instead of investing, then bank A would be a better choice!!

Preview.

- Next, we study the concepts of progression or a sequence and a series (or their sum).
Of special interest are the Arithmetic series and the Geometric series; **a must study** for all students of mathematics!
- Afterwards, we tackle a problem of annuity. Such problems have three types.
- We discuss how to borrow a large sum and pay back with periodic payments (mortgage).
- We discuss how much money can be withdrawn on a periodic basis from set up funds which are earning interest until drawn (sinking funds).
- We also discuss how to build up future reserves by periodic saving.