Lecture on section 2.3

Ma 162 Fall 2009

Ma 162 Fall 2009

September 9, 2009

Terminology for Gauss-Jordan Elimination.

We now explain the general Gauss-Jordan Elimination process. First, some definitions:

- let M be a matrix. In application, we expect it to be the augmented matrix of a system of linear equations, however, this is not relevant to our definitions.
  Assume that the matrix has m rows and n columns, so it is of type $m \times n$.
- A pivot in a row is the first non zero entry in it. The pivot position is the corresponding column number. The pivot position sequence (p.p. for short) is the sequence of such pivot positions in order.
  If the row is full of zeros, then we declare that it does not have a pivot and the pivot position is declared to be ∞.

- **Examples.** Consider:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

What are the pivots and the pivot positions?
Answer
For A, the p.p. is $(1, 2, 2)$ with pivots being $2, 1, 3$ respectively.
For B, the p.p. is $(2, 1, 3)$ with pivots being $-1, 1, 2$ respectively.
For C, the p.p. is $(1, 2, \infty)$ with pivots being $2, 5$ respectively.

Elementary Operations.

- A matrix is said to be in **Row Echelon Form or REF** if its p.p. is a strictly increasing sequence. **For this definition** we shall consider a sequence of ∞ to be a strictly increasing sequence!
- **Note:** In the above examples, C is in REF, while A, B are not.
- We allow two operations which help us put a matrix in REF.
- The first is a row swap. Thus, swapping the first and second row of B produces

$$B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \text{ p.p. is}(1, 2, 3)$$

The notation for this is $R_1 \leftrightarrow R_2$.

Elementary Operations continued.

- The second operation consists of adding some multiple of one row to another.
  For the matrix A, the p.p. is $(1, 2, 2)$ and we need to make the last row pivot position bigger to get REF.
- It is easy to see that subtracting $3R_2$ from R_3 does the trick. We shall write this operation as $R_3 - 3R_2$ with the convention that the first mentioned row is being replaced!
- The result is:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

The new p.p. is $(1, 2, 3)$ with pivots $(2, 1, -1)$.

Getting to REF

- Now we outline the REF procedure in general. First find the p.p. of the given matrix.
- Use row swaps as needed to get the pivot positions increasing, but they may not be strictly increasing yet! For example, the matrix A above had p.p. $(1, 2, 2)$.
- If two successive rows have the same pivot position, use the earlier row to push the pivot position of the latter row as described below.
- In the example of matrix A, the pivot in R_2 was 1, while the pivot in R_3 was 3 in the same column. Call this entry 3 to be the target, which needs to become zero!
- If the target is in R_i and pivot in R_j, then the operation can be described as $R_i - \frac{target}{pivot} R_j$.
  Here it becomes $R_3 - \frac{3}{1} R_2$ or $R_3 - 3R_2$.

RREF or alternate procedure after REF.

- We explained how to make an augmented matrix of a system of linear equations be in REF. We also explained how we can finish the solution process by a "back substitution" method. Now we explain an alternate procedure which is essential for the upcoming Simplex algorithm method.
  It also has the advantage that the final solution can be simply **read** from its display, without further manipulations.
- This form is called **Reduced Row Echelon Form or RREF**. The book calls it **the row-reduced form.**
- We shall first illustrate how to reach this RREF and then give its formal definition.

RREF example.

- To get RREF, we must first get REF. So, we shall start with the already worked example:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 0 & -3/5 & \frac{49}{5} \end{array}\right]$$

  Let us note that the vertical bar separator and the variable names are for understanding only and not part of calculations at this stage.

- We start with the pivot in the last row, namely, $-\frac{3}{5}$. The operation to perform is to make the pivot 1 by multiplying the row by a suitable number and then cleaning up all entries above it to zero.

RREF Example continued.

- The first operation is denoted as $-\frac{5}{3}R_3$ and gives a new matrix:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 0 & -3/5 & \frac{49}{5} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right]$$

- Next we cleanup the entry 2 above the pivot (now made 1) and this is done with $R_2 - 2R_3$.
- It yields:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 2 & 4 \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right]$$

Comtinued Example.

- Next, we cleanup the entry 2 in row 1 column 2. We use the pivot $-5/3$ so the operation shall be $R_1 - \frac{2}{-5/3} R_2 = R_1 + \frac{6}{5} R_2$.
- Thus we get:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 2 & 0 & 0 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 0 & 0 & 44 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right]$$

- Finally, we make all pivots 1, i.e. we make $\frac{1}{3} R_1$ and $-\frac{3}{5} R_2$. This produces:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 3 & 0 & 0 & 44 \\ 0 & -5/3 & 0 & \frac{110}{3} \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 1 & 0 & 0 & \frac{44}{3} \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right]$$

Reading the Answer.

- Recall that we have the final form:

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & RHS \\ 1 & 0 & 0 & \frac{44}{3} \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & -\frac{49}{3} \end{array}\right]$$

- Now we see the main advantage of RREF. The final solution is clearly visible on the RHS. To read the value of x_1, find the pivot under it and read off the value of RHS, namely $\frac{44}{3}$. Similarly, $x_2 = -22, x_3 = -\frac{49}{3}$.
- Thus, RREF takes care of the back substitution without writing the equations again.

How to solve equations in RREF.

- We now give an example of a system in RREF with infinitely many solutions.
  Consider an augmented matrix:

$$\left[\begin{array}{cccc|c} x & y & z & w & RHS \\ 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

- The pivot variables are x, y, w while z is a non pivot variable. Thus we solve the three equations in order for x, y, w in terms of the non pivot (or free) variable z.
- Thus, the answer is: $x = 2 - 2z, \; y = 5 - 3z, \; w = 3$. Here z is arbitrary.
- All we did was to move all the z terms to the RHS and then read off the solutions as before.

RREF defined.

- An augmented matrix is said to be in RREF if it is in REF and satisfies the following additional conditions:
  1. The pivot in each row is equal to 1.
  2. All entries in the pivot column below **or above** the pivot are equal to zero. The book describes this as the pivot column being a unit column, this being a typical column of the identity matrix.
- Of course, it can happen that in RREF, the equations are inconsistent. This happens when the pivot of some row only appears on RHS. Such equations have no solutions.

Some Treminology.

- The number of pivots in RREF only depends on the starting matrix and is called its rank.
- For obvious reasons, rank of a matrix is less than or equal to its number of rows as well as number of columns.
- The system either has no solutions (if an inconsistent equation is present) or a unique solution (if all variables are pivot variables) or infinitely many, if there is at least one non pivot variable.
  This is the so-called $0 - 1 - \infty$ principle of linear algebra.

Comments.

- We can make some general observations based on the above.
- Given a system of m equations in n variables, let r be the rank (i.e. the number of pivots) in the RREF of its augmented matrix.
- $r \leq \min\{m, n\}$.
- The system has a unique solution iff $r = n$.
- The system is consistent (i.e. has at least one solution) iff $r = m$ and no pivot occurs on the RHS.
- The number of free (arbitrary) variables is $n - r$.
- The general solution expresses the pivot variables as suitable constants plus certain combinations of the non pivot variables.